Homework 4

Math 471

1. Prove that if $\{g_n(x)\}$ is any system of L^2 functions, then $f \in \overline{\text{span}}\{g_n\}$ if and only if there is a sequence of functions $\{f_k\} \subseteq \text{span}\{g_n\}$ such that $\lim_{k\to\infty} \|f - f_k\|_2 = 0$. [Hint: For the "only if" part, choose $\{f_k\} \subseteq \text{span}\{g_n\}$ such that $\|f - f_k\|_2 < 1/k$].

2. Prove that if $\{g_n\} \subseteq L^2(I)$ is an orthonormal system on I, then for any $f \in L^2(I)$, the sequence
$$\left\{ \left\| f - \sum_{n=1}^{N} \langle f, g_n \rangle g_n \right\|_2 \right\}$$
is a decreasing sequence.

3. Let $\{g_n\} \subseteq L^2(I)$ be a complete orthonormal system of $L^2(I)$. Show that the orthonormal expansion (generalized Fourier series) of $f \in L^2(I)$ can be integrated term-by-term in the following sense. For any finite numbers $a < b$ such that $[a, b] \subseteq I$,
$$\int_a^b \sum_{n=1}^{\infty} \langle f, g_n \rangle g_n(x) dx = \sum_{n=1}^{\infty} \langle f, g_n \rangle \int_a^b g_n(x) dx.$$

4. If $f \in L^1$ is even, then $\hat{f}(\gamma) = \displaystyle\int_{-\infty}^{\infty} f(x) \cos(2\pi\gamma x) dx$. What happens if f is odd?

5. Excercise 3.5, and Excercise 3.6, [DW], page 63.

6. Let $f \in L^1$. What property of f will ensure that its Fourier transform $\hat{f}$ is real (not complex)?

7. Let $f,\ g \in L^1$, show that
$$|\hat{f}(\gamma)| \leq |\hat{g}(\gamma)| + \|f - g\|_1.$$

8. Exercise 3.10, [DW], page 64. [Hint: need to make use of Corollary 2.37].