Homework 4
Math 471

1. Prove that if \( \{g_n(x)\} \) is any system of \( L^2 \) functions, then \( f \in \text{span}\{g_n\} \) if and only if there is a sequence of functions \( \{f_k\} \subseteq \text{span}\{g_n\} \) such that \( \lim_{k \to \infty} \|f - f_k\|_2 = 0 \). [Hint: For the “only if” part, choose \( \{f_k\} \subseteq \text{span}\{g_n\} \) such that \( \|f - f_k\|_2 < 1/k \).]

2. Prove that if \( \{g_n\} \subseteq L^2(I) \) is an orthonormal system on \( I \), then for any \( f \in L^2(I) \), the sequence

\[
\left\{ \left\| f - \sum_{n=1}^{N} \langle f, g_n \rangle g_n \right\|_2 \right\}
\]

is a decreasing sequence.

3. Let \( \{g_n\} \subseteq L^2(I) \) be a complete orthonormal system of \( L^2(I) \). Show that the orthonormal expansion (generalized Fourier series) of \( f \in L^2(I) \) can be integrated term-by-term in the following sense. For any finite numbers \( a < b \) such that \( [a, b] \subseteq I \),

\[
\int_a^b \sum_{n=1}^{\infty} \langle f, g_n \rangle g_n(x)dx = \sum_{n=1}^{\infty} \langle f, g_n \rangle \int_a^b g_n(x)dx.
\]

4. If \( f \in L^1 \) is even, then \( \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) \cos(2\pi \gamma x)dx \). What happens if \( f \) is odd?

5. Exercise 3.5, and Exercise 3.6, [DW], page 63.

6. Let \( f \in L^1 \). What property of \( f \) will ensure that its Fourier transform \( \hat{f} \) is real (not complex)?

7. Let \( f, g \in L^1 \), show that

\[
|\hat{f}(\gamma)| \leq |\hat{g}(\gamma)| + \|f - g\|_1.
\]

8. Exercise 3.10, [DW], page 64. [Hint: need to make use of Corollary 2.37].