Homework 3

1. Let U be the coefficient map defined by a Bessel sequence $\{x_n\} \subseteq \mathcal{H}$. Show that the null space (kernel) of U, $\mathcal{N}(U) = \overline{span}\{x_n\}^{\perp}$.

2. Let $\mathcal{P}_{X,Y}$ be a nonorthogonal projection onto X along a complementary subspace Y (of X) in $\mathcal{H}$. Show that $\mathcal{P}^*_{X,Y} = \mathcal{P}_{\mathcal{Y}^\perp,\mathcal{X}^\perp}$, i.e., $\mathcal{P}^*_{X,Y}$ is also a projection.

3. Let $\mathcal{X} \equiv span\{u_1, u_2\} \subset \mathbf{C}^3$, where $u_1 =< 1, 0, 1 >$ and $u_2 =< 1, 0, 0 >$. Find three non-colinear vectors $\{x_1^*, x_2^*, x_3^*\}$ in $\mathcal{X}$. Find then a parametric formula for a set of PFFS-duals $\{x_n\}_{n=1}^3$.

4. (Optional) Can you find the most general formula for all PFFS-duals of the above problem.

5. Problem 1 is a PFFS example where $\{x_n^*\}$ is in $\mathcal{X}$, but $\{x_n\}$ is not. We could find an example where even $\{x_n^*\}$ is not in a subspace $\mathcal{X}$.

   Let $\mathcal{X} = span\{u_1\} \subset \mathbf{C}^3$ where u_1 is a line with the direction vector $< 1, -1, 1 >$ and passes through the point $(1, 2, 1)$.

   (a) Find a plane Q containing $\mathcal{X}$, and then select 3 vectors $\{x_n^*\}_{n=1}^3$ in plane Q.

   (b) Find the orthogonal complement of $\mathcal{X}$ in $\mathbf{C^3}$. [Hint: it should be a plane].

   (c) Find a parametric formula for PFFS-duals of $\{x_n^*\}$. [Hint: using the relationship theorem between PFFS for $\mathcal{X}$ and a frame of $\mathcal{X}$. Find a frame of $\mathcal{X}$ first, then find its dual, etc.]