Homework 3

1. Suppose that the 1-periodic function \( f(t) \) has a Fourier Series expansion, show that

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + \sum_{n=1}^{\infty} b_n \sin 2\pi nt,
\]

where

\[
a_n = 2 \int_{0}^{1} f(t) \cos 2\pi nt \, dt, \quad n = 0, 1, \ldots,
\]

and

\[
b_n = 2 \int_{0}^{1} f(t) \sin 2\pi nt \, dt, \quad n = 1, 2, \ldots.
\]

2. Show that the Dirichlet Kernel \( D_n(x) \) satisfies \( \int_{0}^{1} D_n(x) \, dx = 1 \). Show also that \( D_n(x) \) is real, and

\[
D_n(x) = \frac{\sin(2n + 1)\pi x}{\sin \pi x}.
\]

3. Let \( f(t) = \chi_{[-\frac{1}{4}, \frac{1}{4})}(t) \). Find the Fourier Series expansion of \( f \), both in terms of \( \{c(n)\} \) and \( \{a(n), b(n)\} \). [Compare with Example 2.13(a), [DW]]

4. Let \( f(t) = t\chi_{[-\frac{1}{4}, \frac{1}{4})}(t) \). Find the Fourier Series expansion of \( f \), both in terms of \( \{c(n)\} \) and \( \{a(n), b(n)\} \). [Compare with Example 2.13(c), [DW]]

5. The Fejer kernel is defined as

\[
K_n(t) = \frac{1}{n} \left( D_0(t) + D_1(t) + \cdots + D_{n-1}(t) \right).
\]

Show that

\[
K_n(t) = \frac{1}{n} \left( \frac{\sin \pi nt}{\sin \pi t} \right)^2.
\]

6. Let \( \delta \in (0, 1/2) \). Show that the Fejer Kernel \( K_n(t) \) \to 0 uniformly for \( \delta \leq |t| \leq 1/2 \). [Hint: It may be useful to use the estimate \( |\sin x| \leq |x| \).]