Homework 3

1. Suppose that the 1-periodic function $f(t)$ has a Fourier Series expansion, show that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + \sum_{n=1}^{\infty} b_n \sin 2\pi nt,$$

where

$$a_n = 2\int_0^1 f(t)\cos 2\pi nt dt, \qquad n = 0, 1, \cdots,$$

and

$$b_n = 2\int_0^1 f(t)\sin 2\pi nt dt, \qquad n = 1, 2, \cdots.$$

2. Show that the Dirichlet Kernel $D_n(x) \equiv \sum_{k=-n}^{n} e^{2\pi i n x}$ satisfies $\int_0^1 D_n(x)dx = 1$. Show also that $D_n(x)$ is real, and

$$D_n(x) = \frac{\sin(2n+1)\pi x}{\sin \pi x}.$$

3. Let $f(t) = \chi_{[-\frac{1}{4},\frac{1}{4})}(t)$. Find the Fourier Series expansion of f, both in terms of $\{c(n)\}$ and $\{a(n), b(n)\}$. [Compare with Example 2.13(a), [DW]]

4. Let $f(t) = t\chi_{[-\frac{1}{4},\frac{1}{4})}(t)$. Find the Fourier Series expansion of f, both in terms of $\{c(n)\}$ and $\{a(n), b(n)\}$. [Compare with Example 2.13(c), [DW]]

5. The Fejer kernel is defined as

$$K_n(t) = \frac{1}{n}\left(D_0(t) + D_1(t) + \cdots + D_{n-1}(t)\right).$$

Show that

$$K_n(t) = \frac{1}{n}\left(\frac{\sin \pi n t}{\sin \pi t}\right)^2.$$

6. Let $\delta \in (0, 1/2)$. Show that the Fejer Kernel $K_n(t) \to 0$ uniformly for $\delta \le |t| \le 1/2$. [Hint: It maybe useful to use the estimate $|\sin x| \le |x|$.]