# Homework 2

1. Show that if $\{x_n\}$ is an exact frame, then $\{x_n\}$ is a (Riesz) basis and $\{S^{-1}x_n\}$ is the unique biorthogonal sequence to $\{x_n\}$. [Hint: use Theorem 5.3.9 (or 5.4.7 in the old book)]

2. Show that if $\{x_n\}$ is a frame sequence with frame bounds $A$ and $B$, and $T : \mathcal{H} \to \mathcal{H}$ is a unitary operator, then $\{Tx_n\}$ is a frame sequence with the same frame bounds $A$ and $B$.

3. In Example 5.5.6, page 120 (or Example 5.5.6, page 98 in the old book), show that the unique biorthogonal sequence is given by

$$g_k = \begin{cases} \sum_{j=1}^{k} (-1)^j e_j, & k = \text{even}, \\ \sum_{j=1}^{k} (-1)^{j+1} e_j, & k = \text{odd} \end{cases}$$

   [You must provide necessary and/or complementary argument than that in the book for a really detailed proof.]

4. Let $\{e_k\}$ be an ONB of $\mathcal{H}$. Show that $\left\{ \frac{1}{\sqrt{2}} (e_{2k-1} + e_{2k}) \right\}$ is an ONB for $\overline{\text{sp}} \{e_{2k-1} + e_{2k}\}$, and that $\left\{ \frac{1}{\sqrt{2}} (e_{2k} + e_{2k+1}) \right\}$ is an ONB for $\overline{\text{sp}} \{e_{2k} + e_{2k+1}\}$. [Note: It is to be noted, however, we have already shown that $\{(e_{2k} + e_{2k+1})\}$ is neither a basis nor a frame of $\mathcal{H}$, though it is complete in $\mathcal{H}$. This example demonstrates that the union of basis for subspaces may not form a basis or a frame.]

5. Let $U : \mathcal{H} \to l^2$ be a coefficient map of a frame $\{x_n\}$, and let $V^*$ be a bounded left inverse of $U$. Show that the corresponding dual frame $\{\tilde{x}_n\}$ is given by

$$x_n^* = V^* e_n,$$

   where $\{e_n\}$ is the standard ONB of $l^2$.

6. Part (i) of Exercise 5.22,p136, (Exercise 3.4, p76 in the old book); Part (ii) is optional.