

Homework 2

1. Prove the following corollary: Suppose that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ where the series is uniformly convergent on $[a, b]$ and each f_k is integrable. Then f is integrable on $[a, b]$, and

$$\int_a^b f(x)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

2. Prove the following theorem: Suppose that $s(x) = \sum_{k=1}^{\infty} f_k(x)$ converges uniformly on an open interval $I = (a, b)$ and that each f_k is continuous on I . Fix $x_0 \in (a, b)$. Then the function g defined by

$$g(x) = \sum_{k=1}^{\infty} \int_{x_0}^x f_k(t)dt$$

is an anti-derivative for $s(x)$, i.e., $g'(x) = s(x)$ for all $x \in (a, b)$.

3. Prove the term-by-term derivative theorem for an infinite series of functions (as presented in class).
4. Consider a power series $\sum_{n=0}^{\infty} a_n x^n$. Let R be the extended real number defined by $R = 1/\limsup \sqrt[n]{|a_n|}$, where we treat $1/0 = \infty$ and $1/\infty = 0$.
 - (a) Show that $\limsup \sqrt[n]{|a_n x^n|} = |x|/R$.
 - (b) Show that the series converges uniformly absolutely on every bounded subinterval of $(-R, R)$ and diverges whenever $|x| > R$.
5. Show that the radius of convergence of a power series, defined above, is unchanged when the series is differentiated termwise.
6. Prove that a power series is uniformly absolutely convergent on every closed subinterval of $(-R, R)$ where R is the radius of convergence.