Homework 1

1. Verify that $u_1 = (0, 1)$, $u_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and $u_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ are a tight frame with frame bounds $A = B = \frac{3}{2}$.

2. Let $\{e_n\}$ be an orthonormal basis (ONB) of a separable Hilbert space $\mathcal{H}$. Verify that $\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}$ is a tight frame in $\mathcal{H}$ with $A = B = 1$.

3. Let $\{x_n\}$ be a frame for a separable Hilbert space $\mathcal{H}$. Let S be the corresponding frame operator defined by $Sf = \sum_n \langle f, x_n \rangle x_n$ for all $f \in \mathcal{H}$. Given that S is invertible, show that

$$\forall f \in \mathcal{H}, \quad f = \sum_n \langle f, x_n \rangle S^{-1} x_n = \sum_n \langle f, S^{-1} x_n \rangle x_n.$$

4. Let $\{x_n\}$ be a Bessel sequence in $\mathcal{H}$ (i.e., $\forall f \in \mathcal{H}$, $\sum_n |\langle f, x_n \rangle|^2 \leq B\|f\|^2$ for some $0 < B < \infty$). Define $U : \mathcal{H} \to l^2$ by $\forall f \in \mathcal{H}$, $Uf = \{\langle f, x_n \rangle\}_n$. Verify that the adjoint $U^* : l^2 \to \mathcal{H}$ of U is given by

$$\forall c \in l^2, \quad U^* c = \sum_n c(n) x_n.$$

5. Let $\{e_1, e_2\}$ be an orthonormal basis (ONB) of $\mathbf{C}^2$, and let $\{x_n\}_1^3 = \{e_1, e_2, -(e_1 + e_2)\}$. Find the coefficients $\{c_n\}_1^3$ of the minimum l^2-norm such that $e_1 = \sum_n c_n x_n$.