Homework 1

1. Verify that $u_1 = (0, 1)$, $u_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and $u_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ are a tight frame with frame bounds $A = B = \frac{3}{2}$.

2. Let $\{e_n\}$ be an orthonormal basis (ONB) of a separable Hilbert space $\mathcal{H}$. Verify that $\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}$ is a tight frame in $\mathcal{H}$ with $A = B = 1$.

3. Let $\{x_n\}$ be a frame for a separable Hilbert space $\mathcal{H}$. Let S be the corresponding frame operator defined by $Sf = \sum_n \langle f, x_n \rangle x_n$ for all $f \in \mathcal{H}$. We know that S is invertible, self-adjoint and positive.

   (a) Explain why S^{-1} is also self-adjoint.

   (b) Verify that $\langle S^{-1}f, f \rangle = \sum_n |\langle f, S^{-1}x_n \rangle|^2$.

4. Let $\{x_n\}$ be a frame for a separable Hilbert space $\mathcal{H}$. Let S be the corresponding frame operator defined by $Sf = \sum_n \langle f, x_n \rangle x_n$ for all $f \in \mathcal{H}$. Given that S is invertible, show that

$$\forall f \in \mathcal{H}, \quad f = \sum_n \langle f, x_n \rangle S^{-1}x_n = \sum_n \langle f, S^{-1}x_n \rangle x_n.$$

5. Let $\{e_1, e_2\}$ be an orthonormal basis (ONB) of $\mathbf{C}^2$, and let $\{x_n\}_1^3 = \{e_1, e_2, -(e_1 + e_2)\}$. Find the coefficients $\{c_n\}_1^3$ of the minimum l^2-norm such that $e_1 = \sum_n c_n x_n$.

6. Let $\{x_n\}$ be a frame for a finite dimensional vector space $\mathcal{H}$. Let S be the corresponding frame operator.

   (a) Find a matrix representation of S (assume that each element x_n is a column vector in $\mathcal{H}$), i.e., for each $f \in \mathcal{H}$, write $Sf = Mf$ for some matrix M.

   (b) Suppose that $\{x_n\}_1^m$ is a tight frame in $\mathcal{H}$, and $\|x_n\| = 1$ for all $n = 1, 2, \cdots, m$. Find the frame bound A. [Hint: read Section 1.2 and related linear algebra material.]

   (c) What are the eigenvalues of S in part (b)?