Homework 1

(1) Suppose $\{a_n\}$ and $\{b_n\}$ are positive sequences such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Prove that $\sum a_n$ converges if $\sum b_n$ converges.

(2) Let $\sum a_n$ and $\sum b_n$ be convergent series. Show that

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n.$$

(3) Suppose $\{f_n\}$ is a uniformly convergent sequence of continuous functions on $[0, 1]$. Show that $\{f_n\}$ must be bounded.

(4) Let $f_n(x) = e^{x^2/n}$. Show $\{f_n\}$ is uniformly convergent on every bounded interval, but is not uniformly convergent on $\mathbb{R}$.

(5) WADE, PG 191, #3] Suppose that $f_n \to f$ and $g_n \to g$ as $n \to \infty$, uniformly on some set $E \subseteq \mathbb{R}$.

  (a) Prove that $f_n + g_n \to f + g$ and $\alpha f_n \to \alpha f$ as $n \to \infty$ uniformly on E for all $\alpha \in \mathbb{R}$.
  (b) Prove that $f_n g_n \to fg$ pointwise on E.
  (c) Prove that if f and g are bounded on E, then $f_n g_n \to fg$ uniformly on E.
  (d) Show that (c) may be false when g is unbounded.