Homework 1

(1) Suppose \( \{a_n\} \) and \( \{b_n\} \) are positive sequences such that \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \). Prove that \( \sum a_n \) converges if \( \sum b_n \) converges.

(2) Let \( \sum a_n \) and \( \sum b_n \) be convergent series. Show that
\[
\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n.
\]

(3) Suppose \( \{f_n\} \) is a uniformly convergent sequence of continuous functions on \([0, 1]\). Show that \( \{f_n\} \) must be bounded.

(4) Let \( f_n(x) = e^{x^2/n} \). Show \( \{f_n\} \) is uniformly convergent on every bounded interval, but is not uniformly convergent on \( \mathbb{R} \).

(5) WADe, pg 191, #3] Suppose that \( f_n \to f \) and \( g_n \to g \) as \( n \to \infty \), uniformly on some set \( E \subseteq \mathbb{R} \).

(a) Prove that \( f_n + g_n \to f + g \) and \( \alpha f_n \to \alpha f \) as \( n \to \infty \) uniformly on \( E \) for all \( \alpha \in \mathbb{R} \).

(b) Prove that \( f_n g_n \to fg \) pointwise on \( E \).

(c) Prove that if \( f \) and \( g \) are bounded on \( E \), then \( f_n g_n \to fg \) uniformly on \( E \).

(d) Show that (c) may be false when \( g \) is unbounded.