FRAME FUNDAMENTAL SENSOR MODELING AND STABILITY OF ONE-SIDED FRAME PERTURBATION

SHIDONG LI AND DUNYAN YAN

Abstract. We demonstrate that for all linear devices and/or sensors, signal requisition and reconstruction is naturally a mathematical frame expansion and reconstruction issue, whereas the measurement is carried out via a sequence generated by the exact physical response function (PRF) h of the device, termed *sensory frame* $\{h_n\}$. The signal reconstruction, on the other hand, will be carried out using the dual frame $\{\tilde{h}_n^a\}$ of the estimated sensory frame $\{h_n^a\}$. This consequently results in an one-sided perturbation to a frame expansion. We show that the stability of such a one-sided frame perturbation exists. Such an one-sided perturbation to a frame expansion exists in each and every signal and image reconstruction problem. Examples of image reconstructions in de-blurring are demonstrated.

1. Introduction: frame fundamental description of signal requisition and reconstruction and one-sided frame perturbation

A frame in a separable Hilbert space $\mathcal{H}$ stands for a sequence $\{x_n\} \subseteq \mathcal{H}$ with which

$$\forall f \in \mathcal{H}, \qquad A\|f\|^2 \le \sum_n |\langle f, x_n\rangle|^2 \le B\|f\|^2,$$

where $0 < A \le B < \infty$ are some constants unrelated to specific $f \in \mathcal{H}$. A and B are called the lower and the upper frame bounds, respectively. Evidently, frames function much like a basis though redundancy is generally presented in frame systems, quite reminiscent of just about all natural devices and systems in physics and engineering. Mathematical fundamentals of frames, frame extensions and applications can be found in, e.g., [1], [2], [3], [4], [5], [6], [11], [12], [19], [20], [13], [15], [18], [21], [22], [23], [25] and [28].

Frame fundamental description of signal requisition and reconstruction. In almost all practical applications, the measurement of a signal is a projection of the function onto a subspace spanned by the (linear) measuring device. The spanning functions/vectors

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of such a device can be precisely and naturally modeled by the mathematical frame governed by the physics of the device, say a *sensory frame* $\{h_n\}$. This fact is a result of the mathematical Riesz representation theorem, e.g., [14], [27]. It can also be derived from the convolution output of a linear device. One can show that the sensory frame $\{h_n\}$, often times, is formed by translations of the spatial reversal of the sensor's impulse response function, e.g., [26]. Specifically, let r be the impulse response function of a linear device. Let f be the input signal. Then the output of the device is known to be

$$y(t) = \int f(\tau)r(t-\tau)d\tau = \int f(\tau)\overline{h(\tau-t)}d\tau = \langle f, h(\cdot - t)\rangle,$$

where $h \equiv \overline{r(-t)}$ is termed the *physical response function* (PRF) of the device. Since samples of a measurement are often of concern, the variable t is discrete, say, $t_n \in D$, the measurement of the sensor is therefore given by, with the sensory frame $h_n \equiv h(\cdot - t_n)$,

$$y(t_n) = \langle f, h_n\rangle, \quad t_n \in D$$

Here, we have also assumed that the device is shift-invariant, otherwise, the convolution kernel can be generally written as $r(t,\tau)$, and the resultant sensory frame $\{h_n = h(\cdot, t_n)\}$

Consequently, all measurement of a signal f out of a linear device is given by $\{\langle f, h_n\rangle\}$. Signal reconstructions in such a description follows naturally the *frame expansion.* It involves finding a dual frame sequence $\{\tilde{h}_n\}$ to the sensory frame $\{h_n\}$, followed by a linear combination as governed by the frame expansion

$$\sum_n \langle f, h_n\rangle \tilde{h}_n, \tag{1.1}$$

which gives rise to an optimal approximation of f in the linear span of $\{h_n\}$ - the best we can recover as governed by the sensor's physical principles. This is the frame fundamental description of signal requisition and reconstructions, which is applicable to each and every linear device, be it sampling gadgets, cameras, or various sensors etc, when signal recovery is concerned.

One-sided frame perturbation. However, it is safe to say that the sensory frame $\{h_n\}$ is never known precisely, even though the measurements are given by $\{\langle f, h_n\rangle\}$ (with the actual $\{h_n\}$). Training and measurement procedures are possible to learn merely an estimation of $\{h_n\}$, say $\{h_n^a\}$. The signal reconstruction process will then have to use the associated dual frame $\{\tilde{h}_n^a\}$ evaluated from the estimated sensory frame $\{h_n^a\}$. As a result, the reconstruction involves a pair of the actual (but unknown) sensory frame $\{h_n\}$ and an "unmatched" dual frame $\{\tilde{h}_n^a\}$, resulting in an *unbalanced frame expansion* of a function $f \in \mathbb{X} \equiv \overline{sp}\{h_n\}$ with *one-side perturbation*, namely,

$$\forall f \in \mathbb{X}, \qquad f^a = \sum_n \langle f, h_n\rangle \tilde{h}_n^a, \tag{1.2}$$

where we have assumed $\{\tilde{h}_n^a\}$ is a dual frame of the perturbed/estimated frame $\{h_n^a\}$ of the unknown frame $\{h_n\}$, and $\{h_n^a\}$ can be viewed as resulting from a sufficiently small perturbation of $\{h_n\}$ so that $\{h_n^a\}$ remains a frame in $\mathbb{X} = \overline{sp}\{h_n\}$. Unlike (1.1), f^a is understandably merely an approximation of f via the unbalanced frame expansion (1.2).

The stability of (1.2) is therefore of concern. The rest of this article studies stability conditions with which f^a, as in (1.2), is guaranteed a reasonable approximation to the original signal f.

Also presented is the study and numerical experiments in an image reconstruction/recovery problem that heavily involve such unbalanced stability issues. These examples support such stability well.

Note that the stability issues studied here are largely different from the stability of frame perturbation theories which concerns with conditions where a perturbed frame sequence remains a frame in the underlying space. Details on such studies can be found in, e.g., [7], [8], [9], [10], [17] and [16], etc. We shall be assuming that those conditions are warranted so that the estimated $\{h_n^a\}$ remains a frame in $\mathbb{X} \equiv \overline{span}\{h_n\}$ in this study.

2. Stability of one-sided perturbation to frame expansions

As we noted above, the creation of one-sided perturbation to frame expansions (1.2) (or unbalanced frame expansions) lies entirely in practical applications where the requisition of a signal f is given a priori by the sensory frame $\{h_n\}$ in the form of the "sampling value" $c_n = \langle f, h_n\rangle$. Though $\{c_n\}$ is observed via the exact sensory frame $\{h_n\}$, $\{h_n\}$ is never exactly known - causing the unbalance. Reconstruction can never be based on the exact $\{h_n\}$, but on an estimated frame $\{h_n^a\}$, a perturbed version of $\{h_n\}$.

The question is how certain we are that such a reconstruction as in (1.2) is reliable?

As a dual problem, unbalanced frame expansion could have the following form mathematically

$$\forall f \in \mathbb{X}, \qquad \tilde{f} = \sum_n \langle f, \tilde{h}_n^a\rangle h_n, \tag{2.3}$$

where the approximation $\tilde{f}$ in (2.3) needs not necessarily be the same as f^a of (1.2).

We shall begin with the stability study of (2.3) first, followed by the (stability) study of (1.2). We shall also point out in a later remark that the conditions of the next two theorems fit the physical problems.

Theorem 1. *Let $\{h_n\}$ be a frame of $\mathbb{X}$, and let $h_n^a = h_n + \varepsilon_n$, where $\|\varepsilon_n\| < \delta$ and $\delta > 0$ is fixed and sufficiently small so that $\{h_n^a\}$ remains a frame of $\mathbb{X}$. Assume further that $\{\tilde{h_n^a}\}$ is a dual frame of $\{h_n^a\}$, and an unbalanced frame expansion is given by (2.3). Suppose $\{\varepsilon_n\}$ is almost orthogonal in the sense that $\langle \varepsilon_m, \varepsilon_n\rangle = 0$ for $|m-n| > K$ with some positive integer $K \geq 1$. Then*

$$\forall f \in \mathbb{X}, \qquad \|f - \tilde{f}\|_2 < \delta(2K+1)^{1/2}\|f\|_2.$$

Proof. For all $f \in \mathbb{X}$,

$$f = \sum_n \langle f, \tilde{h}_n^a \rangle h_n^a.$$

Consequently,

$$f - \tilde{f} = \sum_n \langle f, \tilde{h_n^a} \rangle \left(h_n^a - h_n \right) = \sum_n \langle f, \tilde{h_n^a} \rangle \varepsilon_n.$$

It follows that

$$\|f - \tilde{f}\|_2^2 = \sum_n \sum_m \langle f, \tilde{h_n^a} \rangle \overline{\langle f, \tilde{h_m^a} \rangle} \langle \varepsilon_n, \varepsilon_m \rangle.$$

Note that $\{\varepsilon_n = h_n^a - h_n\}$ is almost orthogonal, that is, for any $n \in \mathbb{Z}$, there exists a positive integer $K \geq 1$ independent of n such that, whenever $|m - n| > K$,

$$\langle \varepsilon_n, \varepsilon_m \rangle = 0.$$

Consequently, for fixed $n \in \mathbb{Z}$, we have

$$\begin{aligned} \left| \sum_m \langle f, \tilde{h_n^a} \rangle \overline{\langle f, \tilde{h_m^a} \rangle} \langle \varepsilon_n, \varepsilon_m \rangle \right| &= \left| \sum_{|m-n| \leq K} \langle f, \tilde{h_n^a} \rangle \overline{\langle f, \tilde{h_m^a} \rangle} \langle \varepsilon_n, \varepsilon_m \rangle \right| \\ &\leq \frac{\delta^2}{2} \sum_{m=n-K}^{n+K} |\langle f, \tilde{h_m^a} \rangle|^2 + \frac{\delta^2}{2} (2K+1) |\langle f, \tilde{h_n^a} \rangle|^2. \end{aligned}$$

Therefore,

$$\|f - \tilde{f}\|_2^2 \leq \delta^2 (2K+1) \sum_n |\langle f, \tilde{h_n^a} \rangle|^2 \leq (2K+1) \delta^2 \|f\|_2^2.$$

Or

$$\|f - \tilde{f}\|_2 \leq \delta (2K+1)^{1/2} \|f\|_2.$$

□

The stability of such unbalanced frame expansions also holds for ε_n that decays sufficiently fast.

Theorem 2. *Let* $\{h_n\}$ *be a frame of* X*, and let* $h_n^a = h_n + \varepsilon_n$*, where* $\|\varepsilon_n\| < \delta$ *and* $\delta > 0$ *is fixed and sufficiently small so that* $\{h_n^a\}$ *remains a frame of* $\mathbb{X}$*. Assume further that* $\{\tilde{h_n^a}\}$ *is a dual frame of* $\{h_n^a\}$*, and an unbalanced frame expansion is given by (2.3). Suppose* $\{\varepsilon_n\}$ *decays reasonably fast such that*

$$|\langle \varepsilon_n, \varepsilon_m \rangle| \leq \frac{\delta^2}{r^{|n-m|}}$$

for some $r > 1$*. Then, there exists a finite constant* $C > 0$ *such that*

$$\forall f \in \mathbb{X}, \qquad \|f - \tilde{f}\|_2 < C\delta \|f\|_2.$$

Proof. For fixed $n \in \mathbb{Z}$, we have

$$\begin{aligned} \left| \sum_m \langle f, \tilde{h}_n^a \rangle \overline{\langle f, \tilde{h}_m^a \rangle} \langle \varepsilon_n, \varepsilon_m \rangle \right| &\leq \left(\sum_{m=n} + \sum_{m>n} + \sum_{m<n} \right) \left| \langle f, \tilde{h}_n^a \rangle \langle f, \tilde{h}_m^a \rangle \langle \varepsilon_n, \varepsilon_m \rangle \right| \\ &\equiv I(n) + II(n) + III(n). \end{aligned}$$

a) For the first term, $I(n) = \left| \langle f, \tilde{h}_n^a \rangle \right|^2 \|\varepsilon_n\|^2$. Hence,

$$\sum_n I(n) \leq B\delta^2 \|f\|^2.$$

b) For the middle term $II(n)$,

$$\begin{aligned} II(n) &= \sum_{m>n} \left| \langle f, \tilde{h}_n^a \rangle \langle f, \tilde{h}_m^a \rangle \langle \varepsilon_n, \varepsilon_m \rangle \right| \\ &\leq \sum_{m>n} \left| \langle f, \tilde{h}_n^a \rangle \langle f, \tilde{h}_m^a \rangle \right| \frac{\delta^2}{r^{m-n}}. \end{aligned}$$

As a result,

$$\begin{aligned} \sum_n II(n) &\leq \sum_n \sum_{m>n} \left| \langle f, \widetilde{h_n^a} \rangle \right| \left| \langle f, \widetilde{h_m^a} \rangle \right| \frac{\delta^2}{r^{m-n}} \\ &\leq \sum_n \sum_{m>n} \frac{1}{2} \left(\left| \langle f, \widetilde{h_n^a} \rangle \right|^2 + \left| \langle f, \widetilde{h_m^a} \rangle \right|^2 \right) \frac{\delta^2}{r^{m-n}} \\ &= \frac{1}{2} \sum_n \sum_{m>n} \left| \langle f, \widetilde{h_n^a} \rangle \right|^2 \frac{\delta^2}{r^{m-n}} + \frac{1}{2} \sum_n \sum_{m>n} \left| \langle f, \widetilde{h_m^a} \rangle \right|^2 \frac{\delta^2}{r^{m-n}} \\ &\equiv \frac{1}{2} \left(II_1 + II_2 \right). \end{aligned}$$

Here

$$II_1 = \sum_n \sum_{m>n} \left| \langle f, \widetilde{h_n^a} \rangle \right|^2 \frac{\delta^2}{r^{m-n}} \leq \sum_n \left| \langle f, \widetilde{h_n^a} \rangle \right|^2 \frac{\delta^2}{r-1} \leq \frac{B\delta^2}{r-1} \|f\|^2,$$

and,

$$\begin{aligned} II_2 &= \sum_n \sum_{m>n} \left| \langle f, \widetilde{h_m^a} \rangle \right|^2 \frac{\delta^2}{r^{m-n}} = \sum_m \sum_{n<m} \left| \langle f, \widetilde{h_m^a} \rangle \right|^2 \frac{\delta^2}{r^{m-n}} \\ &= \sum_m \left| \langle f, \widetilde{h_m^a} \rangle \right|^2 \frac{\delta^2}{r-1} \leq \frac{B\delta^2}{r-1} \|f\|^2. \end{aligned}$$

Together, it follows that

$$\sum_n II(n) = \frac{1}{2} \left(II_1 + II_2 \right) \leq \frac{B\delta^2}{r-1} \|f\|^2.$$

c) For the last term $III(n)$,

$$\begin{aligned} III(n) &= \sum_{m<n} \left| \langle f, \tilde{h}^a_n \rangle \langle f, \tilde{h}^a_m \rangle \langle \varepsilon_n, \varepsilon_m \rangle \right| \\ &\leq \sum_{m<n} \left| \langle f, \tilde{h^a_n} \rangle \langle f, \tilde{h^a_m} \rangle \right| \frac{\delta^2}{r^{n-m}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_n III(n) &\leq \sum_n \sum_{m<n} \left| \langle f, \tilde{h^a_n} \rangle \right| \left| \langle f, \tilde{h}^a_m \rangle \right| \frac{\delta^2}{r^{n-m}} \\ &\leq \sum_n \sum_{m<n} \frac{1}{2} \left(\left| \langle f, \tilde{h^a_n} \rangle \right|^2 + \left| \langle f, \tilde{h}^a_m \rangle \right|^2 \right) \frac{\delta^2}{r^{n-m}} \\ &= \frac{1}{2} \sum_n \sum_{m<n} \left| \langle f, \tilde{h}^a_n \rangle \right|^2 \frac{\delta^2}{r^{n-m}} + \frac{1}{2} \sum_n \sum_{m<n} \left| \langle f, \tilde{h}^a_m \rangle \right|^2 \frac{\delta^2}{r^{n-m}} \\ &\equiv \frac{1}{2} \left(III_1 + III_2 \right). \end{aligned}$$

Calculating III_1 and III_2 separately, we have

$$\begin{aligned} III_1 &= \sum_n \sum_{m<n} \left| \langle f, \tilde{h}^a_n \rangle \right|^2 \frac{\delta^2}{r^{n-m}} \\ &\leq \sum_n \left| \langle f, \tilde{h}^a_n \rangle \right|^2 \frac{\delta^2}{r-1} \\ &\leq \frac{B\delta^2}{r-1} \|f\|^2. \end{aligned}$$

and,

$$\begin{aligned} III_2 &= \sum_n \sum_{m<n} \left| \langle f, \tilde{h^a_m} \rangle \right|^2 \frac{\delta^2}{r^{n-m}} \\ &= \sum_m \sum_{n>m} \left| \langle f, \tilde{h^a_m} \rangle \right|^2 \frac{\delta^2}{r^{n-m}} \\ &= \sum_m \left| \langle f, \tilde{h^a_m} \rangle \right|^2 \frac{\delta^2}{r-1} \\ &\leq \frac{B\delta^2}{r-1} \|f\|^2. \end{aligned}$$

Together,

$$\sum_n III(n) = \frac{1}{2} \left(III_1 + III_2 \right) \leq \frac{B\delta^2}{r-1} \|f\|^2.$$

Here, the constant B appearing in the computations is the upper frame bound of the sequence involved. Consequently,

$$\|f-\tilde{f}\|_2^2=\sum_n\sum_m\langle f,\tilde{h}_n^a\rangle\overline{\langle f,\tilde{h}_m^a\rangle}\langle\varepsilon_n,\varepsilon_m\rangle\le\frac{r+1}{r-1}B\delta^2\|f\|^2.$$

Or,

$$\|f-\tilde{f}\|_2\le C\delta\|f\|.$$

This proves the desired result. □

Remark In either *Theorem 1* or *Theorem 2*, the conditions about the almost orthogonality or about the decay requirement to ε_n are all practically easy to meet in the sensory frame estimation. Because sensory frames $\{h_n\}$ are typically generated by translations of the physical response function h, namely, $h_n=h(\cdot-n)$. Being an physical (impulse) response function, h decays very fast. As a result, the essential support of h is very small. Consequently, the estimation error ε_n possesses similar support and decay properties of that of h. Relative shifts of ε_m and ε_n can easily cause overlap to (essentially) disappear and thereby satisfy the almost orthogonality requirement (approximately). The decay requirement to ε_n as in *Theorem 2* would be even easier to accomplish, as again, the relative shifts between ε_m and ε_n causes their relative overlaps to diminish fast.

As a result, the stability of the "dual" expression (2.3) of the unbalanced frame expansion occurring in signal reconstructions generally holds.

We are still to establish the stability of the unbalanced frame expansion (1.2). Without having to go through a direct estimation of the stability of (1.2), we may apply the results of *Theorems 1* and *2* to infer the stability results of (1.2).

Theorem 3. *Let $\{h_n\}$ be a frame of $\mathbb{X}$. Let $h_n^a=h_n+\varepsilon_n$, where $\|\varepsilon_n\|<\delta$ and $\delta>0$ is fixed and sufficiently small so that $\{h_n^a\}$ remains a frame of $\mathbb{X}$. Assume further that $\{\tilde{h}_n^a\}$ is a dual frame of $\{h_n^a\}$. Suppose $\{\varepsilon_n\}$ is almost orthogonal as stated in Theorem 1, or it decays sufficiently fast as stated in Theorem 2. Then, the unbalanced frame expansion $f^a=\sum_n\langle f,h_n\rangle\tilde{h}_n^a$ is also stable, i.e., for some finite constant $C>0$,*

$$\forall f\in\mathbb{X},\qquad\|f-f^a\|_2<C\delta\|f\|_2.$$

Proof. With the assumptions on $\{\varepsilon_n\}$, we know that the unbalanced frame expansion $\tilde{g}$ of g as in (2.3) is stable, i.e., for some finite constant $C>0$,

$$\forall g\in\mathbb{X},\qquad\|g-\tilde{g}\|_2<C\delta\|g\|_2.\tag{2.4}$$

Now, for any f, $g\in\mathbb{X}$, and the expressions f^a in (1.2), and $\tilde{g}$ in (2.3), we have

$$f-f^a=\sum_n\langle f,h_n^a\rangle\tilde{h}_n^a-\sum_n\langle f,h_n\rangle\tilde{h}_n^a=\sum_n\langle f,\varepsilon_n\rangle\tilde{h}_n^a,$$

and,

$$\begin{aligned}\langle f-f^a,g\rangle &= \left\langle \sum_n \langle f,\varepsilon_n\rangle \tilde{h}_n^a, g\right\rangle\\ &= \left\langle f, \sum_n \langle g,\tilde{h}_n^a\rangle \varepsilon_n\right\rangle\\ &= \langle f, g-\tilde{g}\rangle.\end{aligned}$$

Thus, for $g=f-f^a$, the last equation shows

$$\begin{aligned}\|f-f^a\|_2^2 &\le \|f\|_2\cdot \left\|(f-f^a)-(\widetilde{f-f^a})\right\|_2\\ &\le \|f\|_2\cdot C\delta\|f-f^a\|_2,\end{aligned}$$

where the assumption (2.4) was applied to the last inequality. Consequently,

$$\|f-f^a\|_2\le \delta C\|f\|_2.$$

□

3. SIGNAL RECONSTRUCTION AND/OR RECOVERY

As an application of the stability study of one-sided frame perturbations, we present in this section an example in signal reconstructions out of, perhaps, distorted measurements of an original signal. Here, the one-sided perturbation can be seen explicitly. We will use "deblurring" as an example - assuming that the signal measurement is blurred, a reconstruction is intended to recover an approximation of the original signal as much as possible.

Blurred measurement of a signal/image is due to an "out-focus" PRF of the camera. Take image f for instance. The blurred image is given by

$$\{f(m,n)\equiv\langle f,h(\cdot-m,\cdot-n)\rangle\},\tag{3.5}$$

where, as described earlier, h is the PRF of the camera. Consequently, the sensory frame in this case is given by $\{\tau_{m,n}h\equiv h(\cdot-m,\cdot-n)\}$. Theoretically, we ought to find a dual frame $\{\tau_{m,n}\tilde{h}\}$ of $\{\tau_{m,n}h\}$, and perform signal reconstruction via a frame expansion

$$\forall f\in\overline{sp}\{\tau_{m,n}h\},\qquad f=\sum_{m,n}\langle f,\tau_{m,n}h\rangle\tau_{m,n}\tilde{h}.\tag{3.6}$$

It is also known that there is a class of infinite many dual frames of $\{\tau_{m,n}h\}$ that are of the translation structure [24]. Consequently, the dual frame used in the previous expression is directly written as $\{\tau_{m,n}\tilde{h}\}$ without pre-qualification.

In practical consideration, h is never known precisely, even though the actual measurement is given by the exact h via (3.5). Signal reconstruction and recovery (thereby deblurring) via (3.6) may only be approximated using "unbalanced frame expansion" (1.2) as described in this article. Translating to image reconstruction scenario, (1.2) becomes the following unbalanced frame expansion

$$\forall f \in \overline{sp}\{\tau_{m,n}h\}, \qquad f^a = \sum_{m,n} \langle f, \tau_{m,n}h\rangle \tau_{m,n}\tilde{h}^a. \tag{3.7}$$

The numerical experiments have the following procedures. First, an estimation h^a of h will be obtained. Assume that h^a is a sufficiently small perturbation to h as specified in *Theorem 1* and *Theorem 2* so that $\{\tau_{m,n}h^a\}$ still forms frame of $\{\tau_{m,n}h\}$. Then a dual frame sequence $\{\tau_{m,n}\tilde{h}^a\}$ to the estimated sensory frame $\{\tau_{m,n}h^a\}$ is evaluated, followed by a signal reconstruction/de-blurring process as in (3.7).

Shown in *Figure 3.1* is a blurred measurement of an image. *Figures 3.2, 3.3* and *3.4* are the dual frame waveforms $\tilde{h}^a$ corresponding to the three estimations h^a of h. One can see visually that the three dual frame waveforms are clearly different. *Figures 3.5, 3.6* and *3.7* are the corresponding reconstructed images through the three different dual frame sequences $\tilde{h}^a$, respectively, via (3.7).

These examples demonstrate that with reasonable approximations h^a of the camera's PRF h, all 3 image recovery results are sound.

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Department of Mathematics, San Francisco State University, San Francisco, CA 94132, U. S. A.

E-mail address: `shidong@sfsu.edu`

Department of Information Sciences, Graduate University of the Chinese Academy of Sciences, Beijing, 100049, P. R. C.

E-mail address: `ydunyan@gucas.ac.cn`

Figure 4.1. A blurred observation of the Mandrill

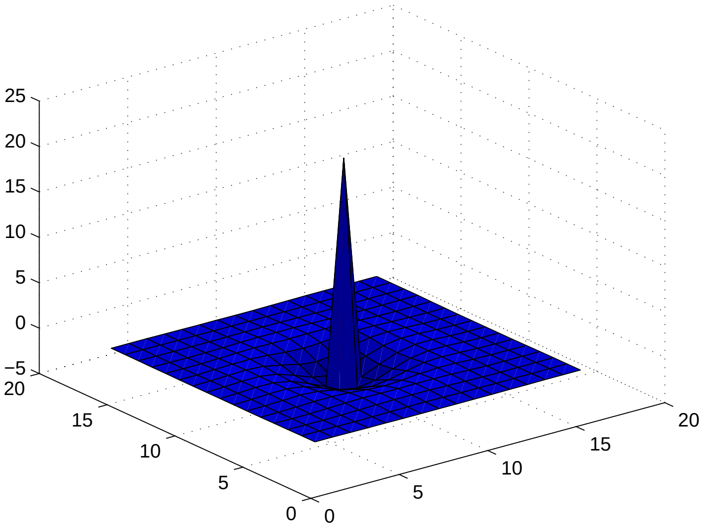

FIGURE 4.2. One dual frame $\tilde{h}^a$ corresponding to an estimated camera frames h^a

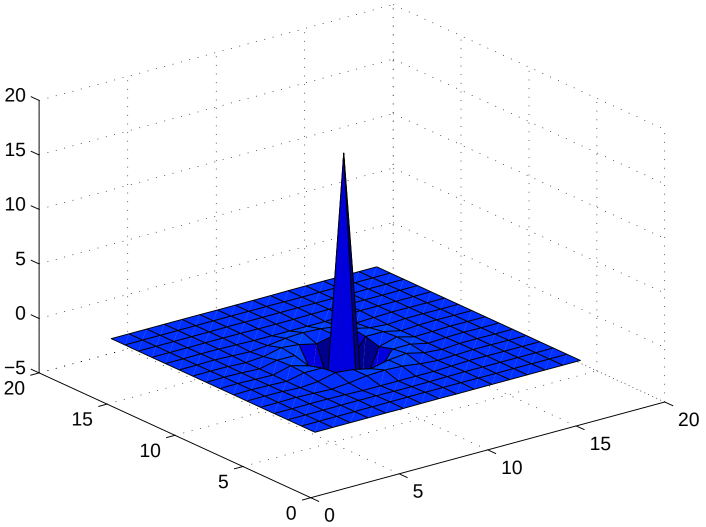

FIGURE 4.3. Another dual frame $\tilde{h}^a$ corresponding to another estimated camera frames h^a

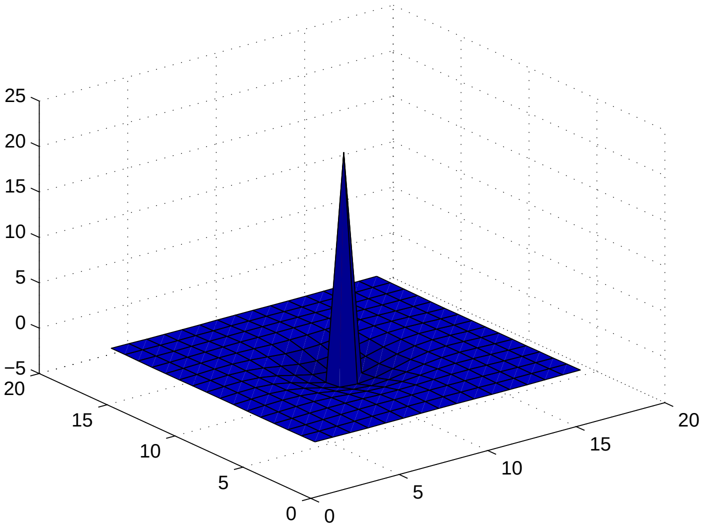

FIGURE 4.4. The 3^{rd} dual frame $\tilde{h}^a$ corresponding to the 3^{rd} estimated camera frames h^a

FIGURE 4.5. A deblurred image using estimated h^a and its dual $\tilde{h}^a$ as in *Figure 3.2.*

FIGURE 4.6. A deblurred image using estimated h^a and its dual $\tilde{h}^a$ as in *Figure 3.3.*

FIGURE 4.7. A deblurred image using estimated h^a and its dual $\tilde{h}^a$ as in *Figure 3.4*.