Chapter 3. **Biorthogonal Wavelets and Filter Banks via PFFS**

§3.0 PFFS applied to shift-invariant subspaces

Definition: X is a shift-invariant subspace if h $\in$ X $\Rightarrow$ h$(\cdot - n) \equiv \tau_n h \in$ X.

Ex: Multiresolution Analysis (MRA) subspaces V_j, and wavelet subspaces W_j. $\forall f \in$ X $\sum_n \langle f, \tau_n \phi \rangle \tau_n \phi^*$.

Theorem: Let X be a shift-invariant subspace. Let $\{\tau_n \phi^*\}$ be a Bessel sequence in H such that $\overline{sp}\{\tau_n \phi^*\} \supseteq$ X. Then there exists PFFS-duals $\{\phi_n\} \subseteq$ H (in the sense that f $= \sum_n \langle f, \phi_n \rangle \tau_n \phi^*$ for all f $\in$ X) such that,

$$\phi_n = \tau_n \phi; \qquad \phi = \phi^0 + z,$$

where $\phi^0 \in X$, & $z \in X^{\perp}$.

Remark: The importance of this result is that we now know that PFFS duals ϕ_n can be generated by the translations of a function ϕ.

$\Rightarrow$ We will be using this result to study the construction of Bi-wavelets via PFFS.

§3.1 Construction of Bi-wavelets (Biorthogonal) wavelets and associated filter banks via PFFS.

1) Assume we have a biorthogonal MRA in which $\{\phi(t-n)\}_n$ is an exact frame (Riesz basis) of $V_0 \equiv \overline{sp}\{\phi(t-n)\}$. Then there exists a unique bi-dual $\{\phi(t-n)\} \subseteq V_0$ s.t.

$$\forall f \in V_0, \qquad f = \sum_n \langle f, \phi^0(t-n) \rangle \phi(t-n)$$

PFFS, other PFFS-biorthogonal duals of $\{\phi(t-n)\}$ can be written as

$$\tilde{\phi} = \phi^0 + z, \qquad z \in {V_0}^{\perp}$$

Assume that the following scaling equations hold:

$$\tilde{\phi} = \sum_n \tilde{h}_n \sqrt{2} \tilde{\phi}(2t-n) = \sum_n \tilde{h}_n \tilde{\phi}_{1n}$$

and

$$\phi^0 = \sum_n {h_n}^0 \sqrt{2}\phi^0(2t-n) = \sum_n {h_n}^0 {\phi_{1n}}^0$$

Using the biorthogonality of ϕ and $\tilde{\phi}$

$$\begin{aligned}
\tilde{h}_m &= \langle \tilde{\phi},\, \phi_{1m} \rangle \\
&= \langle \sum_n \tilde{h}_n \tilde{\phi}_{1n},\, \phi_{1m} \rangle, \qquad \langle \tilde{\phi}_{1n},\, \phi_{1m} \rangle = \delta_{mn} \\
&= \langle \phi^0 + z, \phi_{1m} \rangle \\
&= \langle \phi^0,\, \phi_{1m} \rangle + \langle z,\, \phi_{1m} \rangle \\
&= {h_m}^0 + \Delta h_m.to
\end{aligned}$$

<u>Note</u>: $\Delta h_m = \langle z,\, \phi_{1m} \rangle = \langle\, z_0 + z_1 + z_2 + \cdots,\, \phi_{1m} \rangle$, where $z_i \in W_i$, $(V_i \bigoplus W_i = V_{i+1},\ \ \phi_{1m} \in \text{V},$ and Δh_m is a sequence).

Therefore, $\Delta h_m = \langle\, z_0,\ \phi_{1m} \rangle + \sum_{i=1}^{\infty} \langle z_i,\, \phi_{1m} \rangle$
$= \langle\, z_0,\ \phi_{1m} \rangle \quad Note : \sum_{i=1}^{\infty} \langle z_i,\, \phi_{1m} \rangle = 0$

The secondary component takes information from W_0 ("high pass" subspace) back into the "low-pass" band.

2) Biorthogonal Principle:

<u>Claim</u>: If $\{h_n\}$ is the filter sequence generating the same ϕ then $\Delta\{h_n\}$ satisfy the following biorthgonal principle.

$$\sum_n h_n \overline{\Delta h_{n-2k}} = 0, \qquad \forall k$$

To understand this, let's recall the usual biorthogonal relationship (as illustrated below):

Recall:
In time domain, the biorthogonal filter bank systems with perfect reconstruction requirement is given by

$$\sum_n (h_{k-2n}\tilde{h}_{l-2n} + g_{k-2n}\tilde{g}_{l-2n}) = \delta_{kl} \tag{3}$$

(Cohen, Daubechies, and Feauveau '92)

Working out the Fourier Transform of the identity (3), we will have the equivalent relationship in the frequency domain

$$\begin{aligned} H(\gamma)\overline{\tilde{H}(\gamma)} + G(\gamma)\overline{\tilde{G}(\gamma)} &= 2 \\ H(\gamma)\overline{\tilde{H}(\gamma+\tfrac{1}{2})} + G(\gamma)\overline{\tilde{G}(\gamma+\tfrac{1}{2})} &= 0 \end{aligned} \tag{4}$$

Furthermore,

$$G(\gamma) = e^{-2\pi i\gamma}\overline{\tilde{H}(\gamma+\frac{1}{2})} = e^{-2\pi i\gamma}\overline{(H^0(\gamma+\frac{1}{2}) + \Delta H(\gamma+\frac{1}{2}))} \tag{5}$$

Note: $G^0 = e^{-2\pi i\gamma}H^0(\gamma+\frac{1}{2})$

$$\tilde{G}(\gamma) = e^{-2\pi i\gamma}\overline{H(\gamma+\frac{1}{2})} \tag{6}$$

Recall also that for biorthogonal multiresolution analysis:

$$0 < A \le |H(\gamma)|^2 + |H(\gamma+\frac{1}{2})|^2 \le B.$$

Note that (5) and (6) are derived from the PRFB relationships (4) and (7).

Because of (5) and (6), the biorthogonal relationship of (3) can be written as

$$\sum_n h_n\overline{\tilde{h}_{n-2k}} = \delta_{k0} \tag{9}$$

$$\begin{aligned} \Rightarrow \quad & \sum_n h_n(\overline{h^0{}_{(n-2k)} + \Delta h_{n-2k}}) = \delta_{k0} \\ \Rightarrow \quad & \underbrace{\sum_n h_n\overline{h^0{}_{(n-2k)}}}_{\delta_{k0}} + \underbrace{\sum_n h_n\overline{\Delta h_{n-2k}}}_{0,\ \forall\ k} = \delta_{k0} \end{aligned}$$

therefore,

$$\sum_n h_n \overline{\Delta h_{n-2k}} = 0, \qquad \forall k \tag{10}$$

Take the Fourier Transform of (10): $\sum_k (\sum_n h_n \overline{\Delta h_{n-2k}}) e^{-2\pi i k \gamma} = 0$, we have

$$H(\gamma)\overline{\Delta H(\gamma)} \ + \ H(\gamma + \frac{1}{2})\overline{\Delta H(\gamma + \frac{1}{2})} = 0 \tag{11}$$

where $H(\gamma) = \sum_n h_n e^{-2\pi i n \gamma}$ and $\Delta H(\gamma) = \sum_n \Delta h_n e^{-2\pi i n \gamma}$.

§3.2. Solution to ΔH

Assume we are interested in a finite length sequences $\{h_n\}, \{\tilde{h_n}\}$ etc., $H(\gamma), \tilde{H}(\gamma)$ etc. can be written as a product of Trig. polynomials with a power of $e^{-2\pi i \gamma}$. From (11), we see that $\Delta H(\gamma)$ must be zero whenever $H(\gamma + \frac{1}{2})$ is zero, and because $\Delta H(\gamma)$ is a product of a trig polynomial with a power of $e^{-2\pi i \gamma}$

$$\Delta H(\gamma) = H(\gamma + \frac{1}{2})\hat{q}(\gamma), \tag{12}$$

where $\hat{q}(\gamma)$ is a trig polynomial.

Take (12) into (11), we have

$$H(\gamma)H(\gamma + \frac{1}{2})\hat{q}(\gamma) + H(\gamma + \frac{1}{2})H(\gamma)\hat{q}(\gamma + \frac{1}{2}) = 0 \iff$$

$$H(\gamma)H(\gamma + \frac{1}{2})[\hat{q}(\gamma) + \hat{q}(\gamma + \frac{1}{2})] = 0$$

Assuming that $H(\gamma)$ is such that $H \neq 0$ a.e. expect for at $\gamma = \frac{1}{2}$, we have therefore

$$\hat{q}(\gamma) = \hat{p}(2\gamma)\cos 2\pi N \gamma$$

where $N = odd$. This is because

$$\begin{aligned} \hat{q}(\gamma) + \hat{q}(\gamma + \frac{1}{2}) &= \hat{p}(2\gamma)\cos 2\pi N\gamma + \hat{p}(2\gamma + 1)\cos(2\pi N\gamma + N\pi) \\ &= \hat{p}(2\gamma)\cos 2\pi N\gamma - \hat{p}(2\gamma)\cos(2\pi N\gamma) = 0 \end{aligned}$$

Note: $\hat{p}(\gamma)$ is an arbitrary trig polynomial. In general, $\hat{q}(\gamma) = \hat{p}(2\gamma)e^{-2\pi i N\gamma}$, $N =$ *odd*. Typically, for $N = 1$

$$\begin{aligned} \hat{q}(\gamma) &= \hat{p}(2\gamma)\cos 2\pi\gamma \\ \text{or} \quad \hat{q}(\gamma) &= \hat{p}(2\gamma)e^{-2\pi i\gamma} \end{aligned}$$

3)The changes in the filtering System:

Remark: The corresponding $\tilde{\varphi}, \psi$ and $\tilde{\psi}$ are changed (even though $\tilde{G}$ didn't change).

§3.3 Construction of <u>linear phase</u> FIR $\tilde{H}$ through $\Delta\tilde{H}$

1) Linear phase: $\tilde{H}(\gamma) = e^{-2\pi i\lambda\gamma}|H(\gamma)|, \qquad \lambda \in \mathbf{Z}$

$\Rightarrow$ reduces to "symmetric" $\tilde{H}$

$$\begin{aligned} (1) \qquad & \tilde{H}(-\gamma) = \tilde{H}(\gamma) \leftrightarrow \{\tilde{h}_n\}\text{is real} \\ & \leftrightarrow \tilde{\varphi}(-t) = \tilde{\varphi}(t) \\ (2) \qquad & \tilde{\varphi}(1-t) = \tilde{\varphi}(t) \leftrightarrow \hat{\varphi}(-\gamma) = e^{2\pi i\gamma}\hat{\varphi}(\gamma) \\ & \text{or } \tilde{H}(-\gamma) =^{2\pi i\gamma} \tilde{H}(\gamma) \end{aligned}$$

2) How to increase the vanishing moment of $\tilde{\varphi}$.

<u>**Thm:**</u> Let $\psi(t)$ be a bi-wavelet function generated by a wavelet equation ($\psi(t) = \sum_n g_n\varphi_{1n}(t)$). Assume $\hat{\varphi}(\gamma)$ is N time continuously differentiable at $\gamma = 0$. Then the following are equivalent:

(1) $\qquad \psi$ has N vanishing moments ($\int_{\mathbf{R}} t^n\psi(t)dt = 0,\ \forall_{n=0,1,2,\ldots,N-1}$)

(2) $\quad \hat{\psi}^{(n)}(0) = 0, n = 0, 1, 2, \ldots, N-1$

(3) $\quad \tilde{H}^{(n)}(\frac{1}{2}) = 0, n = 0, 1, 2, \ldots, N-1$

<u>**pf:**</u>(a)

$$\hat{\psi}^{(n)}(\gamma) = \frac{d^n}{d\gamma^n} \int_{\mathbf{R}} \psi(t) e^{-2\pi i \gamma t} dt = \int_{\mathbf{R}} (-2\pi i t)^n \psi(t) e^{-2\pi i \gamma t} dt$$

Therefore

$$\hat{\psi}^{(n)}(\gamma) = (-2\pi i)^n \int_{\mathbf{R}} t^n \psi(t) e^{-2\pi i \gamma t} dt$$

Hence (1) $\Leftrightarrow$ (2).

(b) Since $\hat{\psi}(\gamma) = \frac{1}{\sqrt{2}} G(\frac{\gamma}{2}) \hat{\phi}(\frac{\gamma}{2})$,

$$\begin{aligned} \hat{\psi}(2\gamma) &= \frac{1}{\sqrt{2}} G(\gamma) \hat{\phi}(\gamma) \\ &= \frac{1}{\sqrt{2}} e^{-2\pi i \gamma} \overline{\tilde{H}(\gamma + \frac{1}{2})} \cdot \hat{\phi}(\gamma) \end{aligned} \tag{13}$$

This implies

$$\hat{\psi}(0) = \frac{1}{\sqrt{2}} \tilde{H}(\frac{1}{2}) \hat{\phi}(0).$$

Therefore, $\tilde{H}(\frac{1}{2}) = 0$, because $\hat{\phi}(0) = \int \phi(t) dt \neq 0$.

Now take the derivative of (13):

$$2\hat{\psi}(2\gamma) = \frac{1}{\sqrt{2}} \Big(\overline{\tilde{H}'(\gamma + \frac{1}{2})} \cdot \hat{\phi}(\gamma) e^{-2\pi i \gamma} + \tilde{H}(\gamma + \frac{1}{2}) \big(\hat{\phi} e^{-2\pi i \gamma} \big)' \Big)$$

Then,

$$0 = \hat{\psi}'(0) = \frac{1}{\sqrt{2}} (\tilde{H}'(\frac{1}{2}) \hat{\phi}(0) + 0) \Longleftrightarrow \tilde{H}'(\frac{1}{2}) = 0$$

By induction, $\hat{\psi}^{(n)}(0) = 0 \Longleftrightarrow \tilde{H}^{(n)}(\frac{1}{2}) = 0$. i.e. (2) $\Leftrightarrow$ (3).

Theorem $\Big(\tilde{H}^{(n)}(\frac{1}{2}) = 0, n = 0, 1, 2, \ldots, N-1 \Big) \Leftrightarrow \tilde{H}(\gamma) = (1 + e^{-2\pi i \gamma})^N F(\gamma)$ where $F(\gamma)$ is a trig polynomial such that $F(\frac{1}{2}) \neq 0$.

An outline of the proof: $f(a)=0 \Rightarrow f(x)=(x-a)a_1(x)$

f'(a)=0 $\Rightarrow f'(x)=a_1(x)=(x-a)a_1'(x)$

0=f'(a)=a$_1(a)+0 \Rightarrow a_1(a)=0$

Therefore a$_1(x)=(x-a)a_2(x)$

f(x)=(x-a)$^2 g_2(x)$

etc.

3) General principle of increasing the number of vanishing moments

Assume $H^0(\gamma)$ has l zeros at $\gamma=\frac{1}{2}$, then if $\Delta H(\gamma)=\hat{p}(2\gamma)H(\gamma+\frac{1}{2})e^{-2\pi i\gamma}$ has the same l zeros at $\gamma=\frac{1}{2}$ then $\tilde{H}(\gamma)$ can have at least $(l+1)$ zeros at $\gamma=\frac{1}{2}$. This is because

$$\begin{aligned}\tilde{H}(\gamma) &= H^0(\gamma)+\Delta H(\gamma)\\ &= (\cos\pi\gamma)^l F_1(\gamma)+(\cos\pi\gamma)^l F_2(\gamma)H(\gamma+\frac{1}{2})e^{-2\pi i\gamma}\\ &= (\cos\pi\gamma)^l\Big[F_1(\gamma)+e^{-2\pi i\gamma}F_2(\gamma)\overline{H(\gamma+\frac{1}{2})}\Big]\end{aligned}$$

If $F_2(\frac{1}{2})=\frac{F_1(\frac{1}{2})}{H(0)}$,

then $\Big[$ F$_1(\gamma)+e^{-2\pi i\gamma}F_2(\gamma)H(\gamma+\frac{1}{2})\Big]$ will have an additional zero at $\gamma=\frac{1}{2}$.

Example: Haar Wavelets

$$H(\gamma)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}e^{-2\pi i\gamma}=\frac{2}{\sqrt{2}}\cos\pi\gamma e^{-\pi i\gamma}=\sqrt{2}e^{-\pi i\gamma}\cos\pi\gamma$$

$$\begin{aligned}\hat{p}(\gamma) &= \sin(2\pi\gamma)\cdot\hat{p}_1(\gamma)\\ \hat{p}(2\gamma) &= \sin(4\pi\gamma)\cdot\hat{p}_1(2\gamma)=2\sin(2\pi\gamma)\cos(2\pi\gamma)\hat{p}_1(2\gamma)\\ &= 4\sin(\pi\gamma)\cos(\pi\gamma)\cos(2\pi\gamma)\hat{p}_1(2\gamma)\end{aligned}$$

$$\begin{aligned}\triangle\quad\tilde{H}(\gamma) &= H(\gamma)+\Delta H(\gamma)=\sqrt{2}e^{-\pi i\gamma}\cos\pi\gamma+\hat{p}(2\gamma)\overline{H(\gamma+\frac{1}{2})}e^{-2\pi i\gamma}\\ &= \sqrt{2}e^{-\pi i\gamma}\cos\pi\gamma+4\sin\pi\cos\pi\gamma\cos 2\pi\gamma\cdot\sqrt{2}ie^{-\pi i\gamma}\sin\pi\gamma\hat{p}_1(2\gamma)e^{-2\pi i\gamma}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}e^{-\pi i\gamma}\cos\pi\gamma\Big[1+4i\sin^2\pi\gamma\cos\pi\gamma\cdot\hat{p}_1(2\gamma)e^{-2\pi i\gamma}\Big] \\
&\quad \text{(note that } \hat{p}_1(2\gamma)e^{-2\pi i\gamma} = \frac{i}{4}) \\
&= \sqrt{2}e^{-\pi i\gamma}\cos\pi\gamma\Big[1+\sin^2\pi\gamma\cos 2\pi\gamma\Big] \\
\hat{p}(\gamma) &= \sin 2\pi\gamma\hat{p}_1(\gamma) = \sin 2\pi\gamma(-\frac{i}{4})e^{\pi i\gamma} \\
&= \sum_n (p_n)e^{-2\pi in\gamma}
\end{aligned}
$$

Note: If $H^0(\gamma)$ is to be real and $H^0(\gamma)$ has l zeros at $\gamma = \frac{1}{2}$, then,

$$H^0(\gamma) = (1+\cos 2\pi\gamma)^l F_1(\gamma) = 2^l(\cos 2\pi\gamma)^{2l}F_1(\gamma)$$

$$\hat{p}(\gamma) = \sin 2\pi\gamma\hat{p}_1 = \sin 2\pi\gamma\Big(\frac{i}{4}\Big)e^{\pi i\gamma}$$

$$\text{want: } = \sum_n (p_n)e^{-2\pi in\gamma}$$

$$
\begin{gathered}
\text{solution: } \Delta H(\gamma) = \hat{q}\overline{H(\gamma+\tfrac{1}{2})} = \hat{p}(2\gamma)\overline{H(\gamma+\tfrac{1}{2})}e^{-2\pi i\gamma} \\
\Rightarrow \Delta h(n) = (p_l(n)) * \Big((-1)^n h(1-n)\Big) \\
\tilde{h}(n) = h(n) + \Delta h(n)
\end{gathered}
$$