Chapter 2. Pseudoframes for subspaces (PFFS)

2.1. Motivation and Examples

Example 1 (Sampling). Let $PW_{\frac{1}{4}}$ be the space of bandlimited signals of bandwidth $\Omega = [-\frac{1}{4}, \frac{1}{4})$. Let

$$\hat{\phi}(\gamma) = \begin{cases} 1, & |\gamma| \leq \frac{1}{4} \\ \text{decaying to zero continuously}, & \frac{1}{4} < |\gamma| < \frac{1}{2} \\ 0, & |\gamma| \geq \frac{1}{2} \end{cases}$$

Then by the Shannon Sampling Theorem for sampling interval, $T = 1$, satisfying the Nyquist Rate: $2T \cdot \frac{1}{4} < 1$ we have,

$$\forall f \in PW_{\frac{1}{4}}, \quad f(t) = \sum_n f(n)\phi(t-n) \tag{1}$$

<u>Observations</u>:

1. $\{\phi(t-n)\}$ is not a basis nor a frame for $PW_{\frac{1}{4}}$ since $\phi(t) \notin PW_{\frac{1}{4}}$.
2. Morever, if one puts $\{\phi(t-n)\}$ in its closed linear span, $\overline{sp}\{\phi(t-n)\}$. $\{\phi(t-n)\}$ is not even a frame for its linear span $\overline{sp}\{\phi(t-n)\}$. This is because of the following theorem.

Theorem 0.1 *Let $\phi \in L^2(\boldsymbol{R})$. $\{\phi(t-n)\}_n$ forms a frame for $\overline{sp}\{\phi(t-n)\}$ if and only if there exists $0 < A \leq B < \infty$ s.t.*

$$A \leq \sum_k |\hat{\phi}(\gamma + k)|^2 \leq B \qquad a.e. \quad on \quad [0,1] \setminus \boldsymbol{N}$$

where $\boldsymbol{N} = \{\gamma \in [0,1) \mid \sum_k |\hat{\phi}(\gamma+k)| = 0\}$.

For convenience, we write $\Phi(\gamma) \equiv= \sum_k |\hat{\phi}(\gamma+k)|^2$.

Example 2 (Exact frame / Riesz basis)

Let $\{x_n{}^*\}$ be an exact frame of $\mathcal{X}, \subseteq \mathcal{H}$. Since $\{x_n\}$ is a minimum system, there is a unique biorthogonal dual $\{x_n{}^*\} \subseteq \mathcal{X}$ s.t. $\langle x_m{}^0, x_n{}^* \rangle = \delta_{mn}$, and

$$\forall f \in \mathcal{X}, \quad f = \sum_n \langle f, x_n{}^0 \rangle x_n{}^* = \sum_n \langle f, x_n{}^* \rangle x_n{}^0$$

Now, consider any sequence $\Delta x_n \in X^\perp$, and let $x_n = x_n{}^0 + \Delta x_n$. Claim:

1. $\langle x_m, x_n{}^* \rangle = \langle x_n{}^0 + \Delta x_n, x_n{}^* \rangle = \delta_{mn}$.
2. For all $f \in \mathcal{X}$,

$$\sum_n \langle f, x_n \rangle x_n{}^* = \sum_n \langle f, x_n{}^0 + \Delta x_n \rangle x_n{}^* = \sum_n \langle f, x_n{}^0 \rangle x_n{}^* = f.$$

We notice that there is a frame-type representation for all functions in $\mathcal{X}$. It is nevertherless not a conventional frame representation since $\{x_n\}$ is not in $\mathcal{X}$.

Example 3 Let $\{\phi(t-n)\}$ be an exact frame of $V_0 = \overline{sp}\{\phi(t-n)\}$. Then, $A \leq \Phi(\gamma) \leq B$ a.e. for some $0 < A \leq B < \infty$. Then there exists a unique biorthogonal dual function $\tilde{\phi} \in \overline{V_0}$ s.t. $\{\phi(t-n)\}$. In fact, in terms of the Fourier transform of $\tilde{\phi}$,

$$\hat{\tilde{\phi}}(\gamma) = \frac{\hat{\phi}(\gamma}{\sum_k |\hat{\phi}(\gamma+k)|^2} = \frac{1}{\Phi} \cdot \hat{\phi}(\gamma)$$

Since $\Phi(\gamma) = \sum_k |\hat{\phi}(\gamma+k)|^2$ is not a constant, the Fourier coefficients $\{\tilde{h}_n\}$ of $\dfrac{1}{\Phi}$ is not finite. Therefore, assuming ϕ is symmetric and compactly supported, then

$$\tilde{\phi}(t) = \sum_k \tilde{h_n} \phi(t-n)$$

Therefore, $\tilde{\phi}(t)$ is <u>not</u> compactly supported in V_o.

This suggests that in order to find compactly supported biorthogonal dual functions $\tilde{\phi}$, one must go beyond V_0.

Conclusion In order to analyze a subspace $\mathcal{X}$, frame-like sequence $\{x_n\}$ and $\{x_n{}^*\}$ need not be contained in $\mathcal{X}$.

2.2. Definition of Pseudo frames for subspaces (PFFS) and basic properties

Definition 0.2 (PFFS) Let $\mathcal{X}$ be a closed subspace of a separable Hilbert space. Let $\{x_n\} \subseteq \mathcal{H}$ be a Bessel sequence w.r.t. $\mathcal{X}$, and let $\{x_n{}^*\} \subseteq \mathcal{H}$ be Bessel sequence in $\mathcal{H}$. We say that $\{x_n\}$ is a pseudoframe for the subspace $\mathcal{X}$ w.r.t $\{x_n{}^*\}$ if

$$\forall f \in \mathcal{X}, \quad f = \sum_n \langle f, x_n \rangle x_n{}^* \tag{2}$$

Proposition 0.3 *The following are equivalent:*

1. *For all $f \in \mathcal{X}$, and for all $g \in \mathcal{H}$, $\langle f,\, g\,\rangle \;=\; \sum_n \langle f,\, x_n\,\rangle \langle x_n{}^*, g\rangle$.*
2. *For all $f \in \mathcal{X}$, $f = \sum_n \langle f,\, x_n\,\rangle x_n{}^*$.*

Remarks:

(a) PFFS is not "symmetric." Consequently, $\{x_n\}$ and $\{x_n{}^*\}$ is not commutative.

(b) Because of the non-commutativity of PFFS, the construction of PFFS has two very different directions. One direction is to construct $\{x_n\}$ from given $\{x_n{}^*\}$; the other direction is to construct $\{x_n{}^*\}$ from given $\{x_n\}$.

(c) There are conditions with which $\{x_n\}$ and $\{x_n{}^*\}$ are commutative.

(d) In a special case, let $\{x_n\} \subseteq$ H be a Bessel sequence in H. Then the right hand side of PFFS (2) is well defined in $\mathcal{H}$ (i.e. even for $f \notin \mathcal{X}$, $\sum_n \langle f, x_n \rangle x_n{}^*$ will still be a meaningful approximation of $f \in \mathcal{H}$ in $\mathcal{X}$. We will discuss more about such approximations later.

2.3. Basic Characterization of PFFS and Geometric Properties

Theorem 0.4 *Let $\{x_n\}$ and $\{x_n{}^*\}$ be two sequences in $\mathcal{H}$ (not necessarily in $\mathcal{X}$). Assume $\{x_n\}$ is Bessel w.r.t. $\mathcal{X}$, and $\{x_n{}^*\}$ is Bessel in H. Define $U : \mathcal{X} \to l^2$ and $V : l^2 \to \mathcal{H}$ by:*

$$\forall\, f \,\in\, X, \quad U f = \{\langle f,\, x_n\,\rangle\} \tag{3}$$

and,

$$\forall\, c \,\in\, l^2, \quad V c = \sum_n c(n) x_n{}^* \tag{4}$$

respectively. Suppose that $\mathcal{P}$ is a projection onto $\mathcal{X}$. Then $\{x_n\}$ and $\{x_n{}^*\}$ forms a PFFS for $\mathcal{X}$, if and only if

$$VU\mathcal{P} = \mathcal{P}$$

Proof: Assume that $\{x_n\}$ and $\{x_n{}^*\}$ form a PFFS for $\mathcal{X}$, then for all $f \in \mathcal{H}$,

$$\begin{aligned}\mathcal{P}f &= \sum_n \langle \mathcal{P}f, x_n \rangle x_n{}^* \\ &= \sum_n (U\mathcal{P}f) x_n{}^* \\ &= VU\mathcal{P}f.\end{aligned}$$

Therefore, $VU\mathcal{P} = \mathcal{P}$. Similarly, the other direction holds. ■

Remark: Here $\mathcal{P}$ is any projection from $\mathcal{H}$ onto $\mathcal{X}$. It is typically a non-orthogonal projection.

2.3.1. Orthogonal projections vs. non-orthogonal projections

Orthogonal Projections have the following properties $P^2 = P$, $\mathcal{N}(P) \perp \mathcal{R}(P)$. Hence $\mathcal{N}(P) \oplus \mathcal{R}(P) = \mathcal{H}$. Consequently, $P^* = P$

Nonorthogonal projections satisfy the following: $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{N}(\mathcal{P}) \cap \mathcal{R}(\mathcal{P}) = \{0\}$, and $\mathcal{N}(\mathcal{P}) + \mathcal{R}(\mathcal{P}) = \mathcal{H}$. Here the + stands for linear set addition.

Notation: $\mathcal{P} \equiv \mathcal{P}_{\mathcal{X},\mathcal{N}(\mathcal{P})}$ which is often termed a projection onto $\mathcal{X}$ along the subspace $\mathcal{N}(\mathcal{P})$. One can verify that $\mathcal{P}^* = \mathcal{P}_{\mathcal{N}(\mathcal{P})^\perp,\mathcal{X}^\perp}$.

2.3.2. The consistent principle associated with PFFS

Theorem 0.5 *Let $\{x_n\}$ and $\{x_n{}^*\}$ be a PFFS for $\mathcal{X}$. Assume further that $\{x_n\}$ is Bessel in $\mathcal{H}$. Suppose $\mathcal{N}(\mathcal{P}) = \overline{sp}\{x_n\}^\perp$ Then,*

$$U\mathcal{P} = U.$$

Proof: (a) It is true that $\overline{sp}\{x_n\}^\perp$ is a complement of $\mathcal{X}$.

(b) $\forall f \in H$, $Uf = \{\langle f, x_n \rangle\} UPf = \{\langle Pf, x_n \rangle\} = \{\langle f, P_{N(p)^\perp, x^\perp} x_n \rangle\} \{\langle f, x_n \rangle\}$, since $N(P)^\perp = \overline{sp}\{x_n\}$.

Therefore, $U\mathcal{P} = U$. ■

2.3.3. Meaning of the above theorem?

Assume that the reconstruction of f from $Uf = \{\langle \tilde{f},\, x_n \rangle\}$ is not achievable in the original function space of f. Instead, assume that an approximation of f on a subspace $\mathcal{X}$ is possible. Then the PFFS with $\mathcal{N}(\mathcal{P}) = \overline{sp}\{x_n\}^{\perp}$ provides a consistent approximation of f in $\mathcal{X}$ in which the "measurement" of the original f by $\{\langle f,\, x_n \rangle\}$ is the same/consistent with the "measurement" of the approximation of f by $\{\langle \tilde{f},\, x_n \rangle\} = \{\langle \mathcal{P}f,\, x_n \rangle\}$.
Therefore, even though $\tilde{f} \neq f$, the measurement of f and $\tilde{f}$ in terms of $\langle \cdot,\, x_n \rangle$ are the same/consistent.

2.3.4. Other geometric properties of PFFS

1. If $\mathcal{N}(\mathcal{P}) = \mathcal{X}^{\perp}$ (i.e. $\mathcal{P} = P$ is an orthogonal projection), then PFFS provides a least square approximation of f in $\mathcal{X}$ in the following way

$$\forall f \in \mathcal{H}, \quad Pf = \sum_n \langle Pf, x_n \rangle {x_n}^* = \langle f, Px_n \rangle {x_n}^*$$

   Therefore, simply take the orthogonal projection of x_n onto $\mathcal{X}$, a PFFS provides a least squares approximation of $f \in \mathcal{H}$.

2. Let $\mathcal{P}$ be any non-orthogonal projection onto X. Then PFFS provides a non-orthogonal projection of any $f \in \mathcal{H}$ in the following way

$$\forall f \in \mathcal{H}, \quad \mathcal{P}f = \sum_n \langle f,\, P^* x_n \rangle {x_n}^*$$

   Therefore, starting from a PFFS, one can generate an explicit non-orthogonal projection, while performing the "reconstruction."

3. Intuition on the application of the non-orthogonal projection property of PFFS. For noise removal: by "steering" $\overline{sp}\{x_n\}$ to be perpendicular to the noise subspace, one can remove the noise while doing reconstruction, i.e., if $g = f + n$, $(f \in \mathcal{X})$

$$\sum_n \langle g,\, x_n \rangle {x_n}^* \;=\; \sum_n \langle f + n,\, x_n \rangle {x_n}^* \;=\; \sum_n \langle f,\, x_n \rangle {x_n}^* \;=\; f$$

   Obviously, this is not possible with a basis or conventional frame of $\mathcal{X}$.

Sectin 2.4. The Two Directions of Constructions

The interpolation approach Given $\{x_n\}$ to construct $\{x_n{}^*\}$ that forms a PFFS for $\mathcal{X}$ (i.e., for all $f \in \mathcal{X}$, $f = \langle f, x_n \rangle x_n{}^*$).

This direction of approximation is termed interpolation approach because such problems would correspond to reconstructions of a function from its "samples" $\{\langle f, x_n \rangle$.

The approximation approach Given $\{x_n{}^*\}$ to find $\{x_n\}$.

This direction is called the approximation approach because the given sequence of functions $\{x_n^*\}$ are like "basis" elements for the subspace $\mathcal{X}$, and we need to find a way to express a function f in terms of linear combinations of the elements of $\{x_n^*\}$.

Both directions of constructions will be using the basic characterization

$$VU\mathcal{P} = \mathcal{P}.$$

The first direction is to find the"left inverse" V from $VU\mathcal{P} = \mathcal{P}$. The second direction is to find the "right inverse" U from $VU\mathcal{P} = \mathcal{P}$. (Please see the details in the article).