Chapter 2. Pseudoframes for subspaces (PFFS)

2.1. Motivation and Examples

Example 1 (Sampling). Let $PW_{\frac{1}{4}}$ be the space of bandlimited signals of bandwidth $\Omega = [-\frac{1}{4}, \frac{1}{4})$. Let

$$\hat{\phi}(\gamma) = \begin{cases} 1, & |\gamma| \leq \frac{1}{4} \\ \text{decaying to zero continuously}, & \frac{1}{4} < |\gamma| < \frac{1}{2} \\ 0, & |\gamma| \geq \frac{1}{2} \end{cases}$$

Then by the Shannon Sampling Theorem for sampling interval, $T = 1$, satisfying the Nyquist Rate: $2T \cdot \frac{1}{4} < 1$ we have,

$$\forall f \in PW_{\frac{1}{4}}, \quad f(t) = \sum_{n} f(n) \phi(t - n) \quad (1)$$

Observations:

1. $\{\phi(t - n)\}$ is not a basis nor a frame for $PW_{\frac{1}{4}}$ since $\hat{\phi}(t) \notin PW_{\frac{1}{4}}$.
2. Moreover, if one puts $\{\phi(t - n)\}$ in its closed linear span, $sp\{\phi(t - n)\}$.
   $\{\phi(t - n)\}$ is not even a frame for its linear span $sp\{\phi(t - n)\}$. This is because of the following theorem.

Theorem 0.1 Let $\phi \in L^2(\mathbb{R})$. $\{\phi(t - n)\}_n$ forms a frame for $sp\{\phi(t - n)\}$ if and only if there exists $0 < A \leq B < \infty$ s.t.

$$A \leq \sum_{k} |\hat{\phi}(\gamma + k)|^2 \leq B \quad \text{a.e. on } [0,1] \setminus N$$

where $N = \{\gamma \in [0,1) \mid \sum_{k} |\hat{\phi}(\gamma + k)| = 0\}$. 
For convenience, we write $\Phi(\gamma) \equiv \sum_k |\hat{\phi}(\gamma + k)|^2$.

**Example 2** (Exact frame / Riesz basis)

Let $\{x_n^\ast\}$ be an exact frame of $\mathcal{X} \subseteq \mathcal{H}$. Since $\{x_n\}$ is a minimum system, there is a unique biorthogonal dual $\{x_n^\ast\} \subseteq \mathcal{X}$ s.t. $\langle x_m^0, x_n^\ast \rangle = \delta_{mn}$, and

$$\forall f \in \mathcal{X}, \quad f = \sum_n \langle f, x_n^0 \rangle x_n^\ast = \sum_n \langle f, x_n^\ast \rangle x_n^0$$

Now, consider any sequence $\Delta x_n \in X^\perp$, and let $x_n = x_n^0 + \Delta x_n$.

Claim:

1. $\langle x_m, x_n^\ast \rangle = \langle x_n^0 + \Delta x_n, x_n^\ast \rangle = \delta_{mn}$.
2. For all $f \in \mathcal{X}$,

$$\sum_n \langle f, x_n \rangle x_n^\ast = \sum_n \langle f, x_n^0 + \Delta x_n \rangle x_n^\ast = \sum_n \langle f, x_n^0 \rangle x_n^\ast = f.$$

We notice that there is a frame-type representation for all functions in $\mathcal{X}$. It is nevertheless not a conventional frame representation since $\{x_n\}$ is not in $\mathcal{X}$.

**Example 3** Let $\{\phi(t - n)\}$ be an exact frame of $V_0 = \overline{\mathcal{P}}\{\phi(t - n)\}$. Then, $A \leq \Phi(\gamma) \leq B$ a.e. for some $0 < A \leq B < \infty$. Then there exists a unique biorthogonal dual function $\tilde{\phi} \in V_0$ s.t. $\{\phi(t - n)\}$. In fact, in terms of the Fourier transform of $\tilde{\phi}$,

$$\hat{\phi}(\gamma) = \frac{\hat{\phi}(\gamma) \cdot \hat{\phi}(\gamma)}{\sum_k |\hat{\phi}(\gamma + k)|^2} = \frac{1}{\Phi} \cdot \hat{\phi}(\gamma)$$

Since $\Phi(\gamma) = \sum_k |\hat{\phi}(\gamma + k)|^2$ is not a constant, the Fourier coefficients $\{\hat{h}_n\}$ of $\frac{1}{\Phi}$ is not finite. Therefore, assuming $\phi$ is symmetric and compactly supported, then

$$\tilde{\phi}(t) = \sum_k \hat{h}_n \phi(t - n)$$

Therefore, $\tilde{\phi}(t)$ is not compactly supported in $V_0$.

This suggests that in order to find compactly supported biorthogonal dual functions $\tilde{\phi}$, one must go beyond $V_0$.

**Conclusion** In order to analyze a subspace $\mathcal{X}$, frame-like sequence $\{x_n\}$ and $\{x_n^\ast\}$ need not be contained in $\mathcal{X}$.

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2.2. Definition of Pseudo frames for subspaces (PFFS) and basic properties

**Definition 0.2** (PFFS) Let $\mathcal{X}$ be a closed subspace of a separable Hilbert space. Let $\{x_n\} \subseteq \mathcal{H}$ be a Bessel sequence w.r.t. $\mathcal{X}$, and let $\{x_n^*\} \subseteq \mathcal{H}$ be Bessel sequence in $\mathcal{H}$. We say that $\{x_n\}$ is a pseudoframe for the subspace $\mathcal{X}$ w.r.t $\{x_n^*\}$ if

$$\forall f \in \mathcal{X}, \quad f = \sum_n \langle f, x_n \rangle x_n^* \tag{2}$$

**Proposition 0.3** The following are equivalent:

1. For all $f \in \mathcal{X}$, and for all $g \in \mathcal{H}$, $\langle f, g \rangle = \sum_n \langle f, x_n \rangle \langle x_n^*, g \rangle$.

2. For all $f \in \mathcal{X}$, $f = \sum_n \langle f, x_n \rangle x_n^*$.

**Remarks:**

(a) PFFS is not "symmetric." Consequently, $\{x_n\}$ and $\{x_n^*\}$ is not commutative.

(b) Because of the non-commutativity of PFFS, the construction of PFFS has two very different directions. One direction is to construct $\{x_n\}$ from given $\{x_n^*\}$; the other direction is to construct $\{x_n^*\}$ from given $\{x_n\}$.

(c) There are conditions with which $\{x_n\}$ and $\{x_n^*\}$ are commutative.

(d) In a special case, let $\{x_n\} \subseteq \mathcal{H}$ be a Bessel sequence in $\mathcal{H}$. Then the right hand side of PFFS (2) is well defined in $\mathcal{H}$ (i.e. even for $f \notin \mathcal{X}$, $\sum_n \langle f, x_n \rangle x_n^*$ will still be a meaningful approximation of $f \in \mathcal{H}$ in $\mathcal{X}$. We will discuss more about such approximations later.

2.3. Basic Characterization of PFFS and Geometric Properties

**Theorem 0.4** Let $\{x_n\}$ and $\{x_n^*\}$ be two sequences in $\mathcal{H}$ (not necessarily in $\mathcal{X}$). Assume $\{x_n\}$ is Bessel w.r.t. $\mathcal{X}$, and $\{x_n^*\}$ is Bessel in $\mathcal{H}$. Define $U: \mathcal{X} \to l^2$ and $V: l^2 \to \mathcal{H}$ by:

$$\forall f \in \mathcal{X}, \quad Uf = \{\langle f, x_n \rangle\} \tag{3}$$

and,

$$\forall c \in l^2, \quad Vc = \sum_n c(n)x_n^* \tag{4}$$

respectively. Suppose that $\mathcal{P}$ is a projection onto $\mathcal{X}$. Then $\{x_n\}$ and $\{x_n^*\}$ forms a PFFS for $\mathcal{X}$, if and only if

$$VUP = \mathcal{P}$$

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Proof: Assume that \( \{ x_n \} \) and \( \{ x_n^* \} \) form a PFFS for \( \mathcal{X} \), then for all \( f \in \mathcal{H} \),

\[
\mathcal{P} f = \sum_n (\mathcal{P} f, x_n^*) x_n^* \\
= \sum_n (U \mathcal{P} f) x_n^* \\
= V U \mathcal{P} f.
\]

Therefore, \( V U \mathcal{P} = \mathcal{P} \). Similarly, the other direction holds.

Remark: Here \( \mathcal{P} \) is any projection from \( \mathcal{H} \) onto \( \mathcal{X} \). It is typically a non-orthogonal projection.

2.3.1. Orthogonal projections vs. non-orthogonal projections

**Orthogonal Projections** have the following properties \( \mathcal{P}^2 = \mathcal{P} \), \( \mathcal{N}(\mathcal{P}) \perp \mathcal{R}(\mathcal{P}) \). Hence \( \mathcal{N}(\mathcal{P}) \oplus \mathcal{R}(\mathcal{P}) = \mathcal{H} \). Consequently, \( \mathcal{P}^* = \mathcal{P} \)

**Nonorthogonal projections** satisfy the following: \( \mathcal{P}^2 = \mathcal{P} \), \( \mathcal{N}(\mathcal{P}) \cap \mathcal{R}(\mathcal{P}) = \{0\} \), and \( \mathcal{N}(\mathcal{P}) + \mathcal{R}(\mathcal{P}) = \mathcal{H} \). Here the + stands for linear set addition.

Notation: \( \mathcal{P} \equiv \mathcal{P}_{\mathcal{X},\mathcal{N}(\mathcal{P})} \) which is often termed a projection onto \( \mathcal{X} \) along the subspace \( \mathcal{N}(\mathcal{P}) \). One can verify that \( \mathcal{P}^* = \mathcal{P}_{\mathcal{N}(\mathcal{P})^\perp,\mathcal{X}^\perp} \).

2.3.2. The consistent principle associated with PFFS

**Theorem 0.5** Let \( \{ x_n \} \) and \( \{ x_n^* \} \) be a PFFS for \( \mathcal{X} \). Assume further that \( \{ x_n \} \) is Bessel in \( \mathcal{H} \). Suppose \( \mathcal{N}(\mathcal{P}) = \overline{\mathcal{P}}\{ x_n \}^\perp \) Then,

\[
U \mathcal{P} = U.
\]

Proof: (a) It is true that \( \overline{\mathcal{P}}\{ x_n \}^\perp \) is a complement of \( \mathcal{X} \).

(b) \( \forall f \in \mathcal{H}, \ U f = \{\langle f, x_n \rangle\} U \mathcal{P} f = \{\langle P f, x_n \rangle\} = \{\langle f, P_{\mathcal{N}(\mathcal{P})^\perp,\mathcal{X}^\perp} x_n \rangle\} \{\langle f, x_n \rangle\} \), since \( \mathcal{N}(\mathcal{P})^\perp = \overline{\mathcal{P}}\{ x_n \} \).

Therefore, \( U \mathcal{P} = U \).
2.3.3. Meaning of the above theorem?

Assume that the reconstruction of $f$ from $Uf = \{\langle \tilde{f}, x_n \rangle \}$ is not achievable in the original function space of $f$. Instead, assume that an approximation of $f$ on a subspace $\mathcal{X}$ is possible. Then the PFFS with $\mathcal{N}(\mathcal{P}) = \mathcal{P}(x_n)^\perp$ provides a consistent approximation of $f$ in $\mathcal{X}$ in which the "measurement" of the original $f$ by $\{\langle f, x_n \rangle \}$ is the same/consistent with the "measurement" of the approximation of $f$ by $\{\langle \tilde{f}, x_n \rangle \} = \{\langle \mathcal{P}f, x_n \rangle \}$.

Therefore, even though $\tilde{f} \neq f$, the measurement of $f$ and $\tilde{f}$ in terms of $\langle \cdot, x_n \rangle$ are the same/consistent.

2.3.4. Other geometric properties of PFFS

1. If $\mathcal{N}(\mathcal{P}) = \mathcal{X}^\perp$ (i.e. $\mathcal{P} = P$ is an orthogonal projection), then PFFS provides a least square approximation of $f$ in $\mathcal{X}$ in the following way

$$\forall f \in \mathcal{H}, \quad Pf = \sum_n \langle Pf, x_n \rangle x_n^* = \langle f, Px_n \rangle x_n^*$$

Therefore, simply take the orthogonal projection of $x_n$ onto $\mathcal{X}$, a PFFS provides a least squares approximation of $f \in \mathcal{H}$.

2. Let $\mathcal{P}$ be any non-orthogonal projection onto $\mathcal{X}$. Then PFFS provides a non-orthogonal projection of any $f \in \mathcal{H}$ in the following way

$$\forall f \in \mathcal{H}, \quad \mathcal{P}f = \sum_n \langle f, P^*x_n \rangle x_n^*$$

Therefore, starting from a PFFS, one can generate an explicit non-orthogonal projection, while performing the "reconstruction."

3. Intuition on the application of the non-orthogonal projection property of PFFS. For noise removal: by "steering" $\mathcal{P}(x_n)$ to be perpendicular to the noise subspace, one can remove the noise while doing reconstruction, i.e., if $g = f + n$, $f \in \mathcal{X}$

$$\sum_n \langle g, x_n \rangle x_n^* = \sum_n \langle f + n, x_n \rangle x_n^* = \sum_n \langle f, x_n \rangle x_n^* = f$$

Obviously, this is not possible with a basis or conventional frame of $\mathcal{X}$.

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Sectin 2.4. The Two Directions of Constructions

The interpolation approach Given \( \{x_n\} \) to construct \( \{x_n^*\} \) that forms a PFFS for \( \mathcal{X} \) (i.e., for all \( f \in \mathcal{X} \), \( f = \langle f, x_n \rangle x_n^* \)).

This direction of approximation is termed interpolation approach because such problems would correspond to reconstructions of a function from its “samples” \( \{\langle f, x_n \rangle\} \).

The approximation approach Given \( \{x_n^*\} \) to find \( \{x_n\} \).

This direction is called the approximation approach because the given sequence of functions \( \{x_n^*\} \) are like “basis” elements for the subspace \( \mathcal{X} \), and we need to find a way to express a function \( f \) in terms of linear combinations of the elements of \( \{x_n^*\} \).

Both directions of constructions will be using the basic characterization

\[
VUP = \mathcal{P}.
\]

The first direction is to find the “left inverse” \( V \) from \( VUP = \mathcal{P} \). The second direction is to find the “right inverse” \( U \) from \( VUP = \mathcal{P} \). (Please see the details in the article).