<u>Advances of Frames and Wavelets with Applications</u>

<u>Chapter 1</u>. Frames, why frames, and Frame Properties.

<u>Chapter 2</u>. PFFS: <u>P</u>seudo <u>f</u>rames <u>f</u>or <u>s</u>ubspaces. (Extension of frames).

<u>Chapter 3</u>. Biorthogonal Wavelets (Bi-Wavelets) and Filter Banks via (using) PFFS Bi-wavelets.

<u>Chapter 4</u>. Relationships of bi-wavelet via PFFS and the lifting schemes.

<u>Chapter 5</u>. Open Problems

<u>Preparations</u> l^2 – $\{\{C(n)\} \mid \sum_n |C(n)|^2 \}$
$L^2(R)$

<u>Chapter 1</u>. Frames and Why Frames

Let $\overrightarrow{x}$ be any vector in C^2

$$\overrightarrow{x} = a_1\overrightarrow{u}_1 + a_2\overrightarrow{u}_2$$
$$\overrightarrow{x} = b_1\overrightarrow{u}_1 + b_2\overrightarrow{u}_3$$
$$+ \quad \overrightarrow{x} = c_1\overrightarrow{u}_2 + c_3\overrightarrow{u}_3$$

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$$\overrightarrow{x} = \tfrac{1}{3}(a_1+b_1)\overrightarrow{u}_1 + \tfrac{1}{3}(c_1+a_2)\overrightarrow{u}_2 + \tfrac{1}{3}(b_2+c_3)\overrightarrow{u}_3$$

$\Rightarrow$ the linear combination of $\{\overrightarrow{u_i}\}^3$ is no longer unique.

1. Definition of Frames

Let H be a (separable) Hilbert space. A sequence of vectors $\{x_n\} \subseteq$ H is a frame for H if there exist constants $0 < A \leq B < \infty$ s.t. $\forall f \in$ H, $A\|f\|^2 \leq \sum_n |\langle f, x_n\rangle|^2 \leq B\|f\|^2$.

Basically, we want $\sum_n |\langle f, x_n\rangle|^2 \approx \|f\|^2$

$\{x_n\}$ Here, A and B are called the lower and the upper frame bounds. When A = B, $\{\cancel{x}\}$ is called a tight frame. A frame is called an exact frame if $\{\cancel{x}\}$ if is no longer a frame ~~is only~~ of its elements is removed (i.e. an exact frame is a basis). when any

2. Why Frames?

1) Frames are more flexible and more "natural". (Gabor Frames)

2) Frames are more robust. For instance:

Ex 1 Let $\overrightarrow{u_1} = (0, 1)$, $\overrightarrow{u_2} = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and $\overrightarrow{u_3} = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$

Consider an ONB, $\overrightarrow{e_1} = (0, 1)$, $\overrightarrow{e_2} = (1, 0)$, $\forall \overrightarrow{x} \in C^2$,

$\overrightarrow{x} = \sum_{n=1}^{2} \langle \overrightarrow{x}, \overrightarrow{e_n}\rangle \overrightarrow{e_n}$ (ONB)

Claim:

$$\overrightarrow{x} = \frac{2}{3}\sum_{n=1}^{3}\langle \overrightarrow{x}, \overrightarrow{u_n}\rangle \overrightarrow{u_n} \qquad (Frame)$$

To verify the Frame representation:

$$\sum_{n=1}^{3}\langle \overrightarrow{x}, \overrightarrow{u_n}\rangle \overrightarrow{u_n} = \langle \overrightarrow{x}, \overrightarrow{u_1}\rangle \overrightarrow{u_1} + \langle \overrightarrow{x}, \overrightarrow{u_2}\rangle \overrightarrow{u_2} + \langle \overrightarrow{x}, \overrightarrow{u_3}\rangle \overrightarrow{u_3}$$

$$= \ldots \, rewrite\, into \quad c_1\langle \overrightarrow{x}, \overrightarrow{e_1}\rangle \overrightarrow{e_1} + c_2\langle \overrightarrow{x}, \overrightarrow{e_2}\rangle \overrightarrow{e_2}$$

$$= \ldots = \frac{3}{2}\sum_{n=1}^{3}\langle \overrightarrow{x}, \overrightarrow{e_n}\rangle \overrightarrow{e_n} = \frac{3}{2}\overrightarrow{x}$$

where $\overrightarrow{u_1} = \overrightarrow{e_1}$, and $\overrightarrow{u_2} = -\frac{\sqrt{3}}{2}\overrightarrow{e_2} - \frac{1}{2}\overrightarrow{e_1}$

Now, consider a perturbation to the coefficients by $\{\alpha_n \varepsilon\}$, where $\{\alpha_n$ is an independent and identical distribution (iid random variable). Assume $\{\alpha_n$ has <u>0 mean</u> and <u>variance 1</u>.

Let's study the reconstruction error using ONB and Frames, respectively.

<u>For ONB</u>:

$$\begin{aligned} & \mathrm{E}(\|\overrightarrow{x} - \textstyle\sum_{n=1}^{2}(\langle \overrightarrow{x}, \overrightarrow{e_n}\rangle + \alpha_n\varepsilon)\overrightarrow{e_n}\|^2) \\ &= \mathrm{E}(\|\varepsilon \textstyle\sum_{n=1}^{2} \alpha_n e_n\|^2) = \varepsilon^2 \mathrm{E}({\alpha_1}^2 + {\alpha_2}^2) \\ &= 2\varepsilon^2 \end{aligned} \quad \Leftarrow E_{onb}$$

<u>For ~~ONB:~~</u> Frame

$$\begin{aligned} & \mathrm{E}(\|\overrightarrow{x} - \textstyle\sum_{n=1}^{3} \frac{2}{3}(\langle \overrightarrow{x}, \overrightarrow{u_n}\rangle + \alpha_n\varepsilon)\overrightarrow{u_n}\|^2) \\ &= \mathrm{E}(\|\tfrac{2}{3}\textstyle\sum_{n=1}^{3} \alpha_n \varepsilon u_n\|^2) = \tfrac{4}{9}\varepsilon^2 \mathrm{E}({\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2) \\ &= \tfrac{4}{3}\varepsilon^2 \end{aligned} \quad \Leftarrow E_{frames}$$

$$\frac{E_{frames}}{E_{onb}} = \frac{\frac{4}{3}\varepsilon^2}{2\varepsilon^2} = \frac{2}{3}$$

<u>Ex 2</u>: <u>Gabor Frames</u> and the Uncertainty Principle.

$\{g_{m,n}\} = \{g(t - nT)e^{\frac{2\pi imt}{M}}\}_{\cancel{M}\,m,n}$ — This is exactly the short-time Fourier Transform. A typical choice of g is the Gaussian window!

<u>The Uncertainty Principle</u>: Let g $\in L^2(R)$. Then,

$$\| tg(t) \|_{L^2} \cdot \| \hat{g}(\gamma) \|_{L^2} \geq \frac{1}{4\pi} \| g \|_{L^2}$$

<u>Notation</u>: Fourier Transform of g is defined by:

$$\hat{g}(\gamma) = \int g(t) e^{-2\pi i\gamma t}\, dt$$

$$\Leftrightarrow g(t) = \int \hat{g}(\gamma) e^{2\pi i\gamma t}\, d\gamma$$

Since the Gaussian function achieves the bound of the uncertainty principle, the Gaussian is optimal in terms of "best" time-frequency localization.

Note: Need to understand why $\|tg(t)\|_{L^2}$ reflects the time localization of $g(t)$.

The Balian-Law Theorem: Let g $\in L^2(\mathbf{R})$. Let $\{g_{m,n}\}$ be a Gabor system. If $\{g_{m,n}\}$ forms a basis of $L^2(\mathbf{R})$ then either:

$$\begin{array}{ccc} tg(t) \notin L^2(\mathbf{R}) & or & \gamma\hat{g}(\gamma) \notin L^2(\mathbf{R}) \\ \Updownarrow & & \Updownarrow \\ \|tg(t)\|_{L^2} \text{ is unbounded}, & \text{or} & \|\hat{g}(\gamma)\|_{L^2} \text{ is unbounded} \end{array}$$

Therefore, one cannot form a Gabor basis with Gaussian windows g (because if g is Gaussian, both $\|tg(t)\|_{L^2}$ and $\|\hat{g}(\gamma)\|_{L^2}$ must be finite). But it's possible to form Gabor Frames with Gaussian windows. This is another example when frames are useful.

§1.2 Frame Representation and Frame Properties.

The Frame Operator S: H $\rightarrow$ H (assume $\{x_n\}$ is a frame).

$$\forall f \in H,\ Sf = \sum_{n \in Z} \langle f, x_n \rangle x_n \tag{1}$$

Theorem 1.2.1 Let $\{x_n\}$ be a frame, and let S be the frame operator defined by (1). Then:

(1) S is bounded, invertible, self-adjoint and positive.
(2) $\{S^{-1}x_n\}$ is also a frame for H with frame bounds B^{-1} and A^{-1}.
(3) $\forall f \in H,\ f = \sum_n \langle f, S^{-1}x_n \rangle x_n = \sum_n \langle f, x_n \rangle (S^{-1}x_n)$

Definition (adjoint operator):
Let **X** and **Y** be two Hilbert Spaces with inner product $\langle\, , \rangle_{\mathbf{X}} + \langle\, , \rangle_{\mathbf{Y}}$. Let A: **X** $\rightarrow$ **Y** be a bounded linear operator. An operator: **Y** $\rightarrow$ **X** is called the adjoint of A **if**, $\forall x \in \mathbf{X}$, $y \in \mathbf{Y}$,

$$\langle Ax, y \rangle_{\mathbf{Y}} = \langle x, By \rangle_{\mathbf{X}}$$

Typically, B = A^*, i.e., $\langle Ax, y \rangle_{\mathbf{Y}} = \langle x, A^* \rangle_{\mathbf{X}}$. A is self-adjoint if $A^* = A$.

Definition (Positive operator):
A bounded, self-adjoint operator is positive if $\langle Ax, \cancel{y}\, x \rangle > 0$, $\forall x \in$ H.
We say $A > B \quad if \quad \langle (A - B), x \rangle > 0$, or, $\langle Ax, x \rangle > \langle Bx, x \rangle$.

Rmk: A positive operator is only defined for self-adjoint operators
(A is self-adjoint **iff** $\langle Ax, x \rangle$ is real for all x.)

Pf of Thm 1.2.1:

(1) Define U: H $\to l^2$ by:
$\forall f \in H$, where Uf = $\{\langle f, x_n \rangle\}_n$ (coefficient map)
Then U is bounded , because $\|Uf\|_{l^2} = \sum_n |\langle f, x_n \rangle|^2 \leq B\|f\|^2$.
The adjoint of U, $U^* : l_2 \to H$ is given by,

$$\forall c \in l^2, \quad U^*c = \sum_{n \in \mathbf{Z}} c(n) x_n$$

$\langle c, Uf \rangle_{l^2} = \langle c, \{\langle f, x_n \rangle\} \rangle_{l^2} = \sum_{n \in \mathbf{Z}} c(n) \overline{\langle f, x_n \rangle} = \cdots = \langle ?, f \rangle$
$U^* \hookleftarrow$

therefore, S = U^*U $\quad (U^*Uf = U^*(\{\langle f, x_n \rangle\}) = \sum_n \langle f, x_n \rangle x_n =$ sf)
$S^* = (U^*U)^* = U^*(U^*)^* = U^*U \to$ self-adjoint.
$\|S\| = \|U^*U\| \leq \|U^*\|\|U\| = \|U\|^2 < \infty \to$ bounded.
Since $\langle sf, f \rangle = \langle \sum_n \langle f, x_n \rangle x_n, f \rangle = \sum_n \langle f, x_n \rangle \langle x_n, f \rangle =$
$= \sum_n \langle f, x_n \rangle \overline{\langle f, x_n \rangle} = \sum_n |\langle f, x_n \rangle|^2$

Therefore, by the frame definition,

$$A\langle f, f \rangle = A\|f\|^2 = \langle AIf, f \rangle \leq \langle Sf, f \rangle \leq \langle BIf, f \rangle \leq B\langle f, f \rangle = B\|f\|^2$$

Recall: $\|f\|^2 = \sqrt{\langle f, f \rangle}$
Therefore, 0I < AI $\leq$ S $\leq$ BI $\Rightarrow$ S is positive.

Now, let's prove that S is invertible.
From S $\leq$ BI $\Rightarrow B^{-1}$ S $\leq$ I or I $- B^{-1}S \geq 0$.
Meantime, from AI $\leq$ S $\Rightarrow \frac{A}{B}$ I $\leq B^{-1}$ S
$\Rightarrow$ I $-$ I + $\frac{A}{B}I \leq B^{-1}$ S $\Rightarrow$ I $- B^{-1}$ S $\leq$ I $-\frac{A}{B}$ I = $\frac{B-A}{B}$ I

Therefore, 0 $\leq$ I $- B^{-1}$ S $\leq \frac{B-A}{B}$ I

~~Therefore~~, $\|$ I $- B^{-1}$ S $\| \leq \frac{B-A}{B} < 1$

Therefore, by the Von Neumann Theorem, since $\|$ I $- B^{-1}$ S $\| < 1$,
we know that B^{-1} S is invertible.

Intuition of the <u>Neumann Theorem</u>.   From calculus,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

Now we want to look at $\frac{1}{x}$.

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n, \qquad \|1-x\| < 1$$

$\Rightarrow x^{-1}$ exists if $|1-x|<1$.

Therefore, S^{-1} exists. $B^{-1}S$ is invertible,

(2) From AI $\leq$ S $\leq$ BI, we have $AS^{-1} \leq$ I $\leq BS^{-1}$ or $B^{-1}I \leq S^{-1} \leq A^{-1}$

Note: Recall that we want to show that $\{S^{-1}X_n\}$ is a frame.

by the defn of positive operator,

Therefore, $\underbrace{\langle B^{-1}f,\, f\,\rangle}_{B^{-1}\langle f,\, f\,\rangle \,=\, B^{-1}\|f\|^2} \leq \langle S^{-1}f,\, f\,\rangle \leq \underbrace{\langle A^{-1}f,\, f\,\rangle}_{A^{-1}\langle f,\, f\,\rangle \,=\, A^{-1}\|f\|^2}$

(This is by the definition of positive operator)

$$\begin{aligned}
\langle S^{-1}f,\, f\,\rangle &= \langle S^{-1}S(S^{-1}f),\, f\,\rangle \\
&= \langle S^{-1}(\textstyle\sum_n \langle S^{-1}f,\, x_n\,\rangle x_n),\, f\,\rangle \\
&= \langle S^{-1} \textstyle\sum_n \langle f,\, S^{-1}x_n\,\rangle x_n,\, f\,\rangle \qquad [\text{Since } \langle Ax,\, y\,\rangle = \langle x,\, A^{*}y\,\rangle] \\
&= \langle \textstyle\sum_n \langle f,\, S^{-1}x_n\,\rangle\, (S^{-1}x_n),\, f\,\rangle \\
&\qquad (\text{T is linear if } T(\alpha x + \beta y) = \alpha Tx + \beta Ty) \\
&= \textstyle\sum_n \langle f,\, S^{-1}x_n\,\rangle \langle S^{-1}x_n,\, f\,\rangle \qquad [\,\langle \alpha x,\, y\,\rangle = \alpha\langle x,\, y\,\rangle\,] \\
&= \textstyle\sum_n |\langle f,\, S^{-1}x_n\,\rangle|^2
\end{aligned}$$

Therefore, $B^{-1}\|f\|^2 \leq \sum_n |\langle f,\, S^{-1}x_n\,\rangle|^2 \leq A^{-1}\|f\|^2$
Therefore, $\{S^{-1}x_n\,\}$ is a frame!

(3) Since $SS^{-1} = S^{-1}S =$ I, we have
f $= S^{-1}Sf = S^{-1}(\sum_n \langle f,\, x_n\,\rangle x_n) = \sum_n \langle f,\, x_n\,\rangle (S^{-1}x_n)$

Also, f $= \sum_n \langle f,\, S^{-1}x_n\,\rangle x_n \quad (\textit{since} \quad SS^{-1} = I \quad \textit{and} \quad S(S^{-1})f = f\,)$

Q.E.D.

Note:

(1) $S^{-1}x_n$ is called the standard dual frame

(2) When $\{x_n\}$ is not exact, there exists infinitely many dual frames, $\{{x_n}^*\}$ s.t.

$$\forall f \in H, \quad f = \sum_n \langle f, {x_n}^* \rangle x_n = \sum_n \langle f, x_n \rangle {x_n}^*$$

(3) If $\{x_n\}$ is exact, $\{S^{-1}x_n\}$ is the unique dual, also $\{x_n\}$ and $\{S^{-1}x_n\}$ are bi-orthogonal, i.e.,

$$\langle x_m, S^{-1}\cancel{x}_n \rangle = \delta_{mn} \quad \equiv \quad \begin{cases} 1, & \text{if m } = \text{ n} \\ 0, & \text{if m } \neq \text{ n} \end{cases}$$

Theorem 1.2.2 (Tight Frames with A = B = 1 —- doesn't imply ONB)

Recall: If $\{x_n\}$ is an ONB, then $\sum_n |\langle f, x_n \rangle|^2 = \|f\|^2$

e.g. Let $\{e_n\}_{n=1}^{\infty}$ be an ONB of H, consider

$$x_n \equiv \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}$$

Then we can show that $\{x_n\}$ is a tight frame with A = B = 1. Obviously, $\{x_n\}$ is not an ONB.

Theorem 1.2.2 Let $\{x_n\}$ be a tight frame with A = B = 1. If $\|x_n\|^2 = 1$ $\forall$ n, then $\{x_n\}$ is an ONB.

Pf: Since $\{x_n\}$ is a frame, $\overline{\underline{\text{sp}}}\ \{x_n\} = $ H. ("$\overline{\underline{\text{sp}}}$" stands for the closure of sp). All we have to show is that $\langle x^n, x^k \rangle = \delta_{mn}$.

$$\begin{aligned} \|x_n\|^2 &= \textstyle\sum_k |\langle x_n, x_k \rangle|^2 = |\langle x_n, x_n \rangle|^2 + \sum_{k \neq n} |\langle x_n, x_k \rangle|^2 \\ &= \|x_n\|^4 + \textstyle\sum_{k \neq n} |\langle x_n, x_k \rangle|^2 \end{aligned}$$

$\langle x_n, x_k \rangle = 0 \quad \forall n \neq k.$ Q.E.D.

Theorem 1.2.3 Let $\{x_n\}$ be a frame of H, and let f $\in$ H. If $\exists\{c_n\}$ s.t. f $= \sum_n c_n x_n$, then,

$$\sum_n |c_n|^2 = \sum_n |\langle f, S^{-1}x_n \rangle|^2 + \sum_n |c_n - \langle f, S^{-1}x_n \rangle|^2$$

Note: *This implies that the standard dual frame yields the least-square-norm coefficients.*

Pf: Let $a_n = \langle f, S^{-1}x_n \rangle$ then

$\langle f, S^{-1}f \rangle = \langle \sum_n c_n x_n, S^{-1}x_n \rangle = \sum_n c_n \langle x_n, S^{-1}f \rangle$

$= \sum_n c_n \langle S^{-1}x_n, f \rangle = \sum_n c_n \overline{a_n}$

Meantime,

$\langle f, S^{-1}f \rangle = \langle \sum_n a_n x_n, S^{-1}f \rangle = \sum_n a_n \langle x_n, S^{-1}f \rangle = \sum_n |a_n|^2$

Therefore, $\sum_n |a_n|^2 + \sum_n \underbrace{|c_n - a_n|^2}_{(c_n - a_n)\overline{(c_n - a_n)}} = \sum_n |c_n|^2$ Q.E.D.

§1.3 Construction of Dual Frames $\{x_n{}^*\}$

Definition: (Bessel Sequence) $\{x_n\}$ is *Bessel* if

$$\sum_n |\langle f, x_n \rangle|^2 \leq \infty \qquad \forall f \; \cancel{in} \in H$$

In fact, if $\{x_n\}$ is Bessel, $\sum_n |\langle f, x_n \rangle|^2 \leq B\|f\|^2$.

Define U:H $\to l^2$ by (assuming $\{x_n\}$ is frame) $\forall f \in H$, Uf $= \{\langle f, x_n \rangle\}_n$ (coefficient map). Let $\{x_n{}^*\}$ be a Bessel sequence and define V:H $\to l^2$ by:

$$\forall f \in H, \quad Vf = \{\langle f, x_n{}^* \rangle\}_n$$

Both U and V are bounded linear operators, $V^* : l^2 \to$ H is given by:

$$\forall c \in l^2, \quad V^*c = \sum_n c(n) x_n{}^*$$

$\langle Vf, c \rangle_{l^2} = \langle f, \underbrace{V^*c}_{?} \rangle$ Consider the operator $V^*U : H \to H$

$$\forall f \in H, V^*Uf = V^*(\{\langle f, x_n \rangle\}) = \sum_n \langle f, x_n \rangle x_n{}^*$$

space

conversely, if $V^*U = I$, one can easily show that $\{x_n^*\}$ is a dual frame of $\{x_n\}$

if $\{x_n{}^*\}$ is a dual frame of $\{x_n\}$, then V^* Uf = f $\quad\Rightarrow\quad V^*$ U = I.

Theorem 1.3.1 $\{x_n{}^*\}$ is a dual frame of $\{x_n\}$, iff $\exists$ a bounded V^* s.t. V^* U = I.

Rmk: (a) $\{x_n\}$ is given and U is given. In order to find duals $\{x_n{}^*\}$, we need to find left inverses of U.

- ◇ A mapping A:X $\rightarrow$ Y is left invertible if $\exists$ B:Y $\rightarrow$ X s.t. BA = I on X.
- ◇ A is left invertible iff A is one-to-one.
- ◇ Left inverse is not unique.

Rmk: (b) If V^* is bounded s.t. V^* U = I, then $\{x_n{}^*\} = V^* e_n$, where $\{e_n\}$ is the standard ONB of l^2 .

Theorem 1.3.2 Let $\{x_n\}$ be a frame, let U be defined earlier. Then the class of all left inverses of U is given by,

$$V^* = (U^*U)^{-1}U^* + W(I - U(U^*U)^{-1}U^*) \tag{2}$$

where W: $l^2 \rightarrow$ H is a free bounded linear operator.

Pf: (a) U^* U = S, the frame operator. Therefore, $(U^*U)^{-1}$ exist.

(b) V^* U $= (U^*U)^{-1}U^*U + W(I - U(U^*U)^{-1}U^*)U$

$= \mathrm{I} + \mathrm{W}(\mathrm{U} - U(U^*U)^{-1}U^*U) = \mathrm{I}.$

(c) Let $V_o{}^*$ bee any left inverse of U. We want to find a W s.t. equation (1) gives rise to $V_o{}^*$. This is easy since if we let W = $V_o{}^*$, then,

$$\begin{aligned} &(U^*U)^{-1}U^* + W(I - U(U^*U)^{-1}U^*) \\ =\ &(U^*U)^{-1}U^* + V_o{}^* - V_o{}^*U(U^*U)^{-1}U^* = V_o{}^* \end{aligned}$$

Q.E.D.

Corollary 1.3.2 Let $\{x_n\}$ be a frame, and let W: $l^2 \rightarrow$ H be defined by any Bessel sequence $\{y_n\}$ s.t.

$$\forall c \in l^2, \quad Wc = \sum_n c(n) y_n$$

Then the class of all dual frames $\{x_n^*\}$ is given by,

$$x_n^* = S^{-1}x_n + y_n - \sum_m \langle S^{-1}x_n, x_m \rangle y_m$$

Pf: $x_n^* = V^*e_n^* = (U^*U)^{-1}U^*e_n + W(I - U(U^*U)^{-1}U^*)e_n$
$= (U^*U)^{-1}U^*e_n + We_n - WU(U^*U)^{-1}U^*e_n$
$= (U^*U)^{-1}x_n + y_n - WU(U^*U)^{-1}x_n$
$= S^{-1}x_n + y_n - WUS^{-1}x_n$
$= S^{-1}x_n + y_n - W\{\langle S^{-1}x_n, x_m\rangle\}$
$= S^{-1}x_n + y_n - \sum_n \langle S^{-1}x_n, x_m\rangle\, y_m$ Q.E.D.

§1.4 Construction of dual Gabor frames

Q1: What's the structure of dual Gabor Frames?
Q2: How? Dual formula.

Theorem 1.4.1 Let S be the frame operator corresponding to a Gabor frame, $\{g_{m,n}\}$. Let E_n and T_n be the modulation and translation operator, i.e., $E_m f(t) = f(t)e^{\frac{2\pi imt}{N}}$, and $T_n f(t) = f(t - nT)$. Then:

$$S(E_mT_nh) = E_mT_n(Sh), \quad \forall h \in H$$

Pf: $S(E_mT_nh) = \sum_{j,k}\langle E_mT_nh, E_jT_kg\rangle E_jT_kg(t)$
$= \sum_{j,k}\langle h(t), g(t - kT + NT)e^{\frac{2\pi i(j-m)(t+nT)}{N}}\rangle g(t - kT)e^{\frac{2\pi ijt}{N}}$
$= \sum_{j,k}\langle h(t), g(t-(k-n)T)e^{\frac{2\pi i(j-m)t}{N}} e^{\frac{2\pi i(j-m)nT}{N}}\rangle g(t-kT)e^{\frac{2\pi ijt}{N}}$
$= \sum_{j,k}\langle h(t), g(t-(k-n)T)e^{\frac{2\pi i(j-m)t}{N}}\rangle e^{\frac{-2\pi i(j-m)nT}{N}} g(t-kT)e^{\frac{2\pi ijt}{N}}$

Note: $E_mT_ng(t - (k-n)T)e^{\frac{2\pi i(j-m)t}{N}}$
$= E_mg(t - (k-n)T - nT)e^{\frac{2\pi i(j-m)(t-nT)}{N}}$
$= g(t - kT)e^{\frac{2\pi i(j-m)(t-nT)}{N}} e^{\frac{2\pi imt}{N}}$
$= g(t - kT)e^{\frac{2\pi i(j-m)t}{N}} e^{\frac{2\pi imt}{N}} e^{\frac{-2\pi i(j-m)nT}{N}}$
$= g(t - kT)e^{\frac{2\pi ijt}{N}} e^{\frac{-2\pi i(j-m)nT}{N}}$

$= \sum_{j,k}\langle h(t), g(t-(k-n)T)e^{\frac{2\pi i(j-m)t}{N}}\rangle E_mT_ng(t-(k-n)T)e^{\frac{2\pi i(j-m)t}{N}}$
$= E_mT_n(\sum_{j,k=-\infty}^{\infty}\langle h, g_{j-m,k-n}\rangle g_{j-m,k-n}$

Let $p = j - m$, and $q = k - n$ to obtain:
$= E_mT_n(\sum_{j,k=-\infty}^{\infty}\langle h, g_{p,q}\rangle g_{p,q}$
$= E_mT_n(Sh)$ Q.E.D.

Corollary 1.4.2 $\quad S_{-1}(E_m T_n g) = E_m T_n (S_{-1} g)$

Pf: Let $h = S_{-1}g$. Then, $S[E_m T_n (S_{-1}g)] = E_m T_n(S(S_{-1}g)) = E_m T_n g \quad \diamond$

Theorem 1.4.3 (Dual Gabor Formula) Let $\{g_{mn}\}$ of $L^2(\mathbf{R})$. Let C $\in L^2(\mathbf{R})$ be s.t. $\{C_{m,n}\}$ is a Bessel sequence. Then the set of all dual Gabor functions is given by

$$\gamma = S^{-1}g + C - \sum_{mn} \langle S^{-1}g, \, g_{mn} \rangle \; c_{mn} \qquad (2)$$

and dual Gabor frames are generated by, $\gamma_{mn}(t) = \gamma(t - nT)e^{\frac{2\pi imt}{N}}$

Pf: (outline of Proof) Use the general dual frame formula.

(1) Select W operator by,
$\forall d \in l^2$, Wd $= \sum_{mn} d(m,n) C_{mn} \qquad (d = \{d(m,n)\})$,

(2) Show that the translation and modulation of γ in (2) is a dual Gabor frame, i.e., we need to go through a similar computation of $S(E_m T_n h) = E_m T_n (Sh)$.

Applications: One can construct compactly supported dual Gabor function γ. (Ex. a discrete case). Let γ be zero at the tails, $\gamma(0) = \gamma(1) = \cdots = \gamma(M-1) = 0, \quad \gamma(L-1) = \gamma(L-2) = \gamma(L-1) \cdots = \gamma(L-1)$.

$$\begin{bmatrix} \gamma(0) - \gamma^0(0) \\ \gamma(1) - \gamma^0(1) \\ \vdots \\ \gamma(M-1) - \gamma^{M-1}(0) \\ \gamma(L-M) - \gamma^{L-M}(0) \\ \vdots \\ \gamma(L-1) - \gamma^{L-1}(0) \end{bmatrix} =$$

$$
= \begin{bmatrix} C(0) \\ C(1) \\ \vdots \\ C(M-1) \\ C(L-M) \\ \vdots \\ C(L-1) \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{14} & a_{15} & \cdots & a_{17} \\ a_{21} & a_{22} & \cdots & a_{24} & a_{25} & \cdots & a_{27} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{41} & a_{42} & \cdots & a_{44} & a_{45} & \cdots & a_{47} \\ a_{51} & a_{52} & \cdots & a_{54} & a_{55} & \cdots & a_{57} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{71} & a_{72} & \cdots & a_{74} & a_{75} & \cdots & a_{77} \end{bmatrix} \begin{bmatrix} | \\ | \\ | \\ C's \\ | \\ | \\ | \end{bmatrix}
$$