PARTIALLY CALIBRATED CAMERAS IN COMPUTER VISION AND MULTIVIEW IDEALS

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *PARTIALLY CALIBRATED CAMERAS IN COMPUTER VISION AND MULTIVIEW IDEALS* by Jose Tanquilut and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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PARTIALLY CALIBRATED CAMERAS IN COMPUTER VISION AND MULTIVIEW IDEALS

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We consider two problems that arise from the intersection of algebraic geometry and computer vision: first, a fundamental problem in computer vision is determining whether or not a set of point correspondences can be produced by n projective cameras. When $n = 2$, the answer to this question depends on the existence of a real 3×3 matrix, called the fundamental matrix, satisfying a set of polynomial constraints. We generalize some known results about the existence of a fundamental matrix, based on certain assumptions about the calibration of the cameras. Second, a set of n cameras defines a rational map ϕ from $\mathbb{P}^3$ into $(\mathbb{P}^2)^n$. This map is an instance of a more general class of rational maps from $\mathbb{P}^{r-1}$ into $(\mathbb{P}^{s-1})^n$ defined by matrices $A_1, \ldots, A_n$. We prove the existence of a universal Gröbner basis of the ideal of this map, and provide a determinantal representation of the generators of this ideal.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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I thank everybody and their mothers. . . (optional).

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Chapter 1

Introduction

Computer vision is a field centered around analyzing and extracting information from image data, and mimicing the processes of biological vision (such as perception) with artificial systems. [8] Solving problems in computer vision requires techniques from many disciplies, including computer science, physics, neuroscience, and mathematics. The intersection of computer vision and mathematics is concerned with understanding the geometry of images and cameras, and the relationship between the two. These relationships can often be stated as systems of polynomial equations, so methods in algebraic geometry end up being very useful in attacking these problems.

1.1 Partially Calibrated Epipolar Matrices

The first step in framing computer vision problems in a mathematical context is to think of a camera as a matrix. When a camera takes a picture, it produces a two-dimensional image of a three-dimensional object, so we can think of the camera as being a map from $\mathbb{R}^3$ to $\mathbb{R}^2$. Moreover, the mapping is a projection from $\mathbb{R}^3$ to $\mathbb{R}^2$, so a camera can be thought of as a *linear map* from $\mathbb{R}^3$ to $\mathbb{R}^2$. In practice, it turns out that the natural environment for these objects is the projective space, so a camera can be thought of as a linear map from three-dimensional projective space $\mathbb{P}^3$, to two-dimensional projective space $\mathbb{P}^2$. Since the camera is a linear map, we can then associate to it a matrix P, which encodes the relevant characteristics of the camera, such as the position of its center, the aspect ratio of the image points, and the focal length of the camera. These characteristics determine the calibration information of the camera.

One class of problems in computer vision is to understand the relationship between the image and the characteristics of the camera that has taken the image. Whenever we look at a picture, we can deduce information about the camera that took it, by looking at the perspective, relative position of objects in the picture, and so on. More generally, given multiple images, taken by multiple cameras, we would like to determine the calibration information of the cameras that took the images.

We consider the case of two images being taken by two cameras in Figure 1.1. Each world point $\mathbf{X}_i$ projects down to two image points, x_i in the first image and y_i in the second image. In this case, corresponding pairs of image points (x_i, y_i) must satisfy a set of polynomial constraints called *epipolar constraints*. These constraints take the form of a real 3×3 matrix, called an *epipolar matrix*. [8]

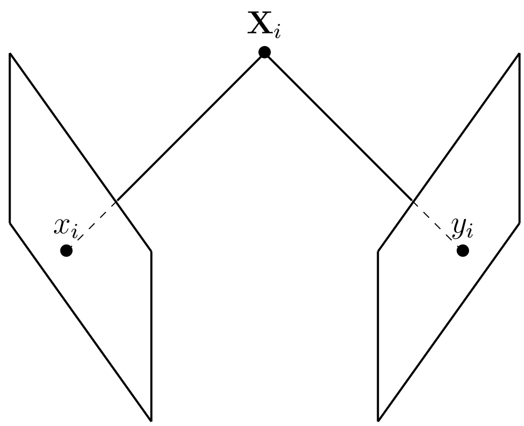

Figure 1.1: A world point $\mathbf{X}_i$ and the corresponding pair of image points $(x_i,\, y_i)$ taken by two cameras.

A slight variation of the problem stated above is determining whether, given a set of corresponding image points, an epipolar matrix exists such that the image points satisfy the corresponding epipolar constraints. For a given set of corresponding image points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, the epipolar constraints translate to a set of polynomial equations in the entries of a 3×3 matrix. Therefore, the question of certifying the existence of an epipolar matrix boils down to the existence of a nontrivial real solution to all of those polynomial equations.

What is already known about this problem can be divided into two broad cases: first, the uncalibrated case, which assumes we know nothing about the calibration of the two cameras, and second, the totally calibrated case, in which we know everything about the calibration of the two cameras. These results are presented in [1]. One area of our research considers an intermediate case, in which we know the calibration information about one camera, but not the other.

1.2 The Multiview Ideal

In the same way that we can think of a camera as a linear map, a set of n cameras can be thought of as a single map, from $\mathbb{P}^3$ to n copies of $\mathbb{P}^2$. The image of this map, called the *multiview variety*, are the solutions to a set of polynomial equations called the *multiview ideal*. As shown in [2], a great deal is known about the structure of this ideal, but it is a specific instance of a more general construction. Since the cameras that produce the multiview variety are defined as 3×4 matrices, we can consider what happens when we let the dimensions of the matrices vary. More specifically, a set of n $s \times r$ matrices define a map from $\mathbb{P}^{r-1}$ to n copies of $\mathbb{P}^{s-1}$. While we lose the connection to computer vision in this case, the points in the image of this map still satisfy polynomial constraints. The second area of our research examines the structure of the multiview ideal in this more general case.

1.3 Results

A paper by Agarwal, Lee, Sturmfels, and Thomas [1] relates the existence of an epipolar matrix, in both the calibrated and uncalibrated case, to the rank of a linear map Z defined by the image points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$. We attempt to extend these results to the partially calibrated case, where the calibration of only one camera is known. From the original map Z we define a family of linear maps Z_K parametrized by the possible calibrations of the camera whose calibration was unknown. The calibration information of a camera can be expressed as an upper triangular matrix, defined up to scale. Therefore, the calibration matrix can then be considered an element of $\mathbb{P}^4$, and we discovered that the set of all such possible calibrations is a subset of $\mathbb{P}^4$ whose Zariski closure is the entirety of $\mathbb{P}^4$. Intuitively, this means that the case of one camera being calibrated is very close to the case of both cameras being completely calibrated, since most random upper triangular matrices will result in a valid calibration.

Another paper, by Aholt, Sturmfels, and Thomas [2], determined that a universal Gröbner basis of the multiview ideal was given by the maximal minors of a particular set of matrices, defined by the cameras $A_1, \ldots, A_n$. This was done by embedding the map defined by the camera matrices into a diagonal map from $\mathbb{P}^3$ to $(\mathbb{P}^3)^n$. Using results from [3], Aholt, Sturmfels and Thomas are able to characterize the initial ideal of the ideal given by the diagonal map, which also gives a determinantal description of the initial ideal of the multiview ideal. In general, we consider a rational map from $\mathbb{P}^{r-1}$ to $(\mathbb{P}^{s-1})^n$ defined by $r \times s$ matrices $A_1, \ldots, A_n$ of rank n. A similar set of minors turns out to give a universal Gröbner basis for the multiview ideal in this case, as well. The same diagonal embedding can be used to give a determinatal representation of the generators of the multiview ideal. Finally, we compute the multigraded Hilbert function of the initial ideal of the multiview ideal certain cases.

1.4 Outline

The outline of this paper is as follows: Chapter 2 provides a overview of algebraic geometry, describing basic properties of ideals, Gröbner bases, and varieties. Chapter 3 is a description of the fundamental algebraic objects in computer vision, the projective camera and the epipolar matrix. We describe the connection between the two objects and include a derivation of the epipolar constraints that provide the motivation for the next chapter. In Chapter 4, we generalize the criteria for certifying the existence of an epipolar matrix to the case where the cameras are partially calibrated. In Chapter 5, we describe the structure of the generalized multiview ideal and its initial ideal. We prove that the maximal minors of a particular set of matrices is a universal Gröbner basis for the multiview ideal. We also compute, for the class of these rational maps from $\mathbb{P}^{r-1}$ into $(\mathbb{P}^1)^n$, the multigraded Hilbert function of the initial ideal with respect to a $\mathbb{Z}^n$-grading. In Chapter 6, we summarize our results and provide possible directions for further research.

Chapter 2

Algebraic Geometry

This chapter provides a brief description of the basic objects in algebraic geometry; this material is covered in more detail, e.g., in [4].

2.1 Polynomial Rings and Ideals

The primary algebraic objects of study in the field of computer vision are ideals of polynomial rings $k[x_1, \ldots, x_n]$, where k is an algebraically closed field (typically $\mathbb{C}$).

Definition 2.1. An **ideal** I of a commutative ring R is a subset of R such that for all $a, b \in I$ and $r \in R$, $a + b \in I$ and $r\, a \in I$.

Example 2.1. Given any commutative ring R, the set $\{0\}$ and the whole ring R are both ideals. An ideal I that is neither is called a **proper** ideal.

Example 2.2. Let F be a field, and I an ideal of F with a nonzero element a. Then $a^{-1}\, a = 1 \in I$, and $b \cdot 1 \in I$ for all $b \in F$. This shows that $I = F$.

Example 2.3. Let $R = \mathbb{Z}$, and I the set of all even integers. Then I is an ideal: given $a, b \in I$, $a + b$ is even, and na is even for all integers $n \in \mathbb{Z}$. The set of all odd integers, however, is not an ideal — 3 and 5 are odd, but their sum 8 is not.

Example 2.4. A common construction is to define an ideal in terms of a generating set. Given a subset $S \subseteq R$, the **ideal generated by** S is the set

$$\langle S \rangle := \left\{ \sum_{i=1}^{n} r_i\, a_i \;:\; n \in \mathbb{N},\, r_i \in R, a_i \in S \right\}.$$

We check that this is an ideal:

(1) two elements of $\langle S \rangle$ are of the form

$$\sum_{i=1}^{n} r_i\, a_i \text{ and } \sum_{i=1}^{m} r_i\, b_i$$

which is still a sum of the form

$$\sum_{i=1}^{n+m} r_i\, a_i$$

and is therefore in $\langle S \rangle$.

(2) given any $r \in R$,

$$r \left[\sum_{i=1}^{n} r_i\, a_i \right] = \sum_{i=1}^{n} r r_i\, a_i$$

is an element of $\langle S \rangle$.

When $S = \{a_1, \ldots, a_n\}$, then we write

$$\langle a_1, \ldots, a_n\rangle = \{r_1\, a_1 + \ldots + r_n\, a_n \,:\, r_i \in R\}$$

to denote the ideal generated by S.

Example 2.5. Given two ideals I and J of R, their sum

$$I + J := \{a + b \,:\, a \in I, b \in J\},$$

their product

$$IJ \,:\, \left\{\sum_{i=1}^{n} a_i\, b_i \,:\, i \in \mathbb{N}, a_i \in I, b_I \in J\right\},$$

and their intersection $I \cap J$ are all ideals of R.

In this thesis we will concern ourselves with some particular classes of ideals:

Definition 2.2. A proper ideal I is a **prime ideal** if for any product $ab \in I$, either $a \in I$ or $b \in I$.

Example 2.6. In $\mathbb{Z}$, the ideal $\langle p\rangle$ is a prime ideal if and only if $|p|$ is a prime number: for any product of integers mn, $\langle p\rangle$ is prime if and only if p divides m or n; in particular, writing the prime factorization of p as

$$p = p_1 \ \ldots \ p_k,$$

we see that if $\langle p\rangle$ is prime, p divides one of the primes p_i in its prime factorization. Then either p or $-p$ is a prime number. Conversely, if $|p|$ is a prime number, then given and product $mn \in \langle p \rangle$, $|p|$, and therefore p, must divide either m or n.

To contrast, $\langle 6 \rangle$ is not prime: $2 \cdot 3 \in \langle 6 \rangle$ but neither 2 nor 3 are in $\langle 6 \rangle$.

Definition 2.3. The **radical** of an ideal I is the set

$$\text{rad } I = \{a : a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

It can be verified that rad I is an ideal that contains I. If $I = \text{rad } I$, then I is a **radical ideal**.

Example 2.7. Let $R = \mathbb{Z}$ and $I = \langle 25 \rangle$. Then rad $I = \langle 5 \rangle$: an element of $\langle 5 \rangle$ is of the form $5n$ for some $n \in \mathbb{Z}$, so $(5n)^2 = 25n^2 \in I$. Conversely, any element in rad I must be divisible by 25, and is therefore in $\langle 25 \rangle$.

Proposition 2.1. *Let I be a prime ideal of R. Then I is a radical ideal: given $a \in$ rad I, there exists some $n \in \mathbb{N}$ such that $a^n \in I$. But since I is prime, this shows that $a \in I$, so rad $I = I$.*

A particularly important structural characteristic of commutative rings is given by its ideal structure:

Definition 2.4. An ideal I of R is **finitely generated** if $I = \langle a_1, \ldots, a_n \rangle$ for some $a_i \in R$.

Definition 2.5. A ring R is **Noetherian** if every ideal I of R is finitely generated.

By appealing to Zorn's lemma, one can prove the following properties of Noetherian rings:

Proposition 2.2. *A ring R is Noetherian if and only if the ideals in R satisfy the* **ascending chain condition***, i.e., for any chain of ideals*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

in R, for some $N \in \mathbb{N}$, $I_n = I_{n+1}$ for all $n \geq N$.

Theorem 2.3 (Hilbert's Basis Theorem)**.** *If a ring R is Noetherian, then the polynomial ring $R[x]$ is Noetherian. In particular, $R[x_1, \ldots x_n]$ is Noetherian.*

In practice, this allows us to define any ideal $I \subseteq \mathbb{C}[x_1, \ldots x_n]$ in terms of a finite generating set $\{f_1, \ldots, f_n\}$. Note that an ideal I can be defined by more than one generating set:

Example 2.8. Let $R = \mathbb{C}[x, y]$ and $I = \langle x, y \rangle$. Then it can be checked that $I = \langle x + y, x - y \rangle$ as well.

Given an ideal I of a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, it is sometimes necessary to work with a generating set with more structure. To understand this structure we must endow our polynomial ring with some additional structure.

2.2 Monomial Ideals and Gröbner Bases

We begin by defining a monomial term order in a polynomial ring:

Definition 2.6. A **monomial** in $\mathbb{C}[x_1, \ldots, x_n]$ is a polynomial of the form $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$.

Definition 2.7. A **monomial ideal** I is an ideal that is generated by monomials; i.e.

$$I = \langle \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_m} \rangle$$

for some monomials $\mathbf{x}^{\alpha_j}$. A **term order** on $\mathbb{C}[x_1, \ldots, x_n]$ is a total ordering $\succ$ on the monomials of $\mathbb{C}[x_1, \ldots, x_n]$ satisfying the following properties:

(1) if $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}$, and $\mathbf{x}^{\gamma}$ is any monomial, then $\mathbf{x}^{\alpha} \cdot \mathbf{x}^{\gamma} \succ \mathbf{x}^{\beta} \cdot \mathbf{x}^{\gamma}$.

(2) $\succ$ is a well-ordering on the set of all monomials, i.e., any set of monomials has a smallest element with respect to $\succ$.

Here are a few typical examples of term orders:

Example 2.9. In $\mathbb{C}[x]$, the only term order is given by $1 \prec x \prec x^2 \prec x^3 \prec \ldots$.

Example 2.10. The **lexicographic** or **dictionary term order** $\succ_{lex}$ on $\mathbb{C}[x_1, \ldots, x_n]$ is given by setting $x_1 \succ_{lex} x_2 \succ_{lex} \cdots \succ_{lex} x_n$, and then defining $\mathbf{x}^{\alpha} \succ_{lex} \mathbf{x}^{\beta}$ if and only if the leftmost nonzero element of $\alpha - \beta$ is positive. With respect to this term order, in $\mathbb{C}[x, y, z]$, $x \succ_{lex} y \succ_{lex} z$, $x^3 \succ_{lex} xyz \succ_{lex} z^5$.

Example 2.11. The **graded lex order** $\succ_{grlex}$ is defined as follows: given $\alpha \in \mathbb{Z}^n_{\geq 0}$, define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

We say that $\mathbf{x}^\alpha \succ_{grlex} \mathbf{x}^\beta$ if and only if $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\mathbf{x}^\alpha \succ_{lex} \mathbf{x}^\beta$. With respect to this term order, $z^5 \succ_{grlex} x^3 \succ_{grlex} xyz$.

Example 2.12. The **graded reverse lex order** $\succ_{grevlex}$ is defined as follows: $\mathbf{x}^\alpha \succ_{grevlex} \mathbf{x}^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\alpha - \beta$ is negative. With respect to this term order, $z^5 \succ_{grevlex} \succ_{grevlex} xyz \succ_{grevlex} xz^2$.

By defining a term order on $\mathbb{C}[x_1, \ldots, x_n]$, we can extend the notion of the degree of univariate polynomial $f(x)$ to the multivariate case:

Definition 2.8. Let $\succ$ be a term order on $\mathbb{C}[x_1, \ldots, x_n]$. Let

$$f = \sum_{a_i=1}^{m} c_i \, \mathbf{x}^{a_i}$$

where $a_i \in \mathbb{Z}^n_{\geq 0}$. Let $\mathbf{x}^a = \max_\succ \{\mathbf{x}^{a_i} \, : \, i = 1, \ldots, m\}$. Then a is the **multidegree of** f with respect to $\succ$, denoted by multideg(f). With this notation, the **leading coefficient of** f is the coefficient c_a of the $\mathbf{x}^a$ term in f, the **leading monomial** of f is $\mathbf{x}^a$, and the **leading term** of f is $c_a \, \mathbf{x}^a$.

Example 2.13. Let $\succ$ be the lexicographic term order on $\mathbb{C}[x, y, z]$ with $x \succ y \succ z$, and let

$$f(x, y, z) = 2x^3 + 3x^2y + 4xy^2 + 5xyz.$$

Then multideg$(f) = (3, 0, 0)$, the leading coefficient of f is 2, and the leading term of f is $2x^3$.

Given an ideal I of a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, an important object of study is the set of leading terms of polynomials in I:

Definition 2.9. Fix a term order $\succ$ on $\mathbb{C}[x_1, \ldots, x_n]$. The **initial ideal** of I with respect to $\succ$, denoted by $\mathrm{in}_{\succ}(I)$, is the ideal

$$\mathrm{in}\ (I) := \langle \mathbf{x}^{\alpha} : \mathbf{x}^{\alpha} \text{ is the leading term of some } f \in I \rangle.$$

Note that the initial ideal of I is dependent on the term order on $\mathbb{C}[x_1, \ldots, x_n]$, since the leading terms of polynomials in I depends on the given term order. Therefore, we must specify that the initial ideal of I is the initial ideal with respect to a term order $\succ$.

It turns out that the initial ideal gives us the appropriate structure to define a particular generating set of I: [4]

Definition 2.10. Let I be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, with a term order $\succ$. A generating set $G = \{g_1, \ldots, g_s\}$ of I is a **Gröbner basis** of I with respect to the term order $\succ$ if

$$\mathrm{in}_{\succ}(I) = \langle \mathbf{x}^{\alpha} : \mathbf{x}^{\alpha} \text{ is a leading term of } g_i \text{ for some } i \rangle.$$

Note that, like the definition of an initial ideal, a set G is a Gröbner basis of an ideal I with respect to a particular term order. However, it is possible for an ideal I to have multiple Gröbner bases with respect to the same term order. This motivates the following specialization of the above definition:

Definition 2.11. A **reduced Gröbner basis** for a polynomial ideal I is a Gröbner basis G for I such that

(a) the leading coefficient of each $g \in G$ is 1,

(b) for all $g \in G$, no monomial of g lies in $\langle \mathbf{x}^\alpha : \mathbf{x}^\alpha$ is a leading term of some $f \in G \setminus \{g\}\rangle$.

Reduced Gröbner bases enjoy a number of nice properties:

Proposition 2.4. *Let I be a nonempty ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Then for a given monomial order $\succ$, I has a unique reduced Gröbner basis.*

Proposition 2.5. *Let I be a nonempty ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Let $\mathcal{G} = \{G_\succ\}_\succ$ be the collection of all reduced Gröbner bases, where $\succ$ ranges over all possible term orders of $\mathbb{C}[x_1, \ldots, x_n]$. Then $\mathcal{G}$ is a finite set.*

This second result is noteworthy because when $n \geq 2$, there are infinitely many possible term orders on $\mathbb{C}[x_1, \ldots, x_n]$, but across all of these term orders, there are only finitely many possible reduced Gröbner bases. This fact motivates the following definition:

Definition 2.12. Let I be a nonempty polynomial ideal, and let $G_1, \ldots, G_k$ be the collection of all reduced Gröbner bases of I. Then

$$G = \bigcup_{i=1}^{k} G_i$$

is a **universal Gröbner basis** of I.

The universal Gröbner basis of an ideal has an equivalent definition that is suggested by its name:

Definition 2.13. A subset G of a nonempty polynomial ideal I is a **universal Gröbner basis** of I if it is a Gröbner basis of I with respect to every term order on $\mathbb{C}[x_1, \ldots, x_n]$.

2.3 Affine Varieties

The fundamental geometric objects of interest are particular subsets of $\mathbb{C}^n$ called affine varieties. These are subsets of affine space:

Definition 2.14. Let n be a positive integer. Then the n**-dimensional affine space** is the set

$$\mathbb{C}^n = \{(a_1, \ldots, a_n) : a_i \in \mathbb{C}\}.$$

Affine varieties are subsets of $\mathbb{C}^n$ that are defined by polynomials:

Definition 2.15. Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. The **affine variety** defined by $f_1, \ldots, f_s$ is the set

$$V = \{a \in \mathbb{C}^n : f_1(a) = \ldots = f_s(a) = 0\}.$$

This set is denoted by $\mathbb{V}(f_1, \ldots, f_s)$.

Example 2.14. The set $V = \{(a, b) \in \mathbb{C}^2 : a = 0\}$ is the variety defined by the polynomial $f(x, y) = x$. In the notation above, $V = \mathbb{V}(x)$.

Example 2.15. As a generalization of the above example, given a point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, the set $\{(a_1, \ldots, a_n)\} = \mathbb{V}(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

Example 2.16. The set $V = \{(x, y) \in \mathbb{C}^2 : y - x^2 = 0\}$ is a variety defined by the polynomial $f(x, y) = y - x^2$; in $\mathbb{R}^2$ the subset V is a parabola defined by the equation $y = x^2$.

Sometimes, we would like to define an affine variety in terms of an ideal:

Definition 2.16. The **affine variety** of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the set

$$\mathbb{V}(I) = \{a \in \mathbb{C}^n : f(a) = 0 \text{ for all } f \in I\}.$$

Note that if $a \in \mathbb{C}^n$, and $f_1, \ldots, f_s$ are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ such that $f_i(a) = 0$ for all i, then $h_1 f_1 + \ldots + h_s f_s$ is a polynomial that vanishes at a for all polynomials h_i. This allows us to interchange the variety of an ideal and the variety of its generating set:

Lemma 2.6. *Let* $I = \langle f_1, \ldots, f_s \rangle$. *Then* $\mathbb{V}(I) = \mathbb{V}(f_1, \ldots, f_s)$.

Proof. If $a \in \mathbb{V}(I)$, then every polynomial in I vanishes at a; in particular, f_i vanishes at a for each i, so $a \in \mathbb{V}(f_1, \ldots, f_s)$. Conversely, if $a \in \mathbb{V}(f_1, \ldots, f_s)$, and every polynomial $g \in I$ is of the form

$$g = h_1\, f_1 + \ldots + h_s\, f_s,$$

then g vanishes at a, which shows that $a \in \mathbb{V}(I)$. □

This lemma is useful in proving the following properties of varieties:

Lemma 2.7. *Let $V = \mathbb{V}(f_1, \ldots, f_l)$ and $W = \mathbb{V}(g_1, \ldots, g_m)$ be affine varieties. Then $V \cap W$ and $V \cup W$ are also affine varieties.*

The foundation of algebraic geometry is the correspondence between these geometric structure of $\mathbb{C}^n$ and the algebraic structure of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. In the same way that a variety is defined in terms of an ideal, an ideal can be defined in terms of a variety:

Definition 2.17. Let S be a subset of $\mathbb{C}^n$. Then the **ideal defined by** S is the set

$$\mathbb{I}(S) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \,:\, f(a) = 0 \text{ for all } a \in S\}.$$

It is clear that $\mathbb{I}(S)$ is an ideal: given $f, g \in \mathbb{I}(S)$ and $h \in \mathbb{C}[x_1, \ldots, x_n]$, $f + g$ and hf will vanish on S as well.

Example 2.17. Let $S = \{(a_1, \ldots, a_n)\} \subseteq \mathbb{C}^n$. Then $\mathbb{I}(S) = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$.

Note that in this case, S is a variety: in particular, $S = \mathbb{V}(I)$, where $I = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$. With this notation, we have that $\mathbb{I}(\mathbb{V}(I)) = I$, which suggests that this is true in general. However, this isn't the case:

Example 2.18. Let $S = \{0\} \subseteq \mathbb{C}$. Then $S = \mathbb{V}(x^2)$, but $\mathbb{I}(S) = \langle x\rangle$.

It is however possible to say the following about the relationship between the operations $\mathbb{V}$ and $\mathbb{I}$:

Proposition 2.8. *For any ideal* $I \subseteq \mathbb{C}[x_1, \ldots x_n]$, $I \subseteq \mathbb{I}(\mathbb{V}(I))$.

Proof. Let $f \in I$. Then f vanishes on every point in $\mathbb{V}(I)$, so $f \in \mathbb{I}(\mathbb{V}(I))$. □

Another important property relating $\mathbb{V}$ and $\mathbb{I}$ is reverse inclusion:

Proposition 2.9.

(1) *Let* $I \subseteq J$ *be two ideals in* $\mathbb{C}[x_1, \ldots, x_n]$. *Then* $\mathbb{V}(J) \subseteq \mathbb{V}(I)$.

(2) *Let* $V \subset W$ *be two varieties in* $\mathbb{C}^n$. *Then* $\mathbb{I}(W) \subseteq \mathbb{I}(V)$.

Proof.

(1) If a is a point in $\mathbb{V}(J)$, then every polynomial in J will vanish on a; in particular, every polynomial in I will vanish on a, so $a \in \mathbb{V}(I)$.

(2) Every polynomial that vanishes on W will vanish on V, so $\mathbb{I}(W) \subseteq \mathbb{I}(V)$.

□

Based on this information, it is clear that there is some sort of correspondence between affine varieties in $\mathbb{C}^n$ and ideals in $\mathbb{C}[x_1, \ldots, x_n]$. The nature of this correspondence is laid out in Hilbert's Nullstellensatz:

Theorem 2.10 (Hilbert's Nullstellensatz)**.** *Let I be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Then $\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad} I$; in particular, there is a one-to-one correspondence*

$$\{\textit{affine varieties of } \mathbb{C}^n\} \longleftrightarrow \{\textit{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

given by the operations $\mathbb{I}$ and $\mathbb{V}$.

As a consequence of the Nullstellensatz, geometric information about a variety V can be encoded as algebraic information about an ideal I, and vice versa: one such example follows from a class of varieties called irreducible varieties:

Definition 2.18. A variety V is called **irreducible** if V cannot be written as a union $V = V_1 \cup V_2$ where $\emptyset \subsetneq V_i \subsetneq V$.

Theorem 2.11. *A variety V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal.*

2.4 Projective Varieties

The results in the previous section illustrate the correspondence between ideals of $\mathbb{C}[x_1, \ldots, x_n]$ and subsets in $\mathbb{C}^n$. Similar results can be obtained when considering a different ambient space from $\mathbb{C}^n$, called projective space. Before defining complex projective space, we first consider its real counterpart, the real projective space:

Definition 2.19. The **real projective plane**, denoted by $\mathbb{P}^2(\mathbb{R})$, is the set of all lines in $\mathbb{R}^3$ that pass through the origin $(0, 0, 0)$.

The elements of $\mathbb{P}^2(\mathbb{R})$ are lines, but a more useful characterization of these elements comes from the following idea: we let $H = \{(x, y, 1) \in \mathbb{R}^3\}$, and $H_\infty = \{(x, y, 0) \in \mathbb{R}^3\}$. A line L in $\mathbb{P}^2(\mathbb{R})$ will either pass through H in exactly one point, or will be contained in H_∞. In each case, we can associate L to a point in $\mathbb{R}^3$ in a particular way:

(a) if L intersects H, then it intersects H in a unique point $(a, b, 1)$. We can identify $(a, b, 1)$ with the point (a, b) in $\mathbb{R}^2$. Conversely, given any $(a, b) \in \mathbb{R}^2$, there exists a unique line L in $\mathbb{R}^3$ that passes through the origin and $(a, b, 1)$; this gives us a bijective correspondence

$$\{\text{lines in } \mathbb{P}^2(\mathbb{R}) \text{ that intersect } H\} \longleftrightarrow \mathbb{R}^2.$$

(b) if $L \subseteq H_\infty$, then L is uniquely determined the origin and another point $(a, b, 0)$ on L where a and b are not both 0. In particular, if L intersects the line $y = 1$ in H_∞, then it does so at exactly one point $(a, 1, 0)$. Conversely, exactly one line L in H_∞ that passes through the origin will pass through the point $(a, 1, 0)$. We can identify these points with the real line $\mathbb{R}$. This leaves out one line in $\mathbb{P}^2(\mathbb{R})$ in H_∞: the line that passes through the origin and $(1, 0, 0)$. All together, this gives us a bijective correspondence

$$\{\text{lines in } \mathbb{P}^2(\mathbb{R}) \text{ contained in } H_\infty\} \longleftrightarrow \mathbb{R} \cup \{(1, 0, 0)\}.$$

Therefore, we can more concretely represent the elements of $\mathbb{P}^2(\mathbb{R})$ as follows:

$$\mathbb{P}^2(\mathbb{R}) = \{(a, b, 1) \in \mathbb{R}^3\} \cup \{(a, 1, 0) \in \mathbb{R}^3\} \cup \{(1, 0, 0)\} \equiv \mathbb{R}^2 \cup \mathbb{R} \cup \{(1, 0, 0)\}$$

The points in $\mathbb{R} \cup \{(1, 0, 0)\}$ are often called the *points at infinity*, which alludes to the geometric realization of the real projective plane. However, our ultimate goal is to define complex projective space, so we must consider another, purely algebraic construction of the projective plane:

Definition 2.20. Define an equivalence relation $\sim$ on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ as follows:

$$(a,\, b,\, c) \,\sim\, (a',\, b',\, c') \iff (\lambda\, a,\, \lambda\, b,\, \lambda\, c) \,=\, (a',\, b',\, c') \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

Define the **real projective plane** $\mathbb{P}^2(\mathbb{R})$ to be the set of all equivalence classes $(\mathbb{R}^3 \setminus \{(0, 0, 0)\})/\sim$. The elements of $\mathbb{P}^2(\mathbb{R})$ are called **homogeneous coordinates** and are denoted by $[a : b : c]$ where $(a, b, c) \neq (0, 0, 0)$.

Naturally, we must check that these two definitions produce the same set:

Proposition 2.12. *There exists a bijective correspondence*

$$\{\textit{lines in } \mathbb{R}^3 \textit{ that pass through the origin}\} \longleftrightarrow \{[a : b : c] : (a,\, b,\, c) \neq (0,\, 0,\, 0)\}.$$

Proof. We correspond to each line in $\mathbb{R}^3$ that passes through the origin a point in $(a, b, c) \in \mathbb{R}^3$, where $(a, b, c) \neq (0, 0, 0)$. This gives a map

$$\begin{array}{ccc} \{\text{lines in } \mathbb{R}^3 \text{ that pass through the origin}\} & \longrightarrow & \{[a : b : c] : (a,\, b,\, c) \neq (0,\, 0,\, 0)\} \\ (a,\, b,\, c) & \longmapsto & [a : b : c] \end{array}$$

The inverse map is defined as follows: given a homogeneous coordinate $[a : b : c]$, if

$c \neq 0$, map to the line corresponding to $(\frac{a}{c}, \frac{b}{c}, 1)$; otherwise, if $c = 0$ and $b \neq 0$, map to the line corresponding to $(\frac{a}{b}, 1, 0)$, and if $c, b = 0$, map to the line corresponding to $(1, 0, 0)$. Under the equivalence relation $\sim$, this is a well-defined inverse, which exhibits the bijective correspondence. □

Note that the algebraic construction of $\mathbb{P}^2(\mathbb{R})$ is readily generalizable to both different fields and different dimensions. We will mimic this construction to define the complex projective space that will serve as the ambient space for the results in this thesis:

Definition 2.21. Let $n \geq 1$. Define an equivalence relation $\sim$ on the set $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ by setting

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff (\lambda a_0, \ldots, \lambda a_n) = (b_0, \ldots, b_n) \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

The set $(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\sim$ of equivalence classes under $\sim$ is the n**-dimensional projective space**, denoted by $\mathbb{P}^n$.

Example 2.19. Let $n \geq 1$, and let $U_0 = \{[a_0 : a_1 : \ldots : a_n] \in \mathbb{P}^n : a_0 \neq 0\}$. Note that given $[a_0 : a_1 : \ldots : a_n] \in U_0$,

$$[a_0 : a_1 : \ldots : a_n] \sim \left[1 : \frac{a_1}{a_0} : \ldots : \frac{a_n}{a_0}\right],$$

so we can write $U_0 = \{[1 : a_1 : \ldots : a_n] \in \mathbb{P}^n\}$. Then $\mathbb{P}^n$ can be written as a disjoint union

$$\mathbb{P}^n = U_0 \cup \{[0 : a_1 : \ldots : a_n] \in \mathbb{P}^n\},$$

where the elements are in U_0 are in bijective correspondence with the elements of n-dimensional affine space $\mathbb{C}^n$, and the elements of $\{[0 : a_1 : \ldots : a_n] \in \mathbb{P}^n\}$ are in bijective correspondence with the elements of $\mathbb{P}^{n-1}$; then we can write

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}.$$

Note that we can mimic this construction with

$$U_i = \{[a_0 : \ldots : a_{i-1} : 1 : a_{i+1} : \ldots : a_n] \in \mathbb{P}^n\}$$

for $0 \leq i \leq n$, and

$$\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n.$$

This shows that $\mathbb{P}^n$ can be thought of as the union of $n+1$ n-dimensional affine spaces.

We would like to define a projective analog to the affine varieties in the previous section, but we cannot define them in terms of polynomials in exactly the same way: for a general polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ and a general element $[a_0 : a_1 : \ldots : a_n] \in \mathbb{P}^n$,

$$f(a_0, \ldots, a_n) \neq f(\lambda\, a_0, \ldots, \lambda\, a_n)$$

for $\lambda \in \mathbb{C}$. Then a general polynomial f will not be well-defined as a function on $\mathbb{P}^n$. To make sense of how a projective variety should be defined, we require that our polynomials are homogeneous:

Definition 2.22. Write a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ as

$$f(x_0, \ldots, x_n) = \sum_{\alpha} c_\alpha \, \mathbf{x}^\alpha.$$

Then f is **homogeneous of degree** d if $|\alpha| = d$ for all monomials $\mathbf{x}^\alpha$ in f.

Note that given a homogeneous polynomial f of degree d,

$$f(\lambda\, a_0, \ldots, \lambda\, a_n) = \lambda^d\, f(a_0, \ldots, a_n)$$

which shows that while f is still not a well-defined function on $\mathbb{P}^n$, $f(a_0, \ldots, a_n) = 0$ for every representation of the homogeneous coordinate $[a_0 : \ldots : a_n]$. Therefore, it makes sense to define a variety in terms of these homogeneous polynomials:

Definition 2.23. Let $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomials. Then the **projective variety** V defined by $f_1, \ldots f_m$ is the set

$$\mathbb{V}(f_1, \ldots, f_m) = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n : f_1(a_0, \ldots, a_n) = \ldots = f_m(a_0, \ldots, a_n) = 0\}.$$

Given an ideal I of $\mathbb{C}[x_0, \ldots, x_n]$ generated by homogeneous polynomials, the **projective variety** V defined by I is defined as the set

$$\mathbb{V}(I) = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

Definition 2.24. Given a projective variety V in $\mathbb{P}^n$, the **ideal of** V is

$$\mathbb{I}(V) = \langle f \in \mathbb{C}[x_0, \ldots, x_n] : f(a_0, \ldots, a_n) = 0 \text{ for all } [a_0 : \ldots : a_n] \in V \rangle.$$

An important invariant of both projective and affine varieties is the **degree**:

Definition 2.25. The **degree** of a projective variety $V \subseteq \mathbb{P}^n$ is the number of points in the intersection of V with a generic linear subspace L whose codimension is the dimension of V.

The following is a useful result regarding the degree of a projective variety:

Lemma 2.13. *If V_1 and V_2 are two projective varieties in $\mathbb{P}^n$ of dimension d and $n-d$, respectively, then $V_1 \cap V_2 \neq \emptyset$.*

2.5 The Hilbert Function

Finally, we define the Hilbert function, an important function associated to a homogeneous ideal of a polynomial ring.

Definition 2.26. Let R be a commutative ring and G an abelian semigroup. R is a G-**graded ring** if

$$R = \bigoplus_{g \in G} R_g$$

where for all $g, h \in G$, $r \in R_g$ and $s \in R_h$, $rs \in R_{gh}$.

Example 2.20. Let $R = \mathbb{C}[x]$ and $G = \mathbb{N}$. Then the degree gives an $\mathbb{N}$-grading on R:

$$\mathbb{C}[x] = R_0 \oplus R_1 \oplus \ldots$$

where R_i is the set of all polynomials in $\mathbb{C}[x]$ with degree i.

Example 2.21. The polynomial ring $\mathbb{C}[x_{ij} \,:\, i \in [r],\, j \in [n]]$ admits an $\mathbb{N}^n$-grading in the following way: let $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, $\alpha = (\alpha_{ij})$, and define

$$R_u = \left\{ \mathbf{x}^\alpha \,:\, \sum_{i=1}^{r} \alpha_{ij} = u_j \text{ for each } j \right\}.$$

Then

$$\mathbb{C}[x_{ij} \,:\, i \in [r],\, j \in [n]] = \bigoplus_{u \in \mathbb{N}^n} R_u.$$

The Hilbert function is defined in terms of a homogeneous ideal of a graded polynomial ring:

Definition 2.27. Let $\mathbb{C}[x_1, \ldots, x_n]$ be a G-graded polynomial ring, and I a homogeneous ideal. Then the **Hilbert function** of this ideal is a function $\mathcal{H} : G \longrightarrow \mathbb{N}$ where $\mathcal{H}(g)$ counts the number of elements of R_g not in I (equivalently, the number of elements of R_g that do not reduce to 0 in the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I$).

Chapter 3

Computer Vision

3.1 The Camera Model

The description of the camera model and epipolar geometry are taken from Hartley and Zisserman's book on computer vision [8].

The process of image formation is one of constructing a two-dimensional representation of a three-dimensional space. It is therefore natural to think of a camera as a projection from three-dimensional space onto a two-dimensional image. The camera model is formed by *central projection*, in which a ray is drawn from a 3D world point through a fixed point in space, called the *center of projection*. This ray intersects a fixed plane in space, called the *image plane*. The intersection of the ray with the image plane represents the image of the world point. This model is in accord with our intuitive model of a camera, in which a ray of light from a world point passes through the lens of a camera and is captured as a single point on the image.

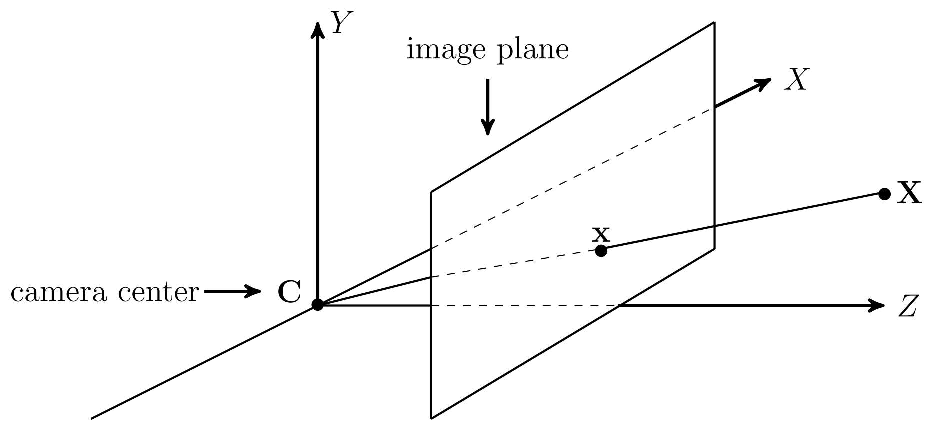

Figure 3.1: A camera model formed by central projection. A world point **X** projects down to an image point **x** in the image plane.

In this setting, world points will be represented by points in the projective space $\mathbb{P}^3$ of the form $[x : y : z : 1]$, and image points by points in the projective plane $\mathbb{P}^2$ of the form $[x : y : 1]$. Central projection is therefore a projection map from $\mathbb{P}^3$ to $\mathbb{P}^2$. We can realize a **projective camera** as a matrix as follows:

Example 3.1. Fix the center of projection to be the origin $[0 : 0 : 0 : 1]$ in $\mathbb{P}^3$. For a given $x, y, z \in \mathbb{R}$, the world line $\{[x : y : z : t] : t \in \mathbb{R}\}$ in $\mathbb{P}^3$ projects down to the image point $[x : y : z] \in \mathbb{P}^2$ under this mapping. This is in fact a linear map $P : \mathbb{P}^3 \longrightarrow \mathbb{P}^2$ that can be represented by a real 3×4 matrix with block structure $P = [I_3 \,|\, \mathbf{0}_3]$, where I_3 is the 3×3 identity matrix and $\mathbf{0}_3$ is the zero vector in $\mathbb{R}^3$. We can verify this by observing that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In general, the center of projection and other intrinsic features of the camera, like the aspect ratio or skew, might change. But these changes can be expressed via a linear transformation on the set of image coordinates:

Example 3.2. If we consider a more general 3×4 matrix

$$P = \begin{bmatrix} 1 & 0 & p_x & 0 \\ 0 & 1 & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $p_x, p_y \in \mathbb{R}$, then

$$P \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x + z\,p_x \\ y + z\,p_y \\ z \end{bmatrix},$$

which has a natural interpretation as a projection map paired with a translation: the first example assumes that the origin $[0 : 0 : 0 : 1]$ is mapped to the origin in the image plane; in general, it might be mapped to some other point of the form $[p_x : p_y : 1]$. This corresponds to a projection and a translation

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \longmapsto \begin{bmatrix} x + z\,p_x \\ y + z\,p_y \\ z \end{bmatrix}$$

which is a transformation given by the matrix P above.

Relaxing the conditions on other intrinsic properties of the camera give rise to similar relaxations on the form of P. A general projective camera is defined as follows:

Definition 3.1. A **projective camera** P is a real 3×4 matrix of rank 3 of the form $P = [A \,|\, \mathbf{t}]$, where A is an invertible 3×3 matrix and $\mathbf{t} \in \mathbb{R}^3$, and P is defined up to scale; i.e., we consider $[A \,|\, \mathbf{v}]$ and $\lambda\,[A \,|\, \mathbf{t}]$ to be the same for nonzero $\lambda \in \mathbb{R}$.

A more useful characterization of the projective camera matrix is given in chapter 6 of Hartley and Zisserman [8]:

Proposition 3.1. *A general projective camera P can be written as*

$$P = K[R \,|\, \mathbf{t}]$$

where R is a 3×3 rotation matrix and K is an invertible matrix that encodes the calibration of the camera.

3.2 Epipolar Constraints

One of the fundamental problems in computer vision is to understand the intrinsic geometry of a system of n projective cameras. We are interested in the case where $n = 2$: given two projective cameras P_1 and P_2, we wish to understand the intrinsic geometry between P_1 and P_2, given a set of m pairs of image points taken by the two cameras. More precisely, given m world points $X_1, \ldots, X_m$, let $P_1 X_i = x_i$ and $P_2 X_i = y_i$ for each $1 \leq i \leq m$, and consider the pairs $(x_1, y_1), \ldots, (x_m, y_m)$. It turns out that for $m \geq 7$, these pairs of image points give rise to a set of constraints on the entries of P_1 and P_2 called **epipolar constraints**. These constraints are derived from the setting illustrated in Figure 3.2. In the figure, $\mathbf{X}$ is a world point whose images under P_1 and P_2 are $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. The centers of these cameras are at C_1 and C_2. The **epipolar plane** π formed by $\mathbf{x}_1$ and $\mathbf{x}_2$ is the plane formed by the lines passing through the points $\{\mathbf{X}, \mathbf{x_1}, \mathbf{C_1}\}$ and $\{\mathbf{X}, \mathbf{x_2}, \mathbf{C_2}\}$. In this setting we can define the following:

Definition 3.2. The **baseline** between two cameras P_1 and P_2 is the line between their centers. The **epipole** is the point of intersection of the baseline with the image plane. In Figure 3.2, the epipoles are $\mathbf{e}_1$ and $\mathbf{e}_2$ — note that these are the image of $\mathbf{C}_2$ in the first view and $\mathbf{C}_1$ in the second, respectively.

Definition 3.3. An **epipolar plane** is a plane containing the baseline.

Definition 3.4. An **epipolar line** is the intersection of an epipolar plane with the

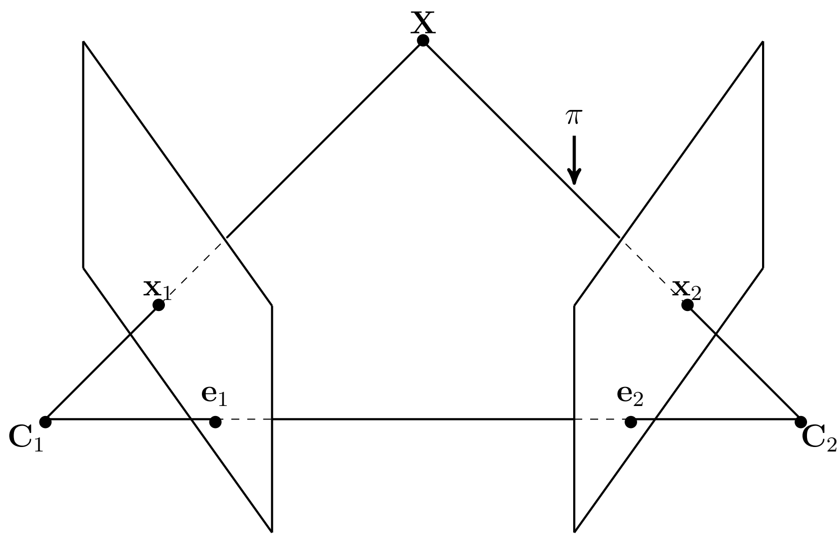

Figure 3.2: The lines between a world point **X** and the camera centers $\mathbf{C}_1$ and $\mathbf{C}_2$ form an epipolar plane π.

image plane. These lines will define the correspondence between points on the image plane.

From Figure 3.2, to each point $\mathbf{x}$ in one of the image planes, we can construct an epipolar line in the second image by taking the line that passes through the corresponding image point $\mathbf{x}'$ and the epipole in the other image plane. This correspondence is encoded in a real 3×3 matrix called the **fundamental matrix**, which turns out to encode the geometry relating the two cameras. Hartley and Zisserman outline a derivation of the fundmamental matrix in [8, Chapter 9]:

Let $\mathbf{x}$ be a point in the first image plane. The first camera center $\mathbf{C}_1$ and this point

$\mathbf{x}$ form a ray that back-projects into the set of world points, points $\mathbf{X}$ that satisfy the equation $P_1\,\mathbf{X} = \mathbf{x}$. The solutions to this equation are of the form

$$\{P_1^+\,\mathbf{x} + \lambda\,\mathbf{C}_1 \,:\, \lambda \in \mathbb{R}\} \cup \{\mathbf{C}_1\}$$

where P_1^+ is the psuedo-inverse of P_1 (so $P_1\,P_1^+ = I$). The images of the points $P_1^+\,\mathbf{x}$ and $\mathbf{C}_1$ in the second image plane are $P_2\,P_1^+\,\mathbf{x}$ and $P_2\,\mathbf{C}_1$, respectively. The epipolar line $\mathbf{l}$ in the second image plane is the line joining these points; therefore,

$$\mathbf{l}' = (P_2\,\mathbf{C}_1) \times (P_2\,P_1^+\mathbf{x})$$

which, since $P_2\,\mathbf{C}_1$ is the epipole in the second image plane $\mathbf{e}_2$, we can rewrite as

$$\mathbf{l}' = [\mathbf{e}_2]_\times\,(P_2\,P_1^+)\,\mathbf{x},$$

and from this we obtain the fundamental matrix F:

Definition 3.5. Let $P_1 = K_1[R_1\,|\,\mathbf{t}_1]$ and $P_2 = K_1[R_2\,|\,\mathbf{t}_2]$ be two projective cameras, and let e_1 and e_2 be the epipoles in the image planes of P_1 and P_2, respectively. Then the **fundamental matrix** F of P_1 and P_2 is the matrix

$$F = [e_2]_\times\,P_2\,P^+,$$

where for a given vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$[v]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

and P^+ is the psuedo-inverse of P_1.

Since projective cameras are defined up to scale, the same applies to the fundamental matrix; we can think of it as an element of $\mathbb{P}^8$. For our purposes, there are several results in [8, Chapter 9] which provide more useful characterizations of the fundamental matrix:

Proposition 3.2. *Let $\mathcal{F}$ be the set of all fundamental matrices in $\mathbb{P}^8$. Then $\mathcal{F}$ is the set of all real 3×3 matrices of rank 2, up to scale.*

Proposition 3.3. *With the same notation as in Definition 3.5, $F = K_2^{-T}\, B\, K_1$, where B is a real 3×3 matrix of rank 2 such that both of its nonzero singular values are equal.*

Example 3.3. Let $P_1 = [I_3 \,|\, \mathbf{0}]$ and $P_2 = [R_2 \,|\, \mathbf{t}_2]$. Then $K_2 = K_1 = I_3$, and the fundamental matrix is the matrix B, in the notation of Proposition 3.3. In this case, the cameras P_1 and P_2 are said to be **calibrated** and the fundamental matrix F is called an **essential matrix**, usually denoted by B.

3.3 The Set of Fundamental and Essential Matrices

We can think of a 3×3 matrix, defined up to scale, as an element of the projective space $\mathbb{P}^8$. In particular, a fundamental matrix F can be thought of as an element of $\mathbb{P}^8$. We can consider the set of fundamental matrices:

Definition 3.6. The set of fundamental matrices in $\mathbb{P}^8$ is denoted by $\mathcal{F}$.

By Proposition 3.2,

$$\mathcal{F} = \{F \in \mathbb{P}^8 \, : \, F \text{ has rank } 2\},$$

which is the difference of two projective varieties, or a quasi-projective variety: if we consider $\mathbb{P}^8$ to be the set of all 3 $\times$ 3 matrices, defined up to scale, then

$$\mathcal{F} = \mathbb{P}^8 \setminus \{F \in \mathcal{P}^8 \, : \, F \text{ has rank } 1\}$$

where the latter set is the variety defined by the 2-minors of a 3 $\times$ 3 matrix.

Similarly, essential matrices can be thought of as real 3 $\times$ 3 matrices, defined up to scale, and therefore as elements of $\mathbb{P}^8$. Let $\mathcal{E}$ be the set of all essential matrices in $\mathbb{P}^8$. Then $\mathcal{E} \subseteq \mathcal{F}$. In fact, in [5], Demazure showed that $\mathcal{E}$ can be described as the solution set of polynomial equations:

Proposition 3.4. *Let $\mathcal{E}$ be the set of all essential matrices in $\mathbb{P}^8$. Then*

$$\mathcal{E} = \{E \in \mathbb{P}^8 \, : \, 2E\,E^T\,E - \operatorname{tr}(E\,E^T)\,E = \mathbf{0}, \, \det E = 0\}.$$

This is a projective variety in $\mathbb{P}^8$: let

$$X = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}.$$

Then $\mathcal{E}$ *is defined by the polynomials* $p_1, \ldots p_{10} \in \mathbb{C}[x_0, \ldots x_8]$*, where*

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix} = 2X\,X^T\,X - \mathrm{tr}(X\,X^T)\,X$$

and $p_{10} = \det\, X$.

Chapter 4

Partially Calibrated Epipolar Matrices

4.1 Motivation

A fundamental problem in computer vision is the following: given m world points $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ and m pairs of corresponding image points $\{(x_1, y_1), \ldots, (x_m, y_m)\}$, does there exist a pair of projective cameras P_1, P_2 such that $P_1 \mathbf{X}_i = x_i$ and $P_2 \mathbf{X}_i = y_i$ for $i = 1, 2, \ldots, m$? In Chapter 3, it was shown that the existence of these cameras is equivalent to the existence of a real 3×3 matrix F, called the **fundamental matrix**, satisfying the equations

$$y_i^T F x_i = 0 \tag{4.1}$$

for $i = 1, 2, \ldots, m$. These equations are called the *epipolar constraints.* In [1], Agarwal, Lee, Sturmfels and Thomas reframe these constraints in the context of linear algebra: given the m pairs of point correspondences above, where $x_i = [x_{i1} :$

$x_{i2} : 1]^T$ and $y_i = [y_{i1} : y_{i2} : 1]^T$, define the matrix Z to be the $m \times 9$ matrix whose i^{th} row is

$$y_i^T \otimes x_i = [y_{i1}\, x_{i1} : y_{i1}\, x_{i2} : y_{i1} : y_{i2}\, x_{i1} : y_{i2}\, x_{i2} : y_{i2} : x_{i1} : x_{i2} : 1].$$

Then Z defines a linear transformation on $\mathbb{P}^8$, which encodes the epipolar constraints in the following way:

Lemma 4.1.

(a) there exists a fundamental matrix F if and only if $\ker Z \cap \mathcal{F}$ is nonempty.

(b) there exists an essential matrix E if and only if $\ker Z \cap \mathcal{E}$ is nonempty.

Proof. We represent a 3×3 matrix F as a column vector by concatenating the rows and taking the transpose: $F = [a_{11} : a_{12} : a_{13} : a_{21} : a_{22} : a_{23} : a_{31} : a_{32} : a_{33}]^T \in \mathbb{P}^8$. Then the i^{th} row of $Z\, F$ is

$$(y_i^T \otimes x_i)\, F = \sum_{j=1}^{3} \sum_{k=1}^{3} y_{ij}\, x_{ik}\, a_{jk}$$

where we let $y_{i3} = x_{i3} = 1$. The corresponding epipolar constraint is

$$y_i^T\, F\, x_i = [y_{i1}\ y_{12}\ 1] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ 1 \end{bmatrix} = \sum_{j=1}^{3} \sum_{k=1}^{3} y_{ij}\, x_{ik}\, a_{jk},$$

which shows that $Z\, F = \mathbf{0}$ if and only if F satisfies each epipolar constraint $y_i^T\, F\, x_i = 0$. Equivalently, a fundamental matrix F (or essential matrix E) exists if and only if $\ker Z \cap \mathcal{F} \neq \emptyset$ (or $\ker Z \cap \mathcal{E} \neq \emptyset$). □

As a consequence of this observation, the problem of certifying the existence of a fundamental matrix reduces to examining the intersection between the linear subspace $\ker Z$ and the quasi-projective variety $\mathcal{F}$. Similarly, to determine whether an essential matrix exists, we consider the intersection between $\ker Z$ and the projective variety $\mathcal{E}$. Intuitively, the lower the rank of Z, the more likely it is that this intersection is nonempty. The strategy presented in [1] is to condition on the rank of Z:

Theorem 4.2. *If* $\operatorname{rank} Z \leq 5$*, then a fundamental matrix* F *exists.*

Theorem 4.3. *If* $\operatorname{rank} Z \leq 4$*, then an essential matrix* E *exists.*

4.2 Generalizations — Partially Calibrated Epipolar Matrices

One way the difference between Theorem 4.2 and 4.3 can be interpreted is as a function of how much is known about the projective cameras — recall that an essential matrix E represents a pair of projective cameras that are totally calibrated. It is therefore natural to consider when a fundamental matrix where some of the calibration information is known. We will call these matrices *partially calibrated epipolar* *matrices*. In general, we assume that a pair of cameras is partially calibrated when we know some information about the calibration of the cameras, but not complete information. To precisely express what this means in terms of the corresponding epipolar matrix, recall that a general fundamental matrix F admits a factorization

$$F = K_1^{-T}\, B\, K_2, \tag{4.2}$$

where B is an essential matrix, and K_1 and K_2 are invertible upper triangular matrices that encode information about the position and skew of each camera. When we assume no knowledge about the calibration of the cameras, we can only assume that the corresponding calibration matrices K_1 and K_2 are invertible and upper triangular. When the cameras are both fully calibrated, we can assume that K_1 and K_2 are both the identity matrix, which means that the position and skew of the cameras are normalized.

Naturally, we can examine the case where K_1 and K_2 are not necessarily the identity matrix, but we have information about their structure. There are a variety of assumptions we can make about the structure of K_1 and K_2, but we will focus on the case where one of K_1 or K_2 is the identity, and the other is an invertible upper triangular matrix of the form

$$K = \begin{bmatrix} x & 0 & a \\ 0 & y & b \\ 0 & 0 & 1 \end{bmatrix}$$

This particular characterization is, in fact, a general characterization of partial calibration:

Theorem 4.4. *Suppose F is a fundamental matrix. Then there exist an essential matrix B and a calibration matrix K such that $F = K^{-T} B$.*

Proof. Let F be a fundamental matrix. Then

$$F = K_1^{-T} B K_2$$

where K_1 and K_2 are invertible upper triangular matrices, and B is an essential matrix, so

$$B = [t]_\times R$$

for some vector $t \in \mathbb{R}^3$ and rotation matrix R. Then using the fact that

$$[t]_\times M = M^{-T} [M^{-1} t]_\times$$

for any invertible matrix M,

$$F = K_1^{-T} B K_2 = K_1^{-T} [t]_\times (R K_2) = K_1^{-T} (R K_2)^{-T} [(R K_2)^{-1} t]_\times.$$

The matrix $(R K_2)^{-T}$ admits a unique LQ decomposition $(R K_2)^{-T} = LQ$ where L

is a lower triangular matrix with positive diagonal and Q is a rotation matrix. Then letting $U = L^{-T}$,

$$K_1^{-T}(R\,K_2)^{-T} = K_1^{-T}\,LQ = K_1^{-T}U^{-T}Q = (K_1\,U)^{-T}Q,$$

so $F = (K_1\,U)^{-T}\,Q\,[(R\,K_2)^{-1}\,t]_\times$. Since Q, and therefore, Q^T is invertible, there exists some vector $t' \in \mathbb{R}^3$ such that $Q^T\,t' = (R\,K_2)^{-1}\,t$. Then

$$Q[(R\,K_2)^{-1}\,t]_\times = Q[Q^T\,t']_\times = [t']_\times\,Q,$$

so

$$F = (K_1\,U)^{-T}\,[t']_\times\,Q,$$

where $K_1\,U$ is an invertible upper triangular matrix, and Q is a rotation matrix, so $[t']_\times\,Q$ is an essential matrix. This shows that every fundamental matrix F can be written as a partially calibrated matrix of the form $K^{-T}\,B$, where K is an invertible upper triangular matrix and B is an essential matrix. □

Note that the matrix $K_1\,U$ is not necessary a matrix of the form

$$\begin{bmatrix} x & 0 & a \\ 0 & y & b \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, a computation can show that this only happens if K_1 and K_2 are both matrices of these form. Therefore, this is a natural generalization of an essential matrix.

4.3 A Diagonal Calibration Matrix

As an example, let $F = K\,B$ where B is an essential matrix and K be a diagonal matrix:

$$K = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $a, b \neq 0$. The epipolar constraints require that

$$y_i^T\, F\, x_i = [y_{i1}\ y_{i2}\ 1] \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} B\, x_i = [ay_{i1}\ by_{i2}\ 1]\, B\, x_i = 0$$

for $i = 1, 2, \ldots, m$. For each i, let $y_i^{a,b} = [ay_{i1} : by_{i2} : 1]$ and define a new matrix $Z_{a,b}$ where the i^{th} row is given by

$$(y_i^{a,b})^T \otimes x_i = [ay_{i1}\, x_{i1} : ay_{i1}\, x_{i2} : ay_{i1} : by_{i2}\, x_{i1} : by_{i2}\, x_{i2} : by_{i2} : x_{i1} : x_{i2} : 1].$$

We are able to rewrite the epipolar constraint in terms of this new matrix $Z_{a,b}$:

Lemma 4.5. *A fundamental matrix $F = K\,B$, where $K = \operatorname{diag}(a, b, 1)$ and B is essential, exists that satisfies the epipolar constraints given by Z if and only if an essential matrix B exists that satisfies the epipolar constraints given by $Z_{a,b}$.*

Proof. Based on the computation above, $F = K\,B$, $Z\,F = \mathbf{0}$ if and only if $Z_{a,b}\,B = 0$, so $\ker Z \cap \mathcal{F}$ is nonempty if and only if $\ker Z_{a,b} \cap \mathcal{E}$ is nonempty. □

To understand the relationship between Z and $Z_{a,b}$, we write Z in terms of its columns:

$$Z = \begin{bmatrix} \vdots & \vdots & & \vdots \\ C_1 & C_2 & \dots & C_9 \\ \vdots & \vdots & & \vdots \end{bmatrix}.$$

This allows us to write $Z_{a,b}$ in terms of the same columns:

$$Z_{a,b} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 & C_7 & C_8 & C_9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

To compute the rank of $Z_{a,b}$, we wish to compute the minors of $Z_{a,b}$, which are in fact characterized by the minors of Z in the following way:

Proposition 4.6. *For $k = 1, \dots, 9$, a k-minor of $Z_{a,b}$ will be a scalar multiple of the determinant of the corresponding k-minor in Z.*

Proof. In general, a k-minor of $Z_{a,b}$ is formed by taking k colums of $Z_{a,b}$ and removing $m - k$ rows. Each column of $Z_{a,b}$ is a scalar multiple of the corresponding column in Z, so the determinant of the minor will be a scalar multiple of the determinant of the corresponding minor in Z. □

Example 4.1. Consider a 6-minor of $Z_{a,b}$ formed by taking the first six columns

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and removing $m-6$ rows. The determinant of this matrix is $a^3\, b^3 A$, where A is the determinant of the corresponding 6-minor in Z (i.e., the 6-minor formed taking the first six columns of Z and removing the same $m-6$ rows).

Therefore, we know the following about rank $Z_{a,b}$:

Corollary 4.7. $\operatorname{rank} Z_{a,b} = \operatorname{rank} Z$.

Proof. Since a and b are nonzero, any given k-minor in $Z_{a,b}$ will have zero determinant if and only if the corresponding k-minor in Z (i.e., the one formed by intersecting the same rows and columns as the ones used to form the k-minor in $Z_{a,b}$) has zero determinant.

If rank $Z = k$, then there exists a k-minor of Z with zero determinant. The corresponding k-minor in $Z_{a,b}$ will also have zero determinant, so rank $Z_{a,b} \geq k$. Additionally, all j-minors of $Z_{a,b}$, where $j > k$, will have nonzero determinant, since the corresponding j-minors of Z will have nonzero determinant. Therefore, rank $Z_{a,b} = k =$ rank Z. □

Now we can characterize when a fundamental matrix $F = K\,B$ exists:

Theorem 4.8. *A fundamental matrix of the form $F = K\,B$, where K is a diagonal matrix and B is an essential matrix, exists with respect to Z if and only if an* *essential matrix B exists with respect to Z. In particular, when* $\operatorname{rank} Z \leq 4$*, such a fundamental matrix F exists.*

4.4 The Parameter Space

We now relax the constraints on the calibration matrix K to our original definition of a partially calibrated epipolar matrix: let $F = K\,B$, where B is an essential matrix and K is of the form

$$K = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{bmatrix},$$

where $a, b \neq 0$. The epipolar constraints require that for $i = 1, \ldots, m$,

$$y_i^T\,F\,x_i = [y_{i1} \,:\, y_{i2} \,:\, 1] \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{bmatrix} B\,x_i = [ay_{i1} + c \,:\, by_{i2} + d \,:\, 1]\,B\,x_i = 0$$

so by letting

$$(y_i^K)^T = [ay_{i1} + c \,:\, by_{i2} + d \,:\, 1],$$

the constraint above is equivalent to requiring that $(y_i^K)^T\,B\,x_i = 0$ for each i. Define Z_K to be the $m \times 9$ matrix whose i^{th} row is given by $(y_i^K)^T \otimes x_i$. Then the columns of Z_K can be written in terms of the columns of Z:

$$Z_K = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ aZ_1 + cZ_7 & \dots & bZ_4 + dZ_7 & \dots & Z_7 & Z_8 & Z_9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We take the same approach as in the previous section: let $\sigma \subseteq [9]$, $\tau \subseteq [m]$, and let $M_{Z,\sigma,\tau}$ be the minor of Z formed by intersecting the columns in σ and the rows in τ. Then in general,

$$M_{Z_K,\sigma,\tau} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & aC_i + cC_{i+6} & \vdots & bC_j + dC_{j+3} & \vdots & C_k & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$, and $k \in \{7, 8, 9\}$. Suppose $|\sigma| = |\tau| = n$, where σ contains e_1 elements from $\{1, 2, 3\}$, e_2 elements from $\{4, 5, 6\}$, and e_3 elements from $\{7, 8, 9\}$ (so $e_1 + e_2 + e_3 = n$). Note that $\det M_{Z_k,\sigma,\tau}$ is a homogeneous polynomial of degree $n - e_3$. Moreover, by the multilinearity of the determinant, the coefficients of this polynomial are of the form $\det M_{Z,\sigma',\tau}$, where σ' ranges over the possible combinations of columns that can be formed from the columns in $M_{Z_K,\sigma,\tau}$. This lets us prove the following:

Proposition 4.9. $\operatorname{rank} Z_K \leq \operatorname{rank} Z$.

Proof. The determinant of any k-minor $M_{Z_K,\sigma,\tau}$ can be thought of as a polynomial in $\mathbb{C}[a, b, c, d]$ whose coefficients are the determinants of certain k-minors of Z. Therefore, if all the k-minors of Z have zero determinant, so do all the k-minors of Z_K, so rank $Z_K \leq$ rank Z. $\square$

Naturally, we would like to know when this inequality is strict, or whether the inequality is actually an equality as in the previous section. To answer this question we take an alternative approach: let

$$K' = \begin{bmatrix} a & & & & & & & & \\ & a & & & & & & & \\ & & a & & & & & & \\ & & & b & & & & & \\ & & & & b & & & & \\ & & & & & b & & & \\ c & & & d & & & 1 & & \\ & c & & & d & & & 1 & \\ & & c & & & d & & & 1 \end{bmatrix}.$$

K' is an invertible 9×9 matrix (since $a, b, 1 \neq 0$) and $Z_K = Z\,K'$. Since K' is invertible,

$$\text{rank } Z_K = \text{rank } Z\,K = \text{rank } Z$$

and $\ker Z_K = \ker(Z\,K')$. The condition in Lemma 4.1 is therefore equivalent to checking

$$\ker Z_K \cap \mathcal{E} = \ker(Z\,K') \cap \mathcal{E}.$$

A matrix B is in this intersection if and only if

$$(Z\,K')\,B = Z\,(K'\,(B)) = \mathbf{0}.$$

If we let $K'(\mathcal{E}) = \{K'(E)\,:\,E \in \mathcal{E}\}$, then this means we must check the intersection $\ker Z \,\cap\, K'(\mathcal{E})$.

Lemma 4.10. *For fixed values of a, b, c, d such that K' is invertible, $K'(\mathcal{E})$ is a projective variety in $\mathbb{P}^8$.*

Proof. Recall that $\mathcal{E}$ is a variety given by the equations

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix} = 2\,E\,E^T\,E - \operatorname{tr}(E\,E^T)\,E$$

and $p_9 = \det(E)$. For $i = 0, \ldots, 9$, Let $q_i(\mathbf{x}) = p_i((K')^{-1}\,\mathbf{x})$, where $\mathbf{x} = \{x_1, x_2, \ldots, x_9\}$. Then $q_i\,(K'\,E) = p_i\,(E)$, so the q_i vanish on $K'\,E$ if and only if the p_i vanish on E, or equivalently, if and only if $E \in \mathcal{E}$. This shows that $K'(\mathcal{E}) = \mathbb{V}(q_0,\,\ldots,\,q_9)$. □

For fixed values of a, b, c, d, it makes sense to think of $K'(\mathcal{E})$ as a variety in $\mathbb{P}^8$, but in reality we are considering a family of projective varieties $K'(\mathcal{E})$ that are parametrized by lower triangular matrices

$$K = \begin{bmatrix} a & & & & & & & & \\ & a & & & & & & & \\ & & a & & & & & & \\ & & & b & & & & & \\ & & & & b & & & & \\ & & & & & b & & & \\ c & & & d & & & e & & \\ & c & & & d & & & e & \\ & & c & & & d & & & e \end{bmatrix},$$

which can be thought of as points $K = [a : b : c : d : e] \in \mathbb{P}^4$. In this case, we are interested in the *parameter space*

$$\mathcal{C} = \{(K, E) : K' \in \mathbb{P}^4, E \in K'(\mathcal{E})\}.$$

We are therefore interested in the existence of a pair $(K, E) \in \mathbb{P}^4 \times \mathbb{P}^8$, where K is defined by a matrix of the form above and $E \in K(\mathcal{E})$. In particular, we are interested in determining whether the set

$$\mathcal{K} = \mathcal{C} \setminus \{(K, E) : K \in \mathbb{P}^4, abe = 0, E \in K'(\mathcal{E})\}$$

is empty. To answer this question we use the following property of parameter spaces from [7]:

Proposition 4.11. *Suppose $X \subseteq \mathbb{P}^n$ is a projective variety, and $\{V_b\}$ is a family of* *projective varieties in* $\mathbb{P}^n$ *parametrized by* $b \in \mathbb{P}^m$*. Then*

$$\{b \in B \,:\, X \cap V_b \neq \emptyset\}$$

is a subvariety of $\mathbb{P}^m$*.*

In particular, the set

$$\{K \in \mathbb{P}^4 \,:\, \ker Z \cap K(\mathcal{E}) \neq \emptyset\}$$

is a subvariety of $\mathbb{P}^4$, and we are interested in finding an element in the quasi-projective variety

$$\mathcal{B} = \{K \in \mathbb{P}^4 \,:\, \ker Z \cap K(\mathcal{E}) \neq \emptyset\} \setminus \{K \in \mathbb{P}^4 \,:\, \ker Z \cap K(\mathcal{E}) \neq \emptyset, abe = 0\},$$

which corresponds to the set of invertible upper triangular matrices K. A computation in *Macaulay2* [6] reveals the following:

Proposition 4.12. *For generic matrices* Z *of rank 5, the Zariski closure of* $\mathcal{B}$ *is* $\mathbb{P}^4$*.*

The computation involves taking the ideal generated by $\ker Z$, for a generic Z of rank 5, and the equations defining the ideal $\mathbb{I}(K(\mathcal{E}))$, and computing the elimination ideal in $\mathbb{C}[a, b, c, d, e]$. Therefore, we can conclude the following:

Theorem 4.13. *For generic matrices* Z*, when* $\operatorname{rank} Z \leq 5$*, a fundamental matrix of the form* $F = K\,B$*, where* B *is an essential matrix and*

$$K = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{bmatrix},$$

exists with respect to Z.

Chapter 5

The Multiview Ideal in $\mathbb{P}^r$

5.1 Motivation

In this chapter, we generalize results regarding the multiview variety. Let $A_1, \ldots, A_n$ be a system of projective cameras. These can be thought of as real 3×4 matrices of rank 3. Given these matrices, we can consider the rational map

$$\begin{aligned} \phi : \; \mathbb{P}^3 \; &\longrightarrow \; \mathbb{P}^2 \times \mathbb{P}^2 \times \ldots \times \mathbb{P}^2 \\ \mathbf{x} \; &\longmapsto \; (A_1\mathbf{x}, A_2\mathbf{x}, \ldots, A_n\mathbf{x}) \end{aligned} \tag{5.1}$$

Let $V_A = \overline{\phi(\mathbb{P}^3)}$ be the closure of the image of ϕ and $J_A = \mathbb{I}(V_A)$ its ideal in the polynomial ring $\mathbb{C}[x_i, y_i, z_i \,:\, i \in [n]]$. V_A and J_A are called the *multiview variety* and *multiview ideal*, respectively. In [2], Aholt, Sturmfels and Thomas characterize the generators and the Hilbert function of the initial ideal $\text{in}_{\succ} J_A$, with respect to the term order

$$x_1 \succ x_2 \succ \ldots \succ x_n \succ y_1 \succ \ldots \succ y_n \succ z_1 \succ \ldots \succ z_n$$

for sufficiently generic cameras $A_1, \ldots, A_n$. Their method was to express V_A as the diagonal embedding of $\mathbb{P}^3$ into $(\mathbb{P}^3)^n$, obtain a set of generators for $\mathrm{in}_{\succ} J_A$ via elimination, and prove that the resulting generators have a determinantal representation. We will generalize these results to the class of rational maps

$$\begin{aligned} \phi : \quad \mathbb{P}^{r-1} & \longrightarrow \mathbb{P}^{s-1} \times \mathbb{P}^{s-1} \times \ldots \times \mathbb{P}^{s-1} \\ \mathbf{x} & \longmapsto (A_1\mathbf{x},\, A_2\mathbf{x}, \ldots, A_n\mathbf{x}) \end{aligned} \tag{5.2}$$

where $r \geq s$ and the A_i are sufficiently generic real $s \times r$ matrices of rank s.

5.2 An Example: $\mathbb{P}^2$ into $(\mathbb{P}^1)^n$

The prototype for our analysis will be based on rational maps of the form

$$\begin{aligned} \phi : \quad \mathbb{P}^{2} & \longrightarrow \mathbb{P}^{1} \times \mathbb{P}^{1} \times \ldots \times \mathbb{P}^{1} \\ \mathbf{x} & \longmapsto (A_1\mathbf{x},\, A_2\mathbf{x}, \ldots, A_n\mathbf{x}) \end{aligned} \tag{5.3}$$

where the A_i are real 2×3 matrices of rank 2. In this case, the multiview ideal M_n is an ideal in $\mathbb{C}[x_i\,, y_i \,:\, i = 1, \ldots, n]$, and is prime because V_A is an irreducible variety. Given a set $\sigma = \{\sigma_1, \ldots, \sigma_s\} \subseteq [n]$, we consider the matrix

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \dots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_s} & \mathbf{0} & \dots & \mathbf{0} & p_{\sigma_s} \end{bmatrix} \tag{5.4}$$

where $p_i := [x_i \;\; y_i]^T$. This is a $2s \times (s+3)$ matrix. Similar to the analysis in [2], we begin by giving a determinantal representation of the multiview ideal:

Lemma 5.1. *The maximal minors of A_σ for $|\sigma| \geq 3$ lie in the prime ideal J_A.*

Proof. If $\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{C}^2)^n$ is a point in $\phi(\mathbb{P}^2)$, then there are some $q \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ and $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$ such that $A_i\, q = c_i\, p_i$ for $i = 1, 2, \dots, n$. Then the columns of A_σ are linearly dependent. For $|\sigma| \geq 3$, A_σ has at least as many rows as columns, so the maximal minors of A_σ must vanish at $\mathbf{p}$. □

This lemma gives us a useful characterization of some of the elements of J_A, and by extension, its initial ideal $\text{in}_\succ J_A$. To see this, we endow $\mathbb{C}[x_i,\, y_i]$ with the lexicographic term order

$$x_1 \succ x_2 \succ \dots \succ x_n \succ y_1 \succ y_2 \succ \dots \succ y_n$$

and assume that our matrices $A_1, \dots A_n$ are *generic* in the sense that the maximal minors of the $3 \times 2n$ matrix

$$[\, A_1^T \, A_2^T \, \dots \, A_n^T \,]$$

are invertible. We focus on a particular set of monomials: define the monomial ideal

$$M_n := \langle\, x_i\, x_j\, x_k \,:\, i,\, j,\, k \text{ distinct indices in } [n] \,\rangle\,. \tag{5.5}$$

For distinct $i,\, j,\, k \in [n]$, consider the 6×6 matrix

$$A_{\{ijk\}} = \begin{bmatrix} A_i & p_i & \mathbf{0} & \mathbf{0} \\ A_j & \mathbf{0} & p_j & \mathbf{0} \\ A_k & \mathbf{0} & \mathbf{0} & p_k \end{bmatrix}. \tag{5.6}$$

Since $A_{\{ijk\}}$ is a square matrix, its only maximal minor is the determinant of the entire matrix. If we write A_t^r for the rth row of A_t, then

$$\det A_{\{ijk\}} = \det \begin{bmatrix} A_i^2 \\ A_j^2 \\ A_k^2 \end{bmatrix} x_i\, x_j\, x_k + \text{ lex. lower order terms}$$

where the coefficient of $x_i\, x_j\, x_k$ is nonzero due to the genericity assumption on $A_1, \ldots, A_n$. Then $x_i\, x_j\, x_k \in \text{in}_{\succ} J_A$. This proves the following result:

Lemma 5.2. $M_n \subseteq \text{in}_{\succ} J_A$.

From here, we can mimic the construction in [2] to express V_A as the projection of a diagonal embedding of $\mathbb{P}^3$: extend each camera matrix A_i to an invertible 3×3 matrix

$$B_i = \begin{bmatrix} b_i \\ A_i \end{bmatrix} \tag{5.7}$$

by adding a row b_i to the top. The corresponding diagonal map is

$$\begin{aligned} \psi_B : \ \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \ldots \times \mathbb{P}^2 \\ \mathbf{x} &\longmapsto (B_1\mathbf{x},\, B_2\mathbf{x}, \ldots ,\, B_n\mathbf{x}) \end{aligned} \tag{5.8}$$

Let $V_B = \overline{\psi_B(\mathbb{P}^3)}$, $J_B = \mathbb{I}(V_B) \subseteq \mathbb{C}[w_i,\, x_i,\, y_i \,:\, i \in [n]]$, and consider the coordinate projection

$$\begin{array}{rccc} \pi : & \mathbb{P}^2 \times \mathbb{P}^2 \times \ldots \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \\ & (w_i \,:\, x_i \,:\, y_i) & \longmapsto & (x_i \,:\, y_i) \end{array} . \tag{5.9}$$

The composition $\pi \circ \psi_B$ is a rational map that coincides with ϕ where ϕ is defined. Then $V_A = \pi(V_B)$ and

$$J_A = J_B \cap \mathbb{C}[x_i,\ y_i \,:\, i \in [n]]. \tag{5.10}$$

The polynomial ring $\mathbb{C}[w_i,\, x_i,\, y_i \,:\, i \in [n]]$ admits the $\mathbb{Z}^n$-grading

$$\deg(w_i) = \deg(x_i) = \deg(y_i) = e_i \tag{5.11}$$

where e_i is the i^{th} unit vector in $\mathbb{R}^n$. With respect to this grading, the multigraded Hilbert function of $\mathbb{C}[w_i,\, x_i,\, y_i \,:\, i \in [n]]/J_B$ is

$$\begin{array}{rccc} \mathcal{H}: & \mathbb{N}^n & \longrightarrow & \mathbb{N} \\ & (u_1, \dots u_n) & \longmapsto & \binom{u_1 + \dots + u_n + 2}{2} \end{array} \tag{5.12}$$

The multigraded Hilbert scheme $H_{3,n}$, which parametrizes $\mathbb{Z}^n$-homogeneous ideals in $\mathbb{C}[w_i, x_i, y_i : i \in [n]]$, has a unique Borel-fixed ideal $Z_{3,n}$ that, under the correct genericity conditions on $B_1, \dots, B_n$, is the initial ideal $\text{in}_{\succ} J_B$. We obtain the following lemma by making use of the results from from Cartwright and Sturmfels in [3]:

Lemma 5.3. *$Z_{3,n}$ is generated by the following monomials, where i, j and k are distinct indices in $[n]$:*

$$w_i w_j, \; w_i x_j, \; x_i x_j x_k.$$

Also, when $B_1, \dots, B_n$ are sufficiently generic, then $Z_{3,n} = \text{in}_{\succ} J_B$ with respect to the lexicographic term order

$$w_1 \succ w_2 \succ \dots \succ w_n \succ x_1 \succ \dots \succ x_n \succ y_1 \succ \dots \succ y_n.$$

From this, we can prove the following:

Theorem 5.4. *If $A_1, \dots, A_n$ are generic, then $M_n = \text{in}_{\succ} J_A$, where $\succ$ is the lexicographic term order induced by*

$$x_1 \succ x_2 \succ \dots \succ x_n \succ y_1 \succ \dots \succ y_n$$

Proof. Fix the term order above on $\mathbb{C}[w_i, x_i, y_i : i \in [n]]$. From Lemma 5.3,

$M_n = Z_{3,n} \cap \mathbb{C}[x_i, y_i : i \in [n]]$. With respect to the lexicographic term order, the operation of taking initial ideals and intersections commute, so

$$\begin{aligned} \mathrm{in}_\succ(J_A) &= \mathrm{in}_\succ(J_B \cap \mathbb{C}[x_i, y_i : i \in [n]]) \\ &= \mathrm{in}_\succ(J_B) \cap \mathbb{C}[x_i, y_i : i \in [n]] \\ &= Z_{3,n} \cap \mathbb{C}[x_i, y_i : i \in [n]] = M_n. \end{aligned}$$

□

Finally, this gives us a determinantal description of the generators of $\mathrm{in}_\succ J_A$:

Corollary 5.5. *For generic $A_1, \ldots, A_n$, the generators of* $\mathrm{in}_\succ J_A$ *are given by the leading terms of the maximal minors of A_σ for $|\sigma| = 3$.*

Proof. The generators of $\mathrm{in}_\succ J_A$ are of the form $x_i\, x_j\, x_k$ where i, j, k are distinct indices in $[n]$. From Lemma 5.2, these are the leading monomials of det A_{ijk}. □

5.3 Generalizations: $\mathbb{P}^{r-1}$ into $(\mathbb{P}^1)^n$

The techniques used in the previous section readily generalize to rational maps of the form

$$\begin{aligned} \phi :\ \mathbb{P}^{r-1} &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \\ \mathbf{x} &\longmapsto (A_1\mathbf{x}, A_2\mathbf{x}, \ldots, A_n\mathbf{x}) \end{aligned} \tag{5.13}$$

where $r \geq 3$ and the A_i are real $2 \times r$ matrices of rank 2. Again, we let $V_A = \overline{\phi(\mathbb{P}^r)}$ be the multiview variety, $J_A = \mathbb{I}(V_A)$ the multiview ideal, and for $\sigma = \{\sigma_1, \ldots, \sigma_s\} \subseteq$

$[n]$ we consider the matrix

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \dots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_s} & \mathbf{0} & \dots & \mathbf{0} & p_{\sigma_s} \end{bmatrix} \tag{5.14}$$

where $p_i = [x_i \, y_i]^T$. In this case, A_σ is a $2s \times (r+s)$ matrix. We proceed in the same manner as in Section 5.2:

Lemma 5.6. *The maximal minors of A_σ for $|\sigma| \geq r$ lie in J_A.*

Proof. The proof is identical to the proof of Lemma 5.1, except in order for A_σ to have more rows than columns, we require that $2\,|\sigma| \geq r + |\sigma|$, so $|\sigma| \geq r$. □

Lemma 5.7. *Let M_n be the ideal generated by monomials of the form*

$$x_{i_1} x_{i_2} \dots x_{i_r}$$

for distinct indices $i_1, i_2, \dots, i_r \in [n]$. Assume that $A_1, \dots, A_n$ are generic in the same sense as in section 1. Then with respect to the lexicographic term order

$$x_1 \succ x_2 \succ \dots \succ x_n \succ y_1 \succ \dots \succ y_n,$$

$M_n \subseteq in_\succ J_A$.

Proof. Given r distinct indices $i_1, i_2, \dots, i_r \in [n]$, the corresponding matrix

$$A_{\{i_1,i_2,\ldots,i_r\}} = \begin{bmatrix} A_{i_1} & p_{i_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{i_2} & \mathbf{0} & p_{i_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{i_r} & \mathbf{0} & \ldots & \mathbf{0} & p_{i_r} \end{bmatrix}$$

is an $r \times r$ square matrix, so its determinant

$$\det A_{\{i_1,i_2,\ldots,i_r\}} = \det \begin{bmatrix} A^2_{i_1} \\ A^2_{i_2} \\ \vdots \\ A^2_{i_r} \end{bmatrix} x_{i_1}\, x_{i_2}\, \ldots\, x_{i_r} + \text{ lex. lower order terms}$$

is in J_A. The coefficient of the leading term is nonzero, due to the genericity of $A_1, \ldots, A_n$. Then $x_{i_1}\, x_{i_2}\, \ldots\, x_{i_r}$ is in $\text{in}_{\succ} J_A$. $\square$

The diagonal embedding in this case is similar in principle to the embedding in the previous section, but in this case we need to add $r-2$ rows to each A_i to obtain an invertible $r \times r$ matrix B_i: for each i, let

$$B_i := \begin{bmatrix} b_i^1 \\ b_i^2 \\ \vdots \\ b_i^{r-2} \\ A_i \end{bmatrix}$$

be the matrix A_i with $r-2$ rows added to create an invertible matrix. The corresponding diagonal embedding is

$$\begin{array}{rccc} \psi_B: & \mathbb{P}^{r-1} & \longrightarrow & \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} \\ & \mathbf{x} & \longmapsto & (B_1\mathbf{x},\, B_2\mathbf{x}, \ldots,\, B_n\mathbf{x}) \end{array} . \tag{5.15}$$

As before, we let $V_B = \overline{\psi_B(\mathbb{P}^{r-1})}$. The corresponding ideal $J_B = \mathbb{I}(V_B)$ is an ideal in the polynomial ring $\mathbb{C}[w_{i,j}, x_j, y_j \,:\, i \in [r-2], j \in [n]]$. Let

$$\begin{array}{rccc} \pi: & \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \\ & (w_{1,j} \,:\, w_{2,j} \,:\, \ldots \,:\, w_{r-2,j} \,:\, x_j \,:\, y_j) & \longmapsto & (x_j \,:\, y_j) \end{array} . \tag{5.16}$$

Then ϕ and $\pi \circ \psi_B$ agree wherever ϕ is defined, so $V_A = \pi(V_B)$ and $J_A = J_B \cap \mathbb{C}[x_j, , y_j \,:\, j \in [n]]$. The polynomial ring $\mathbb{C}[w_{i,j}, x_j, y_j \,:\, i \in [r-2], j \in [n]]$ admits the $\mathbb{Z}^n$-grading

$$\deg(w_{1,i}) = \deg(w_{2,i}) = \ldots = \deg(w_{r-2,i}) = \deg(x_i) = \deg(y_i) = e_i$$

where e_i is the i^{th} standard unit vector in $\mathbb{R}^n$. With respect to this grading, the multigraded Hilbert function of $\mathbb{C}[w_{i,j}, x_j, y_j \,:\, i \in [r-2], j \in [n]]/J_B$ is

$$\mathcal{H}(u_1, \ldots, u_n) = \binom{u_1 + \ldots + u_n + r - 1}{r-1}$$

which again puts us in a position to use the results in [3]: now, we consider the multigraded Hilbert scheme $H_{r,n}$ which has a unique Borel-fixed ideal $Z_{r,n}$ that, under similar genericity conditions on $B_1, \ldots, B_n$, is the initial ideal of J_B. With respect to the term order

$$w_{1,1} \succ \ldots w_{1,n} \succ w_{2,1} \succ \ldots \succ w_{r-2,n} \succ x_1 \succ \ldots \succ x_n \succ y_1 \succ \ldots \succ y_n,$$

we have that J_A is the elimination ideal obtained by eliminating the variables $w_{i,j}$ (for $i \in [r-2]$ and $j \in [n]$) from J_B, so the same result follows at the level of initial ideals: $\text{in}_{\succ}(J_A)$ will be the ideal $\text{in}_{\succ}(J_B) \cap \mathbb{C}[x_j,\, y_j \,:\, j \in [n]]$.

For ease of exposition, we rename the variables $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_j\}$ to $\{w_{r-1,1}, \ldots, w_{r-1,n}\}$ and $\{w_{r,1}, \ldots, w_{r,n}\}$, respectively. In this setting, [3, Theorem 2.5] characterizes the generators of $Z_{r,n}$:

Proposition 5.8. *The generators of* $Z_{r,n} \subseteq \mathbb{C}[w_{i,j} \,:\, i \in [r],\, j \in [n]]$ *are monomials of the form*

$$w_{i_1,j_1}\, w_{i_2,j_2}\, \ldots\, w_{i_k,j_k}$$

where

(1) $2 \leq k \leq \min(r, n)$

(2) $1 \leq k-1 \leq i_1,\, i_2,\, \ldots, i_k \leq r-1$

(3) $j_1 < j_2 < \ldots < j_k$

(4) $i_1 + i_2 + \ldots + i_k \leq r(k-1)$.

The next step is to compute the elimination ideal $Z = Z_{r,n} \cap \mathbb{C}[w_{i,j} \,:\, i \in \{r-1, r\}, j \in [n]]$:

Lemma 5.9. *The generators of* Z *are monomials of the form*

$$w_{r-1,j_1}\, w_{r-1,j_2}\, \ldots\, w_{r-1,j_k}$$

where $j_1 < j_2 < \ldots < j_k$.

Proof. A monomial generator of $Z_{r,n}$ is of the form

$$\mathbf{w} = w_{i_1,j_1}\, w_{i_2,j_2}\, \ldots\, w_{i_k,j_k}$$

satisfying conditions (1)-(4) of Proposition 5.8. If $\mathbf{w} \in Z$, then by condition (2), $i_l = r-1$ for all $l \in [k]$, and by condition (4),

$$i_1 + i_2 + \ldots + i_k = k(r-1) \leq r(k-1)$$

$$-k \leq -r$$

$$k \geq r$$

but by condition (2), $k-1 \leq r-1$ implies that $k \leq r$, so we must have that $k = r$, and $\mathbf{w}$ is of the form

$$\mathbf{w} = w_{r-1,j_1}\, w_{r-1,j_2}\, \ldots\, w_{r-1,j_k}.$$

□

Since we have identified x_j and y_j with $w_{r-1,j}$ and $w_{r,j}$, respectively, this shows that the elimination ideal $Z_{r,n} \cap \mathbb{C}[x_j,\, y_j \;:\; j \in [n]]$ is the ideal generated by the monomials

$$x_{j_1}\, x_{j_2}\, \ldots\, x_{j_r}$$

where $j_1, \ldots, j_r$ are distinct indices in $[n]$. This gives us the following generalization of Theorem 1:

Theorem 5.10. *When $A_1, \ldots A_n$ are generic, then $\mathrm{in}_\succ(J_A) = M_n$, where $\succ$ is the lexicographic term order induced by*

$$x_1 \succ x_2 \succ \ldots \succ x_n \succ y_1 \succ \ldots \succ y_n.$$

Proof. The above argument shows that $Z_{r,n} \cap \mathbb{C}[x_j, y_j : j \in [n]] = M_n$. As before, since taking initial ideals commutes with intersections,

$$\begin{aligned} \mathrm{in}_\succ(J_A) &= \mathrm{in}_\succ(J_B \cap \mathbb{C}[x_j, y_j : j \in [n]]) \\ &= \mathrm{in}_\succ(J_B) \cap \mathbb{C}[x_j, y_j : j \in [n]] \\ &= Z_{d,n} \cap \mathbb{C}[x_j, y_j : j \in [n]] \\ &= M_n. \end{aligned}$$

□

In particular, this shows that the generators of $\mathrm{in}_\succ(J_A)$, like in Section 5.1, have a determinantal representation as the leading monomials of maximal minors of A_σ.

5.4 Generalizations: $\mathbb{P}^r$ into $(\mathbb{P}^{r-1})^n$

The rational maps in the previous sections gave rise to multiview ideals J_A where it was relatively straightforward to come up with a determinantal representation for $\mathrm{in}_\succ(J_A)$. In general, this is less straightforward: for $r \geq 3$, consider a rational map

$$\begin{aligned} \phi: \quad \mathbb{P}^r &\longrightarrow \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} \\ \mathbf{x} &\longmapsto (A_1\mathbf{x},\, A_2\mathbf{x}, \ldots ,\, A_n\mathbf{x}) \end{aligned} \tag{5.17}$$

where $A_1, \ldots, A_n$ are real $r \times (r+1)$ matrices of rank r. Let $V_A = \overline{(\phi(\mathbb{P}^r)}$ and $J_A = \mathbb{I}(V_A) \subseteq \mathbb{C}[x_{i,j} : i \in [r],\, j \in [n]]$ be the multiview variety and ideal of ϕ, and for $\sigma = \{\sigma_1, \ldots, \sigma_s\} \subseteq [n]$ and $p_k = [x_{1,k}\, x_{2,k}\, \ldots\, x_{r,k}]^T$, consider the matrix

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_s} & \mathbf{0} & \ldots & \mathbf{0} & p_{\sigma_s} \end{bmatrix}. \tag{5.18}$$

This is an $sr \times (s+r+1)$ matrix, which suggests the following result:

Lemma 5.11. *The maximal minors of A_σ, for $|\sigma| \geq 2$, are in J_A.*

Proof. The argument is the same as in Lemma 5.1, except in order to ensure that A_σ has more rows than columns, we require that $sr \geq s + r + 1$, or equivalently, $s \geq \lceil \frac{r+1}{r-1} \rceil$. For $r \geq 3$, $\lceil \frac{r+1}{r-1} \rceil = 2$, so $s \geq 2$. □

As in the case of $\mathbb{P}^2$, we give the polynomial ring $\mathbb{C}[x_{i,j} : i \in [r], j \in [n]]$ the lexicographic term order

$$x_{a,b} \succ x_{c,d} \iff a < c \text{ or } a = c \text{ and } b < d.$$

First, we will use the results from [3] to give a description of the generators of

$\mathrm{in}_{\succ}(J_A)$:

Lemma 5.12. *For $n \geq r+1$, the generators of $\mathrm{in}_{\succ}(J_A)$ are of the form*

$$x_{i_1 j_1} x_{i_2 j_2} \ldots x_{i_k j_k}$$

where

(1) $2 \leq k \leq r+1$

(2) $j_1 < j_2 < \cdots < j_k$

(3) $1 \leq k-1 \leq i_1, i_2, \ldots, i_k \leq r-1$, *and*

(4) $i_1 + i_2 + \cdots + i_k \leq r(k-1) - 1$.

Proof. This follows from mimicing the construction in Sections 5.2 and 5.3: to each matrix A_i, we add a row b_i to create an $(r+1) \times (r+1)$ invertible matrix

$$B_i := \begin{bmatrix} b_i \\ A_i \end{bmatrix}.$$

Then we let

$$\begin{aligned} \psi : \ \mathbb{P}^{r+1} &\longrightarrow \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} \\ \mathbf{x} &\longmapsto (B_1 \mathbf{x}, \ldots, B_n \mathbf{x}) \end{aligned}$$

and consider the variety $V_B = \overline{\psi(\mathbb{P}^{r-1})}$ and its ideal $J_B = \mathbb{I}(V_B)$. If we think of J_B as an ideal in $\mathbb{C}[w_j, x_{ij} : i \in [r], j \in [n]]$, then [3, Theorem 2.5] characterizes the generators of $\in_{\succ} J_B$ where $\succ$ is the original term order in $\mathbb{C}[x_{ij} \, : \, i \in [r], \, j \in [n]]$ with

$$w_1 \succ w_2 \succ \ldots \succ w_n \succ x_{ij}$$

for all i, j. The result is obtained by eliminating the w_i variables with respect to this term order. □

Here, it is not as clear that these monomials appear as the leading terms of minors of A_σ; one difficulty that we encounter is that the size of σ is allowed to vary, unlike in the previous cases. However, it turns out that a description similar to Theorems 5.4 and 5.10 is possible:

Theorem 5.13. *For generic matrices $A_1, \ldots, A_n$, the monomials described in Lemma 5.12 are the leading terms of maximal minors of A_σ.*

Proof. For a monomial $\mathbf{x} = x_{i_1 j_1} x_{i_2 j_2} \ldots x_{i_k j_k}$ of the form above, consider the corresponding matrix

$$A = \begin{bmatrix} A_{i_1} & p_{i_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{i_2} & \mathbf{0} & p_{i_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{i_k} & \mathbf{0} & \ldots & \mathbf{0} & p_{i_k} \end{bmatrix}.$$

This is a $rk \times (r+k+1)$ matrix, so the maximal minors of this matrix are obtained by removing $rk - r - k - 1$ rows. We will form a maximal minor A_0 of A by removing every row of A containing a variable not in $\mathbf{x}$ and lexicographically greater than a variable in $\mathbf{x}$. This is possible because by condition (4) of Lemma 5.12, for each variable x_{i_l,j_l}, there are at most $i_1 - 1$ lexicographically greater variables in the set $\{x_{1,j_1}, x_{2,j_1}, \ldots, x_{i_l,j_l}, \ldots, x_{r,j_l}\}$, so in the matrix A_σ there are at most $i_1 - 1$ such variables in the same column as x_{i_l,j_l}. Then in total we must remove

$$(i_1 - 1) + (i_2 - 1) + \ldots + (i_k - 1) = i_1 + i_2 + \ldots + i_k - k$$

rows from A_σ, which by condition (4) of Lemma 5.12 is bounded above by $r(k-1) - 1 - k = rk - r - k - 1$. This shows that we are able to remove exactly the rows we need, to obtain a maximal minor A_0 such that

$$\det A_0 = a_0\,\mathbf{x} + \text{lower order terms}$$

where a_0 is the determinant of a subset of the rows of the A_{i_k}, which is nonzero by the assumption that $A_1, \ldots, A_n$ is generic. □

5.5 Generalizations: $\mathbb{P}^{r-1}$ into $(\mathbb{P}^{s-1})^n$

We now consider the most general class of rational maps: let $n \geq r \geq s > 1$, and let

$$\begin{array}{rccc} \phi : & \mathbb{P}^{r-1} & \longrightarrow & \mathbb{P}^{s-1} \times \mathbb{P}^{s-1} \times \ldots \times \mathbb{P}^{s-1} \\ & \mathbf{x} & \longmapsto & (A_1\mathbf{x}, A_2\mathbf{x}, \ldots, A_n\mathbf{x}) \end{array} \tag{5.19}$$

where $A_1, \ldots, A_n$ are real $s \times r$ matrices of rank s. As before, let $V_A = \overline{\phi(\mathbb{P}^{r-1})}$ be the multiview variety, $J_A = \mathbb{I}(V_A)$ the multiview ideal in $\mathbb{C}[x_{i,j} : i \in [s], j \in [n]]$, and for $\sigma = \{\sigma_1, \ldots, \sigma_k\} \subseteq [n]$, let

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_k} & \mathbf{0} & \ldots & \mathbf{0} & p_{\sigma_k} \end{bmatrix} \tag{5.20}$$

where $p_j := [x_{1,j}\, x_{2,j}\, \ldots\, x_{s,j}]^T$. This is a $ks \times r + k$ matrix, so for A_σ to have more rows that columns, we need that $k \geq \frac{r}{s-1}$. By following the same argument as in Lemma 5.1, we can conclude the following:

Lemma 5.14. *The maximal minors of A_σ, for $|\sigma| \geq \lceil \frac{r}{s-1} \rceil$, lie in J_A.*

Our ultimate goal is to give a determinantal representation of $\text{in}_\succ(J_A)$ with respect to the lexicographic term order

$$x_{a,b} \succ x_{c,d} \iff a < c \text{ or } a = c \text{ and } b < d$$

and with the same genericity assumption on the A_i as before.

We add $r - s$ rows, $b_i^1, \ldots b_i^{r-s}$ to A_i to obtain an invertible $r \times r$ matrix

$$B_i := \begin{bmatrix} b_i^1 \\ \vdots \\ b_i^{r-s} \\ A_i \end{bmatrix}$$

for $i = 1, \ldots n$. Define a map

$$\begin{aligned} \psi_B : \ \mathbb{P}^{r-1} &\longrightarrow \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} \\ \mathbf{x} &\longmapsto (B_1\mathbf{x},\, B_2\mathbf{x}, \ldots ,\, B_n\mathbf{x}) \end{aligned} \tag{5.21}$$

and let $V_B = \overline{\psi_B(\mathbb{P}^{r-1})}$ and $J_B = \mathbb{I}(V_B) \subseteq \mathbb{C}[y_{i,j},\, x_{k,j} \,:\, i \in [r-s],\, k \in [s],\, j \in [n]]$. We can realize our original map ϕ as a composition of ψ_B with a projection: let

$$\begin{aligned} \pi : \quad \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \times \ldots \times \mathbb{P}^{r-1} &\longrightarrow \mathbb{P}^{s-1} \times \mathbb{P}^{s-1} \times \ldots \times \mathbb{P}^{s-1} \\ (y^i_1 \,:\, \ldots \,:\, y^i_{r-s} \,:\, x^i_1 \,:\, \ldots \,:\, x^i_s) &\longmapsto (x^i_1 \,:\, \ldots \,:\, x^i_s). \end{aligned} \tag{5.22}$$

Then $\phi = \pi \circ \psi_B$ whenever ϕ is defined. Then $V_A = \pi(V_B)$, and $J_A = J_B \cap \mathbb{C}[x_{i,j} \,:\, i \in [s],\, [j] \in [n]]$. Note that J_B is generated by the 2×2 minors of the matrix

$$\left[B_1^{-1}\, p_1 \;\, B_2^{-1}\, p_2 \; \ldots \; B_n^{-1}\, p_n\right]$$

where $p_i = [y_{1,i}\, y_{2,i}\, \ldots\, y_{r-s,i}\, x_{1,i}\, \ldots\, x_{s,i}]^T$. Now we compute the Hilbert function of $\mathbb{C}[y_{i,j},\, x_{k,j} \,:\, i \in [r-s],\, k \in [s],\, j \in [n]]/J_B$ with respect to the $\mathbb{Z}^n$-grading $\deg(y_{1,j}) = \ldots = \deg(y_{r-s,j}) = \deg(x_{1,j}) = \ldots = \deg(x_{s,j}) = e_j$, where e_j is the j^{th} standard unit vector in $\mathbb{R}^n$.

$$\mathbb{N}^n \longrightarrow \mathbb{N}, \qquad (u_1, \ldots, u_n) \longmapsto \binom{u_1 + \ldots + u_n + d - 1}{d-1}. \tag{5.23}$$

This puts us in a position to use the results from [3]: in particular, that the unique Borel-fixed ideal $Z_{r,n}$ of the multigraded Hilbert scheme $H_{r,n}$, under the correct genericity conditions on the B_i, is the initial ideal of J_B. Our goal is to characterize the generators of $Z_{d,n}$. As in Section 5.3, we rename the variables $x_{i,j}$ to $y_{r-s+i,j}$ and consider $Z_{r,n}$ as an ideal in $\mathbb{C}[y_{i,j} \,:\, i \in [r],\, j \in [n]]$.

Lemma 5.15. *The ideal $Z_{r,n} \subseteq \mathbb{C}[y_{i,j} \,:\, i \in [r],\, j \in [n]]$ is generated by all monomials of the form*

$$y_{i_1,j_1}\, y_{i_2,j_2} \,\ldots\, y_{i_k,j_k}$$

where

(1) $2 \le k \le \min(r,n)$

(2) $1 \le k-1 \le i_1,\, i_2,\, \ldots,\, i_k \le r-1$

(3) $j_1 < j_2 < \ldots < j_k$

(4) $i_1 + i_2 + \ldots + i_k \le r(k-1)$.

The ideal Z that we are interested in is the ideal we obtain by eliminating $y_{i,j}$ for $1 \le i \le r-s$:

Corollary 5.16. *The ideal $Z = Z_{r,n} \cap \mathbb{C}[y_{i,j} \,:\, r-s+1 \le i \le r,\, j \in [n]]$ is generated by all monomials of the form*

$$y_{i_1,j_1}\, y_{i_2,j_2} \,\ldots\, y_{i_k,j_k}$$

where

(1) $\lceil \frac{r}{s-1} \rceil \le k \le r$

(2) $1 \le k-1 \le i_1,\, i_2,\, \ldots,\, i_k \le r-1$

(3) $j_1 < j_2 < \ldots < j_k$

(4) $i_1 + i_2 + \ldots + i_k \leq r(k-1)$.

Proof. The only difference between this statement and the statement of Lemma 5.15 is (1). A monomial in Z is of the form

$$y_{i_1,j_1}\, y_{i_2,j_2}\, \ldots\, y_{i_k,j_k}$$

where $i_1, \ldots, i_k \geq r-s+1$ and $i_1+i_2+\ldots+i_k \leq r(k-1)$. Then $i_1+i_2+\ldots i_k \geq k\,(r-s+1)$ and

$$k\,(r-s+1) \leq r(k-1)$$

$$k(1-s) \leq -r$$

$$k \geq -\frac{r}{1-s} = \frac{r}{s-1} \;\; (\text{since } s > 1,\, 1-s < 0)\;,$$

so $k \geq \lceil \frac{r}{s-1} \rceil$. □

If we identify the $x_{i,j}$ with the $y_{r-s+i,j}$ in the above lemma, we obtain the following result:

Corollary 5.17. *Identifying the $x_{i,j}$ with the $y_{r-s+i,j}$, the above ideal Z in $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]$ is the ideal generated by monomials of the form*

$$x_{i_1,j_1}\, x_{i_2,j_2}\, \ldots\, x_{i_k,j_k}$$

where

(1) $\lceil \frac{r}{s-1} \rceil \leq k \leq r$

(2) $1 \leq i_1, i_2, \ldots, i_k \leq s-1$

(3) $j_1 < j_2 < \ldots < j_k$

(4) $i_1 + i_2 + \ldots + i_k \leq sk - r$.

Proof. (1) and (3) are unchanged from Lemma 5.16 because we identify the variables $x_{i,j}$ with $y_{r-s+i,j}$, where $i \in [s]$. This changes the upper bound in (2) from r to s. Finally, the upper bound in (4) changes because given a monomial

$$x_{i_1,j_1}\, x_{i_2,j_2}\, \ldots\, x_{i_k,j_k}$$

in Z, where $i_l \in [s]$, the corresponding monomial from the setting of Lemma 5.16 is

$$y_{r-s+i_1,j_1}\, y_{r-s+i_2,j_2}\, \cdots\, y_{r-s+i_k,j_k}$$

where

$$(r-s+i_1) + (r-s+i_2) + \ldots + (r-s+i_k) \leq r(k-1)$$

$$k(r-s) + i_1 + i_2 + \ldots + i_k \leq r(k-1)$$

$$i_1 + i_2 + \ldots + i_k \leq sk - r.$$

□

Corollary 5.17 gives us a characterization of the generators of Z. The connection between the ideal Z and the initial ideal of J_A is that Z arises as an elimination ideal of $Z_{d,n}$, which by [3] is the initial ideal of J_B. But J_A itself arises as an elimination ideal of J_B, so the same result will follow at the level of initial ideals:

Theorem 5.18. *When $A_1, \ldots, A_n$ are generic,* $\mathrm{in}_\succ(J_A) = Z$.

Proof. With respect to the lexicographic term order, the operation of taking the initial ideal commutes with intersecting ideals. Therefore,

$$\begin{aligned}
\mathrm{in}_\succ(J_A) &= \mathrm{in}_\succ(J_B \cap \mathbb{C}[x_{i,j} : i \in [s], j \in [n]]) \\
&= \mathrm{in}_\succ(J_B) \cap \mathbb{C}[x_{i,j} : i \in [s], j \in [n]] \\
&= Z_{d,n} \cap \mathbb{C}[x_{i,j} : i \in [s], j \in [n]] \\
&= Z.
\end{aligned}$$

□

Finally, we use Corollary 5.17 to give a determinantal representation of the generators of Z:

Theorem 5.19. *When $A_1, \ldots A_n$ are generic, the generators of Z are leading monomials for a maximal ideal of A_σ for some $\sigma \subseteq [n]$.*

Proof. A monomial generator of Z is of the form

$$x_{i_1,j_1}\, x_{i_2,j_2}\, \ldots\, x_{i_k,j_k}$$

satisfying conditions (1)-(4) of Lemma 5.12. In particular, $i_l \in [s-1]$. Let $\sigma = \{i_1, i_2, \ldots, i_k\}$. Then

$$A_\sigma = \begin{bmatrix} A_{i_1} & p_{i_1} & \mathbf{0} & \dots & \mathbf{0} \\ A_{i_2} & \mathbf{0} & p_{i_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{i_k} & \mathbf{0} & \dots & \mathbf{0} & p_{i_k} \end{bmatrix}$$

is a $ks \times r + k$ matrix. Assuming that $k \geq \lceil \frac{r}{s-1} \rceil$, we have that $ks \geq r + k$, so we must remove $ks - r - k$ rows to produce a maximal minor of A_σ. In each block of variables $x_{1,j_l}, x_{2,j_l}, \dots x_{i_l,j_l} \dots x_{s,j_l}$, there are $i_l - 1$ variables greater than x_{i_l,j_l} with respect to $\succ$, so in total the number of variables in the vectors $p_{i_1}, \dots, p_{i_k}$ that we must eliminate is

$$(i_1 - 1) + (i_2 - 1) + \dots + (i_k - 1)$$

which by condition (3) of Lemma 5.17 is bounded above by $sk - r - k$. This shows that we can form a maximal minor of A_σ in the following way: for each submatrix

$$\begin{bmatrix} A_{i_l} & \dots & p_{i_l} & \dots \end{bmatrix}$$

of A_σ, remove every row containing a variable $x_{i_l,m}$ that is lexicographically greater than x_{i_l,j_l}. By the argument above, the number of rows that we need to remove in this fashion is bounded above by the number of rows that we are allowed to remove to form a maximal minor of A_σ. Therefore, it is possible to form a maximal minor of A_σ of the form

$$a_0 \, x_{i_1,j_1} \, x_{i_2,j_2} \, \dots \, x_{i_k,j_k} \, + \text{ lex. lower order terms}$$

where a_0 is the determinant of some subset of rows of the A_i, which by our genericity assumption will be nonzero. $\square$

5.6 A Universal Gröbner Basis

So far, we have characterized the initial ideal of the multiview ideal defined by the rational map

$$\begin{array}{rccc} \phi: & \mathbb{P}^{r-1} & \longrightarrow & \mathbb{P}^{s-1} \times \mathbb{P}^{s-1} \times \ldots \times \mathbb{P}^{s-1} \\ & \mathbf{x} & \longmapsto & (A_1\mathbf{x},\, A_2\mathbf{x},\, \ldots,\, A_n\mathbf{x}) \end{array}$$

where $n \geq r \geq s > 1$, and the $A_1, \ldots, A_n$ are $s \times r$ matrices of rank s. When the A_i are generic, then the initial ideal of J_A is generated by the leading terms of the maximal minors of the matrix

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_s} & \mathbf{0} & \ldots & \mathbf{0} & p_{\sigma_s} \end{bmatrix}$$

where $p_{\sigma_i} = [x_{1,\sigma_i}\ x_{2,\sigma_i}\ \ldots\ x_{s,\sigma_i}]^T$ and $\sigma = \{\sigma_1, \ldots, \sigma_k\}$ is a k-subset of $[n]$ where

$$\lceil \frac{r}{s-1} \rceil \leq k \leq r.$$

We use this to prove the following:

Theorem 5.20. *Let $A_1, \ldots, A_n$ be generic. Then the set of maximal minors of A_σ,* *where* $\lceil \frac{r}{s-1} \rceil \leq |\sigma| \leq r$ *is a universal Gröbner basis for* J_A*.*

Proof. Let M be the set of all maximal minors of A_σ, for all $\sigma \subseteq [n]$ where $\lceil \frac{r}{s-1} \rceil \leq |\sigma| \leq r$. We already know that $M \subseteq J_A$. Given a polynomial $f \in J_A$, Lemma 9 shows that the leading term of f is an element of $\mathrm{in}_\succ M$. Then there exists a polynomial $g_1 \in M$ such that the leading term of g_1 divides the leading term of f, and there exists some polynomial h_1 such that the leading terms of f and $h_1 \, g_1$ are the same; then $f - h_1 \, g_1$ is a polynomial in J_A whose degree, with respect to $\succ$, is strictly less than f. We can repeat this process with $f - h_1 \, g_1$ and conclude that

$$f - h_1 \, g_1 - h_2 \, g_2 - \; \ldots \; - h_m \, g_m = 0$$

where the g_i are generated by maximal minors of A_{σ_i}, for some subsets $\sigma_i \subseteq [n]$ with $\lceil \frac{r}{s-1} \rceil \leq |\sigma_i| \leq r$. This shows that M generates J_A; since the leading terms of M generate $\mathrm{in}_\succ J_A$, this shows that M is a Gröbner basis of J_A with respect to $\succ$.

To show that M is a universal Gröbner basis, we begin with the following observation. Let $f \in M$. Then there exists some $\sigma \subseteq [n]$ such that f is formed by removing some rows from the matrix

$$A_\sigma := \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_s} & \mathbf{0} & \ldots & \mathbf{0} & p_{\sigma_s} \end{bmatrix}$$

and taking the determinant of the resulting submatrix. With respect to the original term order $\succ$, the leading monomial of f can be read off by looking at each column $p_{\sigma_1}, \ldots, p_{\sigma_r}$ and recording the variable in the first row (for example, if the row containing x_{σ_1} was removed but the row containing y_{σ_1} was not, the leading monomial of f will contain y_{σ_1}). The remaining monomials of f come from taking all possible combinations of taking one variable from each column $p_{\sigma_1}, \ldots, p_{\sigma_r}$. Since the A_i are generic, the coefficients of each monomial, which are the determinants of some subset of the rows of the A_i, will be nonzero, so all such monomials will appear as a monomial in f.

Let $\succ'$ be any lexicographic term order. The set of maximal minors M is invariant under term orders, so it still generates J_A. Then $\text{in}_{\succ'} M \subseteq \text{in}_{\succ'} J_A$. We will first show that the generators of $\text{in}_{\succ'} M$ are in bijection with the generators of $\text{in}_{\succ} M$. Let $f \in M$. There exists some $\sigma \subseteq [n]$ such that f is a maximal minor of

$$A_\sigma = \begin{bmatrix} A_{\sigma_1} & p_{\sigma_1} & \mathbf{0} & \ldots & \mathbf{0} \\ A_{\sigma_2} & \mathbf{0} & p_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{\sigma_k} & \mathbf{0} & \ldots & \mathbf{0} & p_{\sigma_k} \end{bmatrix}.$$

The term order $\succ'$ permutes the ordering of the variables in each column vector p_{σ_i}, in the sense that there exists some $\tau_i \in S_s$ (not necessarily the same for each σ_i) such that

$$x_{\tau_i(1),\sigma_i} \succ' x_{\tau_i(2),\sigma_i} \succ' \ldots \succ' x_{\tau_i(s),\sigma_i}.$$

For each i, let $p'_{\sigma_i} = [x_{\tau_i(1),\sigma_i} \; x_{\tau_i(2),\sigma_i} \; \dots \; x_{\tau_i(s),\sigma_i}]^T$, and let A'_{σ_i} be the matrix A_{σ_i} with the rows permuted with respect to τ_i. Consider the matrix

$$A'_\sigma = \begin{bmatrix} A'_{\sigma_1} & p'_{\sigma_1} & \mathbf{0} & \dots & \mathbf{0} \\ A'_{\sigma_2} & \mathbf{0} & p'_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A'_{\sigma_k} & \mathbf{0} & \dots & \mathbf{0} & p'_{\sigma_k} \end{bmatrix}$$

where the rows of the σ_i^{th} block $[A'_{\sigma_i} \; \dots \; p'_{\sigma_i} \; \dots]$ are permuted by τ_i (so the m^{th} row of the block $[A_{\sigma_i} \; \dots \; p_{\sigma_i} \; \dots]$ is the $\tau_i(m)^{\text{th}}$ row of $[A'_{\sigma_i} \; \dots \; p'_{\sigma_i} \; \dots]$). Let f' be the determinant of the maximal minor fo A'_σ obtained by removing the same rows that were removed from A_σ to form f (i.e., if the j^{th} row of A_σ is removed, then the j^{th} row of A'_σ is removed). Since A'_σ is defined by permuting the rows of A_σ, f', up to a power of (-1), is a maximal minor of A_σ, so $\text{LM}_{\succ'}(f') \in \text{in}_{\succ'} M$. $\text{LM}_{\succ'}(f')$ can be obtained from A'_σ in the same way that $\text{LM}_{\succ}(f)$ can be obtained from A_σ (recording the top variable left in each column p'_{σ_i}. If

$$\text{LM}_{\succ}(f) = x_{i_1,j_1} \, x_{i_2,j_2} \, \dots \, x_{i_k,j_k}$$

then

$$\text{LM}_{\succ'}(f') = x_{\tau_1(i_1),j_1} \, x_{\tau_2(i_2),j_2} \, \dots \, x_{\tau_k(i_k),j_k}$$

which defines a map

$$\begin{array}{ccc} \{\text{generators of } \mathrm{in}_{\succ} M\} & \longrightarrow & \{\text{generators of } \mathrm{in}_{\succ'} M\} \\ x_{i_1,j_1}\, x_{i_2,j_2}\, \ldots\, x_{i_k,j_k} & \longmapsto & x_{\tau_1(i_1),j_1}\, x_{\tau_2(i_2),j_2}\, \ldots\, x_{\tau_k(i_k),j_k} \end{array}$$

and its inverse

$$\begin{array}{ccc} \{\text{generators of } \mathrm{in}_{\succ'} M\} & \longrightarrow & \{\text{generators of } \mathrm{in}_{\succ} M\} \\ x_{i_1,j_1}\, x_{i_2,j_2}\, \ldots\, x_{i_k,j_k} & \longmapsto & x_{\tau_1^{-1}(i_1),j_1}\, x_{\tau_2^{-1}(i_2),j_2}\, \ldots\, x_{\tau_k^{-1}(i_k),j_k} \end{array}$$

which shows that the generators of $\mathrm{in}_{\succ'} M$ and $\mathrm{in}_{\succ} M = \mathrm{in}_{\succ} J_A$ are in bijection. In particular, this means that the Hilbert function of $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/\mathrm{in}_{\succ'} M$ is the same as the Hilbert function of $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/\mathrm{in}_{\succ} J_A$. But the Hilbert function of $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/\mathrm{in}_{\succ} J_A$ is the Hilbert function of $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/J_A$.

Since the Hilbert functions of $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/\mathrm{in}_{\succ'} M$ and $\mathbb{C}[x_{i,j} : i \in [s],\, j \in [n]]/J_A$ coincide, this forces $\mathrm{in}_{\succ'} M = \mathrm{in}_{\succ'} J_A$, so M is a Gröbner basis of J_A with respect to $\succ'$. This shows that M is a Gröbner basis with respect to any lexicographic term order.

Now let $\succ'$ be any term order. Again, the set of maximal minors M is invariant under changing term orders, so it still generates J_A. Let $\succ'_{lex}$ be the lexicographic term order induced by the ordering of the variables $x_{i,j}$ under $\succ'$. Let $f \in M$. Then for some $\sigma \subseteq [n]$, f, up to a power of -1, is a maximal minor of

$$A'_\sigma = \begin{bmatrix} A'_{\sigma_1} & p'_{\sigma_1} & \mathbf{0} & \dots & \mathbf{0} \\ A'_{\sigma_2} & \mathbf{0} & p'_{\sigma_2} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A'_{\sigma_k} & \mathbf{0} & \dots & \mathbf{0} & p'_{\sigma_k} \end{bmatrix}$$

where p'_{σ_i} and A'_{σ_i} are defined above, with respect to the term order $\succ'_{lex}$. $\mathrm{LM}_{\succ'_{lex}}(f)$ can be obtained by recording the top variable left in each column p'_{σ_i}. Any other monomial $\mathbf{x}^\alpha$ that appears in the support of f will differ from $\mathrm{LM}_{\succ'_{lex}}(f)$ by picking variables from the p_{σ_i} that are less than, with respect to $\succ'$, the top remaining variable in p_{σ_i}. Therefore, $\mathrm{LM}_{\succ'_{lex}}(f) \succ' \mathbf{x}^\alpha$ for all other monomials $\mathbf{x}^\alpha$ that appear in the support of f.

This shows that for any $f \in M$, $\mathrm{LM}_{\succ'_{lex}}(f) = \mathrm{LM}_{\succ'}(f)$, so $\mathrm{in}_{\succ'} M = \mathrm{in}_{\succ'_{lex}} M = \mathrm{in}_{\succ'_{lex}} J_A$. In particular, this means that the Hilbert functions of $\mathbb{C}[x_{i,j} : i \in [s], j \in [n]]/\mathrm{in}_{\succ'} M$ and $\mathbb{C}[x_{i,j} : i \in [s], j \in [n]]/\mathrm{in}_{\succ'_{lex}} J_A$ coincide. By the same argument as in the lexicographic case, this forces $\mathrm{in}_{\succ'} M = \mathrm{in}_{\succ'} J_A$. Then M is a Gröbner basis of J_A with respect to $\succ'$, which shows that M is a universal Gröbner basis of J_A. $\square$

5.7 The Primary Decomposition and Hilbert Function of $\mathrm{in}_\succ(J_A)$

In the case of a rational map $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \times \dots \times \mathbb{P}^1$ defined by generic 2×3 matrices $A_1, \dots, A_n$, it is possible to read off the primary decomposition of $\mathrm{in}_\succ(J_A)$:

Proposition 5.21. *The primary decomposition of $\mathrm{in}_\succ J_A$ is*

$$\text{in}_{\succ} J_A = \bigcap_{i,j} P_{i,j}$$

where $P_{i,j} = \langle x_k \,:\, k \in [n] \setminus \{i, j\}\rangle$ *and the intersection runs over all distinct indices* $i, j, \in [n]$*. Here, the* $P_{i,j}$ *are also prime ideals.*

Proof. A generator of J_A is of the form $x_a\, x_b\, x_c$ for distinct indices $a, b,$ and $c \in [n]$. If $x_a\, x_b\, x_c \notin P_{i,j}$ for some $i, j \in [n]$, then x_a, x_b and x_c are all not in $P_{i,j}$, which is a contradiction. This shows that $\text{in}_{\succ} J_A$ is contained in the intersection of the $P_{i,j}$.

Conversely, given a monomial $\mathbf{x}^\alpha$ in the intersection of the $P_{i,j}$, there must exist three distinct indices $a, b,$ and $c \in [n]$ such that $x_a\, x_b\, x_c$ divides $\mathbf{x}^\alpha$; otherwise $\mathbf{x}^\alpha$ is a unit multiple of $x_i\, x_j$ for some $i, j \in [n]$, and then $\mathbf{x}^\alpha \notin P_{i,j}$, a contradiction. This shows that the intersection of the $P_{i,j}$ is contained in $\text{in}_{\succ} J_A$, so the two are equal. □

We can use this primary decomposition to compute the multigraded Hilbert function of the quotient ring $\mathbb{C}[x_i,\, y_i \,:\, i \in [n]]/\text{in}_{\succ} J_A$:

Proposition 5.22. *With respect to the* $\mathbb{Z}^n$*-grading* $\deg(x_i) = \deg(y_i) = e_i$*, where* e_i *is the* i^{th} *standard unit vector in* $\mathbb{R}^n$*,* $\mathbb{C}[x_i,\, y_i \,:\, i \in [n]]/\,\text{in}_{\succ} J_A$ *has the Hilbert function*

$$\mathcal{H}(u_1,\, \ldots\, , u_n) = 1 + \sum_{i=1}^{n} u_i + \sum_{i,j \in [n]} u_i\, u_j \tag{5.24}$$

Proof. Let $u = (u_1, \dots, u_n) \in \mathbb{N}^n$. $\mathcal{H}(u)$ counts the number of monomials of degree $\mathbf{u}$ (with respect to the $\mathbb{Z}^n$-grading) in $\mathbb{C}[x_i, y_i : i \in [n]]/\mathrm{in}_{\succ} J_A$. Such a monomial is of the form $\mathbf{x}^a \mathbf{y}^b$, where $a = (a_1, \dots, , a_n)$, $b = (b_1, \dots, b_n)$, and $a + b = u$. Then $b_i = a_i - u_i$ for all $i \in [n]$, and the choice of a_i determines b_i for each i. Using the primary decomposition of $\mathrm{in}_{\succ} J_A$, $\mathbf{x}^a \mathbf{y}^b \notin P_{i,j}$ for some $i, j \in [n]$. Then $a_k = 0$ for all $k \neq i, j$. To prevent overcounting, we count the number of possible monomials $\mathbf{x}^a \mathbf{y}^b$ by considering 3 cases:

(1) $a = \mathbf{0}$: this contributes one monomial, $\mathbf{y}^u$, to the count.

(2) $a_k = 0$ for all $k \neq i$, for some $i \in [n]$, and $a_i > 0$. Then $1 \leq a_i \leq u_i$ so there are u_i possible choices for a_i. Summing over all $i \in [n]$, this contributes

$$\sum_{i=1}^{n} u_i$$

to the count.

(3) $a_i > 0$ and $a_j > 0$ for two distinct indices $i, j \in [n]$, and $a_k = 0$ for all $k \neq i, j$. Then there are $u_i\, u_j$ possible choices for i and j, and summing over all distinct pairs $i, j \in [n]$, this contributes

$$\sum_{i,j \in [n]} u_i\, u_j$$

to the count.

Adding each of these cases together, we conclude that the number of distinct monomials $\mathbf{x}^a \mathbf{y}^b$ such that $a + b = u$ is

$$1 + \sum_{i=1}^{n} u_i + \sum_{i,j \in [n]} u_i u_j.$$

□

Now we consider the case where $\phi : \mathbb{P}^{r-1} \longrightarrow \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ is a rational map defined by generic $2 \times r$ matrices $A_1, \ldots, A_n$. In this case, we've shown that

$$\text{in}_{\succ}(J_A) = \langle\, x_{j_1} x_{j_2} \ldots x_{j_r} \;:\; j_1,\, j_2,\, \ldots\, j_r \text{ are distinct indices in } [n] \,\rangle$$

which has the following prime decomposition:

Proposition 5.23.

$$in_{\succ}(J_A) = \bigcap_{\sigma} \langle x_i \;:\; i \in \sigma \rangle,$$

where the intersection runs over all $(n - r + 1)$-subsets σ of $[n]$.

Proof. A generator of $\text{in}_{\succ}(J_A)$ is of the form $x_{j_1} x_{j_2} \ldots x_{j_r}$ where $j_1, \ldots, j_r$ are distinct indices in $[n]$. Then for every $(n - r + 1)$-subset $\sigma \subseteq [n]$, there exists some $l \in [r]$ such that $j_r \in \sigma$ (otherwise, σ contains at most $(n - r)$ elements of $[n]$, a contradiction). This shows that

$$\text{in}_{\succ}(J_A) \subseteq \bigcap_{\sigma} \langle x_i \;:\; i \in \sigma \rangle.$$

Conversely, if $\mathbf{x}$ is a monomial in the right side, there must exist r distinct indices $j_1, \ldots, j_r$ in $[n]$ such that $x_{j_1} x_{j_2} \ldots x_{j_r}$ divides $\mathbf{x}$; otherwise, $\mathbf{x}$ is the unit multiple of at most $r-1$ distinct variables $x_{j_1}, \ldots, x_{j_{r-1}}$. In this case, $\mathbf{x} \notin \langle x_i : i \in [n]\backslash\{j_1, \ldots, j_{r-1}\}\rangle$, which is a contradiction since $[n]\backslash\{j_1, \ldots, j_{r-1}\}$ is an $(n-r+1)$-subset of $[n]$. This shows the other inclusion. □

Again, we can use this decomposition of $\text{in}_\succ(J_A)$ to compute the Hilbert function of $\text{in}_\succ(J_A)$:

Proposition 5.24. *With respect to the* $\mathbb{Z}^n$ *grading* $\deg(x_i) = \deg(y_i) = e_i$*, where* e_i *is the* i^{th} *standard unit vector in* $\mathbb{R}^n$*,* $\mathbb{C}[x_i, y_i : i \in [n]]/\,\text{in}_\succ(J_A)$ *has the Hilbert function*

$$\mathcal{H}(u_1, \ldots, u_n) = 1 + \sum_{i=1}^{r-1} \sum_{\{j_1, j_2, \ldots, j_i\} \subseteq [n]} \prod_{l=1}^{i} u_{j_i}. \tag{5.25}$$

Proof. Let $u = (u_1, \ldots, u_n)$. $\mathcal{H}(u)$ counts the number of monomials of degree u (with respect to the $\mathbb{Z}^n$-grading) in $\mathbb{C}[x_i, y_i : i \in [n]]/\text{in}_\succ(J_A)$. Such a monomial is of the form $\mathbf{x}^a\,\mathbf{y}^b$, where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $a + b = u$. Then $b_i = u_i - a_i$ for all $i \in [n]$, and the choice of a_i determines b_i for each i. It suffices to count the number of monomials $\mathbf{x}^a\,\mathbf{y}^b \notin P_\sigma$ for some $(n-r+1)$-subset σ of $[n]$. Equivalently, we want $a_k = 0$ for all but at most $(r-1)$ indices. To prevent overcounting, we count the number of possible monomials $\mathbf{x}^a\,\mathbf{y}^b$ where $a + b = u$, $a_k = 0$ for all but exactly j indices (where $j \in [r-1]$). The number of such monomials is

$$1 + \sum_{i=1}^{r-1} \sum_{\{j_1, j_2, \dots, j_i\} \subseteq [n]} \prod_{l=1}^{i} u_{j_i}.$$

□

Chapter 6

Conclusion

The goal of our research was to generalize the results on the existence of epipolar matrices in and the structure of the multiview ideal. In [1], sufficient conditions for the existence of epipolar matrices were given, based on the rank of a particular linear map Z and whether the projective cameras were assumed to be uncalibrated or totally calibrated. In Chapter 4, we extended the rank condition to the case where the cameras are partially calibrated, using the properties of projective varieties described in Chapter 2.

In Chapter 4, we examined two types of partially calibrated camera systems. In Section 3 of Chapter 4, we considered pairs of cameras where the partially calibrated camera was known to have a diagonal calibration matrix, and we proved that the rank condition in this case is the same as the rank condition for totally calibrated cameras. Then in Section 4 of Chapter 4, we relaxed the condition on the calibration matrix and in this case, the rank condition was exactly the same as in the uncalibrated case.

These results are somewhat surprising, since what seemed to be intermediate cases for calibration behave like the two extreme cases for calibration. It would be interesting to consider other characterizations of partial calibration and determine if a more intermediate case exists.

In Chapter 5, we generalized some of the results in [8] regarding the structure of the multiview ideal by examining the structure of the ideal J_A of the rational map

$$\begin{array}{rccl} \phi: & \mathbb{P}^{r-1} & \longrightarrow & \mathbb{P}^{s-1} \times \mathbb{P}^{s-1} \times \ldots \times \mathbb{P}^{s-1} \\ & \mathbf{x} & \longmapsto & (A_1\mathbf{x},\, A_2\mathbf{x},\, \ldots,\, A_n\mathbf{x}) \end{array}$$

where $n \geq r \geq s > 1$, and $A_1, \ldots, A_n$ are $s \times r$ matrices of rank s. The primary objects of interest here are the monomial ideals and Gröbner bases described in Chapter 2. In Sections 3, 4 and 5 of Chapter 5, we were able to mimic the argument used in [2, Section 2] and use the results from [3] to characterize the initial ideal of J_A as the leading terms of maximal minors of a particular set of matrices A_σ. In Section 6, we proved that these maximal minors actually define a universal Gröbner basis for J_A. Finally, in Section 7, we obtained a prime decomposition for the initial ideal of J_A in the case where ϕ is a map from $\mathbb{P}^{r-1}$ into $(\mathbb{P}^1)^n$. This result is used to compute the multigraded Hilbert function of $\mathrm{in}_{\succ} J_A$.

While we were able to generalize the result regarding the universal Gröbner basis of J_A in [2], we were unable to generalize the primary decomposition of its initial ideal except for the cases described in Section 7. This result, for general $r \geq s > 1$, would allow us to compute the multigraded Hilbert function of the initial ideal.

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