Optimality conditions

Serkan Hoşten
Department of Mathematics, San Francisco State University, San Francisco

September 18, 2003

1 Optimality conditions

We will develop an algorithm for an LP in standard form where we will start with a BFS and move to a *better* BFS (i.e. a BFS which has a better objective function value). We hope that this will take us to an optimal solution (if it exists, of course) that is guaranteed to be a BFS as well. But how would we know that we reached the optimal solution?

Definition 1. Let $\mathbf{x}$ be an element of a polyhedron P. A vector $\mathbf{d}$ is called a *feasible direction* at $\mathbf{x}$, if there exists a positive scalar θ such that $\mathbf{x}+\theta\mathbf{d} \in P$.

Now suppose we are at a BFS $\mathbf{x}$. Suppose the basic variables are $x_{B(1)}, \ldots, x_{B(m)}$ where $B = \left[A_{B(1)}, \ldots, A_{B(m)}\right]$ is the basis. The nonbasic variables are equal to zero and the basic variables are determined by

$$\mathbf{x}_B = B^{-1}b.$$

We will try to find a feasible direction $\mathbf{d}$ where $d_j = 1$ for a single nonbasic variables, and $d_i = 0$ for all other nonbasic variables, and we will move to $\mathbf{x}+\theta\mathbf{d}$. This has the effect of increasing the special nonbasic variable x_j to θ while keeping all the other nonbasic variables at level 0. How about the basic variables? Let's denote the portion of $\mathbf{d}$ corresponding to the basic variables by $\mathbf{d}_B$. Since the new point we will move to needs to be feasible we have $A\mathbf{d} = 0$. But $A\mathbf{d} = B\mathbf{d}_B + A_j = 0$, and we conclude that $\mathbf{d}_B = -B^{-1}A_j$. This is called the $j$the basic direction. This construction guarantees that the new point satisfies the equality constraints. But how about nonnegativity? We need to make sure that $\mathbf{x}_B + \theta\mathbf{d}_B$ stays nonnegative. If $\mathbf{x}_B > 0$, for small

θ the new point will be still feasible. If $\mathbf{x}$ is degenerate we need to be more careful since $\mathbf{d}$ might not be a feasible direction. If it is a feasible direction, the change in the cost is $c_j + c_B\mathbf{d}_B = c_j - c_B B^{-1} A_j$. This quantity is called the *reduced cost* of x_j, and is denoted by $\bar{c}_j$. Observe that if x_j is a basic variable then $\bar{c}_j = 0$.

Theorem 1.1. *Suppose* $\mathbf{x}$ *is a basic feasible solution with the basis matrix* B*. If all the reduced costs are nonnegative then* x *is optimal.*

Proof. We will compare $\mathbf{c}'\mathbf{x}$ against $\mathbf{c}'\mathbf{y}$ where $\mathbf{y}$ is any feasible solution. Let $\mathbf{d} = \mathbf{y} - \mathbf{x}$. We need to show that $\mathbf{c}'\mathbf{d} \geq 0$:

$$\mathbf{c}'\mathbf{d} = c_B\mathbf{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i.$$

But observe that $d_i \geq 0$ for nonbasic variables. □