Basic solutions of polyhedra in standard form

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1 Characterization of basic solutions

We will consider polyhedra in standard form

$$P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0\}$$

where A is $m \times n$ matrix. Without loss of generality we can assume that the rows of A are linearly independent. Now, for a basic solution, we need a total of n active constraints m of which are provided by the matrix A. Then we need to choose $n-m$ variables x_i and set them to zero. Will an arbitrary choice lead to a basic solution? The answer is given by the following theorem:

Theorem 1.1. *Suppose the $m \times n$ matrix A has independent rows. Then $\mathbf{x}^* \in \mathbb{R}^n$ is a basic solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$ if and only if $A\mathbf{x}^* = b$ and there are m indices $B(1), \ldots, B(m)$ such that the columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent and if $i \neq B(1), \ldots, B(m)$ then $x_i^* = 0$.*

Proof. "if": Since the columns are linearly independent $\mathbf{x}^*$ is uniquely determined.
" only if": Let $x^*_{B(1)}, \ldots, x^*_{B(k)}$ are the nonzero components of $\mathbf{x}^*$. In order to get a unique $\mathbf{x}^*$ we need to have corresponding columns to be linearly independent. So $k \leq m$, and we can complete these columns to m linearly independent columns. □

This theorem suggests that in order to form a basic solution, we need to pick m linearly independent columns of A, say $A_{B(1)}, \ldots, A_{B(m)}$, set $x_i = 0$ not corresponding to these columns and solve the system $A\mathbf{x} = b$. If we get a nonnegative solution it is a basic feasible solution. The variables $x_{B(1)}, \ldots, x_{B(m)}$ are called basic variables, the rest nonbasic variables. The corresponding columns of A form a basis, which is an $m \times m$ submatrix B of A which is invertible. Hence the basic variables are given by $\mathbf{x}_B = B^{-1}\mathbf{b}$.