1 Characterization of basic solutions

We will consider polyhedra in standard form

\[ P = \{ x \in \mathbb{R}^n : Ax = b, \ x \geq 0 \} \]

where \( A \) is a \( m \times n \) matrix. Without loss of generality we can assume that the rows of \( A \) are linearly independent. Now, for a basic solution, we need a total of \( n \) active constraints \( m \) of which are provided by the matrix \( A \). Then we need to choose \( n - m \) variables \( x_i \) and set them to zero. Will an arbitrary choice lead to a basic solution? The answer is given by the following theorem:

**Theorem 1.1.** Suppose the \( m \times n \) matrix \( A \) has independent rows. Then \( x^* \in \mathbb{R}^n \) is a basic solution of \( Ax = b \) and \( x \geq 0 \) if and only if \( Ax^* = b \) and there are \( m \) indices \( B(1), \ldots , B(m) \) such that the columns \( A_{B(1)}, \ldots , A_{B(m)} \) are linearly independent and if \( i \neq B(1), \ldots , B(m) \) then \( x^*_i = 0 \).

**Proof.** "if": Since the columns are linearly independent \( x^* \) is uniquely determined.

"only if": Let \( x^*_{B(1)}, \ldots , x^*_{B(k)} \) are the nonzero components of \( x^* \). In order to get a unique \( x^* \) we need to have corresponding columns to be linearly independent. So \( k \leq m \), and we can complete these columns to \( m \) linearly independent columns.

This theorem suggests that in order to form a basic solution, we need to pick \( m \) linearly independent columns of \( A \), say \( A_{B(1)}, \ldots , A_{B(m)} \), set \( x_i = 0 \) not corresponding to these columns and solve the system \( Ax = b \). If
we get a nonnegative solution it is a basic feasible solution. The variables
$x_{B(1)}, \ldots, x_{B(m)}$ are called basic variables, the rest nonbasic variables. The
 corresponding columns of $A$ form a basis, which is an $m \times m$ submatrix $B$
of $A$ which is invertible. Hence the basic variables are given by $x_B = B^{-1}b$. 