Extrem Points, Vertices and basic feasible solutions

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1 Definitions

We will clarify what we mean by a "corner point" of a polyhedron. We will define three objects: *extreme points*, *vertices*, and *basic feasible solutions*. The first two are geometric definitions. The last one is algebraic. Then we will show that they are all the same. But this is convenient since we will have three different ways of knowing and describing "corner points", and we will choose the best depending on the context.

Definition 1. A vector $\mathbf{x}$ of a polyhedron P is called an extreme point if there are no two vectors $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{z} \neq \mathbf{x}$ in P such that $\mathbf{x}$ is a convex combination of $\mathbf{y}$ and $\mathbf{z}$.

This means an extreme point is a vector which does not lie on the line connecting any two vectors in P. Note that we did not use the inequality description of P in this definition.

Definition 2. A vector $\mathbf{x}$ of a polyhedron P is called a vertex if there exists $\mathbf{c}$ such that $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$ for all $\mathbf{y} \neq \mathbf{x} \in P$.

This means P is contained entirely on one side of the hyperplane defined by $\mathbf{c}$, and the only point that is on this hyperplane is $\mathbf{x}$.

A polyhedron is given by $P = \{\mathbf{x} \in \mathbb{R}^n : \ A\mathbf{x} \geq b\}$. If $\mathbf{x}^*$ is a vector in P, then some of these equations could be satisfied with equality: $\mathbf{a}_i'\mathbf{x}^* = b_i$. Such a constraint is called *active* or *binding* at $\mathbf{x}^*$. Note that if there are

n linearly independent constraints active at $\mathbf{x}^*$, the vector $\mathbf{x}^*$ is uniquely determined by these active constraints.

Definition 3. A vector $\mathbf{x}^*$ is called a basic solution if the active constraints at $\mathbf{x}^*$ contain the equality constraints, and among the active constraints there are n linearly independent ones. Moreover, if $\mathbf{x}^*$ is actually in P, then it is called a basic feasible solution.

Example 1.1. Give a two dimensional geometric example with degeneracy.

2 Equivalence of the definitions

Theorem 2.1. *Let $\mathbf{x}^*$ be a point of a polyhedron P. Then the following are equivalent.*

a) $\mathbf{x}^*$ *is a vertex.*

b) $\mathbf{x}^*$ *is an extreme point.*

c) $\mathbf{x}*$ *is a basic feasible solution.*

Proof. a) $\implies$ b). Suppose $\mathbf{c}$ is the supporting hyperplane of the vertex $\mathbf{x}^*$. If $\mathbf{x}^*$ is a convex combination of two vectors $\mathbf{y}$ and $\mathbf{z}$ in P, then $\mathbf{x}^* = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ for some $0 < \lambda < 1$. This means $\mathbf{c}'\mathbf{x}^* = \lambda\mathbf{c}'\mathbf{y} + (1-\lambda)\mathbf{c}'\mathbf{z}$. But $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{y}$ and $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{z}$. A contradiction.
b) $\implies$ c). Suppose $\mathbf{x}^*$ is a not BFS. We will show that $\mathbf{x}^*$ is not an extreme point. Since $\mathbf{x}^*$ is feasible but not a BFS, the active constraints at $\mathbf{x}^*$ cannot have n linearly independent constraints. If we assume that the active constraints are the first k constraints, then the system of equations $\mathbf{a}_i'\mathbf{x} = 0$ for $i = 1, \ldots, k$ has a nontrivial solution $\mathbf{d}$. Now for very small ϵ we define $\mathbf{y} = \mathbf{x}^* + \epsilon\mathbf{d}$ and $\mathbf{z} = \mathbf{x}^* - \epsilon\mathbf{d}$. Both $\mathbf{y}$ and $\mathbf{z}$ are in P for suitably chosen ϵ, and $\mathbf{x}^* = (1/2)\mathbf{y} + (1/2)\mathbf{z}$.
c) $\implies$ a). Let's assume that the first k constraints are the active constraints at $\mathbf{x}^*$. We let $\mathbf{c} = \sum_{i=1}^{k} \mathbf{a}_i$. It is easy to see that $\mathbf{c}$ is a supporting hyperplane. $\square$