1 Graphical solution

When the number of decision variables in a LP is two or three we saw how to visualize the set of all feasible solutions, the \emph{feasible region}. We draw the boundary of this region by drawing the lines or planes given by the constraints. Let’s look at the Example 1.6 from the textbook:

\begin{align*}
\text{min} & \quad -x_1 - x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}

Now we draw the \emph{level sets} of the objective function. I.e. for any given scalar $z$ we consider the set of all points $\mathbf{x}$ whose cost $\mathbf{c}'\mathbf{x}$ is equal to $z$. In our example this is the line $-x_1 - x_2 = z$. Changing $z$ has the effect of shifting this line parallel to itself. Since we are trying to minimize the objective function we will shift the line in that direction where $z$ decreases, as long as we do not leave the feasible region. We see that $\mathbf{x} = (1, 1)$ is the optimal solution.

Depending on the feasible region and the objective one of the following will happen:

- There will be a unique optimal solution (as above).
- There will be multiple optimal solutions (take $-x_1 - 2x_2$ as the objective function in the above problem).
• The problem will have an unbounded objective function value.
• There will be no feasible solutions. The problem is infeasible.

2 Polyhedra and convex sets

Recall that the feasible region of an LP can be described by the constraints $Ax \geq b$ where $A$ is an $m \times n$ matrix. In this case there are $m$ constraints describing a set in $\mathbb{R}^n$. Such set of points is called a polyhedron (plural: polyhedra)

$$\{ x \in \mathbb{R}^n : Ax \geq b \}.$$

A polyhedron may or may not be bounded, and it is the intersection of $m$ halfspaces $\{ x \in \mathbb{R}^n : a'x \geq b \}$. The boundary of such a halfspace is defined by the hyperplane $\{ x \in \mathbb{R}^n : a'x = b \}$.

**Definition 1.** A set $S$ in $\mathbb{R}^n$ is called convex if for all $x$ and $y$ in $S$, and any $0 \leq \lambda \leq 1$ we have $\lambda x + (1 - \lambda)y \in S$.

This means that the line segment joining every pair of points in $S$ must be entirely contained in $S$.

**Theorem 2.1.** The intersection of convex sets is again a convex set. In particular, every polyhedron is a convex set.

**Proof.** The first statement is easily proved using the definition of a convex set. For the second, we observe that every halfspace is a convex set. Now every polyhedron is the intersection of finitely many halfspaces so it must be convex. \hfill \Box

**Definition 2.** Suppose $x^1, \ldots, x^k$ are vectors in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k$ are non-negative numbers such that $\lambda_1 + \cdots + \lambda_k = 1$. Then the vector $\sum_{i=1}^{n} \lambda_i x^i$ is called a convex combination of the $k$ vectors. The set of all convex combinations of $x^1, \ldots, x^k$ is the convex hull of these vectors.

**Theorem 2.2.** The convex combination of finitely many points of a convex set also belongs to this convex set. The convex hull of a finite number of vectors is a convex set.