Graphical solution and Introduction to Polyhedral Convexity

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1 Graphical solution

When the number of decision variables in a LP is two or three we saw how to visualize the set of all feasible solutions, the *feasible region*. We draw the boundary of this region by drawing the lines or planes given by the constraints. Let's look at the Example 1.6 from the textbook:

$$\begin{array}{llll} min & -x_1 & -x_2 & \\ s.t. & x_1 & +2x_2 & \leq 3 \\ & 2x_1 & +x_2 & \leq 3 \\ & & x_1, x_2 & \geq 0 \end{array}$$

Now we draw the *level sets* of the objective function. I.e. for any given scalar z we consider the set of all points $\mathbf{x}$ whose cost $\mathbf{c}'\mathbf{x}$ is equal to z. In our example this is the line $-x_1 - x_2 = z$. Changing z has the effect of shifting this line parallel to itself. Since we are trying to minimize the objective function we will shift the line in that direction where z decreases, as long as we do not leave the feasible region. We see that $\mathbf{x} = (1, 1)$ is the optimal solution.

Depending on the feasible region and the objective one of the following will happen:

- There will be a unique optimal solution (as above).
- There will be multiple optimal solutions (take $-x_1 - 2x_2$ as the objective function in the above problem).

- The problem will have an unbounded objective function value.
- There will be no feasible solutions. The problem is infeasible.

2 Polyhedra and convex sets

Recall that the feasible region of an LP can be described by the constraints $A\mathbf{x} \geq b$ where A is an $m \times n$ matrix. In this case there are m constraints describing a set in $\mathbb{R}^n$. Such set of points is called a polyhedron (plural: polyhedra)

$$\{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{A}\mathbf{x} \geq \mathbf{b}\}.$$

A polyhedron may or may not be *bounded*, and it is the intersection of m *halfspaces* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'\mathbf{x} \geq b\}$. The boundary of such a halfspace is defined by the hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'\mathbf{x} = b\}$.

Definition 1. A set S in $\mathbb{R}^n$ is called convex if for all x and y in S, and any $0 \leq \lambda \leq 1$ we have $\lambda x + (1-\lambda)y \in S$.

This means that the line segment joining every pair of points in S must be entirely contained in S.

Theorem 2.1. *The intersection of convex sets is again a convex set. In particular, every polyhedron is a convex set.*

Proof. The first statement is easily proved using the definition of a convex set. For the second, we observe that every halfspace is a convex set. Now every polyhedron is the intersection of finitely many halfspaces so it must be convex. □

Definition 2. Suppose $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are vectors in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k$ are non-negative numbers such that $\lambda_1 + \cdots + \lambda_k = 1$. Then the vector $\sum_{i=1}^n \lambda_i \mathbf{x}^i$ is called a convex combination of the k vectors. The set of all convex combinations of $\mathbf{x}^1, \ldots, \mathbf{x}^k$ is the convex hull of these vectors.

Theorem 2.2. *The convex combination of finitely many points of a convex set also belongs to this convex set. The convex hull of a finite number of vectors is a convex set.*