Piecewise linear objective functions

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September 2, 2003

1 LPs in standard form

We saw that every linear program can be written as

$$\min \mathbf{c}'\mathbf{x} \ \text{ s.t. } \ A\mathbf{x} \geq b.$$

When we start talking about *solving* LPs it will be better if we can assume that all variables are non-negative and all the constraints are equalities. In other words, it would be great if we could write our LPs in the *standard form*:

$$\min \mathbf{c}'\mathbf{x} \ \text{ s.t. } \ A\mathbf{x} = 0 \ \ \mathbf{x} \geq 0.$$

It turns out that any LP can be turned into this form. If we have $\mathbf{a}'\mathbf{x} \leq b$ then we can add a *slack variable* and turn it into an equality: $\mathbf{a}'\mathbf{x} + s = b$. We make $s \geq 0$. For the inequality of the type $\mathbf{a}'\mathbf{x} \geq b$ we substract a *surplus* variable: $\mathbf{a}'\mathbf{x} - t = b$ and force $t \geq 0$. And a free variable x_j could be written as $x_j = x_j^+ - x_j^-$ where the two new variables are nonnegative (and we substitute them into the cost function and the equations).

Example 1.1.

$$\begin{array}{llllll} \mathit{min} & 2x_1 & +4x_2 & & \\ \mathit{s.t.} & x_1 & +x_2 & \leq & 8 \\ & -x_1 & +2x_2 & = & 2 \\ & 2x_1 & +x_2 & \geq & 3 \\ & x_1 & \geq & 0 & \end{array}$$

We need a slack variable in the first constraint and a surplus variable in the third constraint. Since x_2 is free we replace it by $x_2 = x_2^+ - x_2^-$. So we get an LP in standard form:

$$\begin{array}{lccccccc}
min & 2x_1 & +4x_2^+ & -4x_2^- & & & & \\
s.t. & x_1 & +x_2^+ & -x_2^- & +x_3 & & = & 8 \\
 & -x_1 & +2x_2^+ & -2x_2^- & & & = & 2 \\
 & 2x_1 & +x_2^+ & -x_2^- & & -x_4 & = & 3 \\
 & x_1 & & & & & \geq & 0 \\
 & & x_2^+ & & & & \geq & 0 \\
 & & & x_2^- & & & \geq & 0 \\
 & & & & x_3 & & \geq & 0 \\
 & & & & & x_4 & \geq & 0
\end{array}$$

It is important to note that by doing this we have not changed anything in the problem. More precisely, all the feasible solutions of the original LP and its standard form are in bijection. For instance, the feasible solution $(x_1, x_2) = (4, 3)$ corresponds to $(x_1, x_2^+, x_2^-, x_3, x_4) = (4, 3, 0, 1, 8)$.

How about absolute values? Well, if we have an absolute value in the cost, say, $|x_i|$, then we can replace $x_i = x_i^+ - x_i^-$ in the constraints, but use $c_i \times (x_i^+ + x_i^-)$ in the objective function. Claim: in an optimal solution either $x_i^+ = 0$ or $x_i^- = 0$. If both positive, then substracting an epsilon amount from each variable will not affect their difference (so the constraints will not be altered), but the objective function will be smaller (assuming $c_i > 0$).

2 Piecewise linear objective functions

Suppose $\mathbf{c}_1, \ldots, \mathbf{c}_m$ are vectors in $\mathbb{R}^n$ and $d_1, \ldots, d_m \in \mathbb{R}$. Then we can form the function $f(x) = \max\ (\mathbf{c}_i'\mathbf{x} + d_i)$. This function is called a piecewise linear function.

Example 2.1. A one dimensional example ($n = 1$). Let $f(x) = \max\ (-2x + 5, x + 2, 3x - 4)$. Then this function can be defined piecewise, and each piece is linear: $f(x) = -2x + 5$ for $(-\infty, 1]$, $f(x) = x + 2$ for $[1, 3]$, and $f(x) = 3x - 4$ for $[3, +\infty)$.

Suppose we want to solve min $f(x)$ s.t. $A\mathbf{x} \geq b$. We can write this as a linear program by introducing one extra variable: min z s.t. $z \geq \mathbf{c}_1'\mathbf{x} + d_1, \ldots, z \geq \mathbf{c}_m'\mathbf{x} + d_m,\ \ A\mathbf{x} \geq b$.

Example 2.2. Evilsoft and Goodguy Linux are two companies who need to determine their business strategies for next year. Strategy A is expand and invest into the server software market and Strategy B is expand and invest into the personal software market. Depending on the strategies the market share gained by Evilsoft is given below as a matrix (a negative number indicates the market share gained by Goodguy Linux). Evilsoft will put all its resources into one strategy whereas Goodguy Linux can "mix" his strategies. Write an LP that will minimize the worst case loss of Goodguy Linux.

	Linux A	Linux B
Evilsoft A	3	-2
Evilsoft B	-4	3

min z s.t. $3p_A - 2p_B \leq z,\ \ -4p_A + 3p_B \leq z,\ \ p_A + p_B = 1,\ \ p_A, p_B \geq 0.$