Linear Programming: Examples and Definition

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1 Modelling

Modeling starts with a problem that needs a solution: given the distances between all pairs of major cities in United States what is the shortest way to take to drive from NYC to SF? Given the amount of available parts and labor, and a forcasted demand for passenger jets for the next year how many 737s, 747s and 777s should Boeing Corporation produce at a minimum cost? etc. etc.
Most problems of this sort are *optimization* problems: maximize or minimize something (max: profit, yield, coverage; min: cost, time spent, waste) subject to some constraints. Mathematically, we usually maximize or minimize an *objective function* f subject to a set of constraints. The goal is to find a best possible solution so that f is minimized or maximized within the given constraints. We usually need a precise procedure to achieve this goal: an *algorithm*.

Example 1.1. Let $S \subset \mathbb{R}^2$ be the region defined by inequalities:

$$x \geq 0,\ y \geq 0,\ y - x \leq 2,\ x - 2y \leq 2,\ x + y \leq 5.$$

Problem: Find a circle that is contained in S with the largest possible radius. This does not look like a "real-world" problem (though actually it is...), and I will give a more real-looking example in a minute, however, let's try to model this problem. Observation 1: the center of the circle (x_0, y_0) must be in S, i.e. it needs to satisfy the above equations. This gives us a set of constraints (write the constraints on the board). Observation 2: The circle will touch at least one side of S. In other words, the shortest distance from any side to the center (x_0, y_0) must be at least the actual radius r of the circle. Remember that the distance of the point (x_0, y_0) to any line $ax + by = c$ is $|c - (ax_0 + by_0)|/\sqrt{a^2 + b^2}$. For instance we get $x_0 \geq r$ and $y_0 \geq r$. But also $(5 - x_0 - y_0)/\sqrt{2} \geq r$, etc. These give us a second set of constraints. The goal is to maximize r. This is the objective function. We are after (x_0, y_0, r) that satisfy all the constraints *and* maximizes r.

Example 1.2. Linear Programming models are used by many Wall Street firms to select a desirable portfolio. The following is a simplified version of such a model: Solodrex is considering investing in four bonds; \$ 1,000,000 is available for investment. The expected annual return, the worst case return on each bond, and the "duration" of each bond are given in the table below. The duration of a bond is a measure of the bond's sensitivity to interest rates.

	Exp.Ret.	*Worst − case.Ret.*	*Duration*
*Bond*1	13%	6%	3
*Bond*2	8%	8%	4
*Bond*3	12%	10%	7
*Bond*4	14%	9%	9

Solodrex wants to maximize the expected return from its bonds investments, and wants to respect the following constraints. First of all, the worst-case return of the bond portfolio must be at least 8%. Secondly, the average duration of the portfolio must be at most 6. For example, a portfolio that invested \$600,000 in bond 1 and \$400,000 in bond 4 would have an average duration of

$$\frac{600,000 \cdot 3 + 400,000 \cdot 9}{1,000,000} = 5.4.$$

And finally, because of the diversification requirements, at most 40% of the total amount invested can be invested in a single bond.

In a linear programming problem we have a vector of variables $\mathbf{x} = (x_1, \ldots, x_n)$, and a *cost vector* $\mathbf{c} = (c_1, \ldots, c_n)$. We seek to minimize the *linear function* $c_1x_1 + \cdots + c_nx_n = \sum_{i=1}^n c_ix_i$ subject to a set of linear equality and inequality constraints. Note that the above sum is the dot product of $\mathbf{c}$ and $\mathbf{x}$, and using the book's notation we will denote it by $\mathbf{c}'\mathbf{x}$. There will be constraints of the form $\mathbf{a}'\mathbf{x} \geq b$, $\mathbf{a}'\mathbf{x} \leq b$, and $\mathbf{a}'\mathbf{x} = b$, and some variables will be non-negative $x_j \geq 0$, and some will be nonpositive $x_j \leq 0$. Those variables which are neither are *free* or *unrestricted* variables.

Example 1.3.

$$\begin{array}{lccccc} min & 2x_1 & +3x_2 & -4x_3 & & \\ s.t. & x_1 & +x_2 & & \leq & 6 \\ & -x_1 & +2x_2 & -3x_3 & = & 2 \\ & & +x_2 & -2x_3 & \geq & 8 \\ & x_1 & & & \geq & 0 \\ & & x_2 & & \leq & 0 \end{array}$$

Terminology: $x_1, \ldots, x_n$: decision variables. A vector $\mathbf{x}$ satisfying all constraints is a feasible solution, and the set of all feasible solutions is the feasible set or feasible region. $\mathbf{c}'\mathbf{x}$ is the objective (cost) function. If a feasible solution $\mathbf{x}^*$ has the minimum possible cost then it is called an optimal solution with the optimal solution value $\mathbf{c}'\mathbf{x}^*$. If we can make the optimal solution value as small as we can, i.e. $-\infty$, then the LP is said to be unbounded. We do not need to study maximization problems separately since max $\mathbf{c}'\mathbf{x}$ is equivalent to min $-\mathbf{c}'\mathbf{x}$.
Observe that $\mathbf{a}'\mathbf{x} = b$ is equivalent to $\mathbf{a}'\mathbf{x} \geq b$ and $\mathbf{a}'\mathbf{x} \leq b$ (where the latter is $-\mathbf{a}'\mathbf{x} \geq -b$). So we can write the linear program as: min $\mathbf{c}'\mathbf{x}$ subject to $A\mathbf{x} \geq b$.

Example 1.4. Example 1.3 continued. We have $\mathbf{c} = (2, 3, -4)$, and

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

where $b = (-6, 2, -2, 8, 0, 0,)$.