Final Exam, Math 371

There are five questions on the exam. Each question is worth twenty points.

1. Do TWO of the following:
   
a) State the Heine-Borel Theorem.

b) State Lindelöf’s Theorem.

c) What does it mean for a subset \( A \) of a metric space \( X \) to be compact in \( X \)?

d) State the Cauchy-Schwarz Inequality.

2. Are the following true or false? Justify your answers briefly.
   
a) If \( A, B \subseteq \mathbb{R} \), then \( \overline{A \cap B} = \overline{A} \cap \overline{B} \).

b) If \( x_k = \left( \frac{1}{k}, \frac{1}{k^2}, \frac{1}{k^3} \right) \), then \( x_k \to (0, 0, 0) \) as \( k \to \infty \).

c) Every compact subset of \( \mathbb{R} \) is connected.

d) If \( A \) is an open subset of \( \mathbb{R} \), and \( B \) is any subset of \( \mathbb{R} \), then \( AB = \{ xy \in \mathbb{R} : x \in A, y \in B \} \) is open in \( \mathbb{R} \).

3. For \( x, y \in \mathbb{R}^2 \), define \( \rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \), where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). Show that \( \mathbb{R}^2 \) is a metric space with respect to \( \rho \).

4. Let \( X \) be a metric space. Suppose \( A \subseteq X \) and \( A \) has the property that there is an \( \epsilon > 0 \) such that \( \rho(a, b) \geq \epsilon \) for all \( a, b \in A \) with \( a \neq b \). Show that \( A \) is compact if and only if \( A \) is finite.

5. Let \( X \) be a metric space. Suppose \( A \subseteq X \) and \( A \) has the property that for each \( a, b \in A \), there is a continuous function \( f : [0, 1] \to A \) with \( f(0) = a \) and \( f(1) = b \). Show that \( A \) is connected.