6.1 Introduction.

6.1.0. a) False. $a_k = 1/k$ is strictly decreasing to 0 but $\sum_{k=1}^{\infty} 1/k$ diverges.
   b) False. The series associated with $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$ both diverge, but $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} 0 = 0$.
   c) True. For example, if $\sum_{k=1}^{\infty} (a_k + b_k)$ and $\sum_{k=1}^{\infty} a_k$ converge, then $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k - a_k) = \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{\infty} a_k$ converges by Theorem 6.10.
   d) True. By algebra and telescoping

   \[
   \sum_{k=1}^{\infty} (a_k - a_{k+2}) = \sum_{k=1}^{\infty} (a_k - a_{k+1}) + \sum_{k=1}^{\infty} (a_{k+1} - a_{k+2}) = (a_1 - a) + (a_2 - a).
   \]

6.1.1. a) $\sum_{k=1}^{\infty} (-1)^{k+1}/\sqrt{k} = -\sum_{k=0}^{\infty} (-1)^k = 1/(1 + 1/e) = e/(1 + e)$.
   b) $\sum_{k=0}^{\infty} (-1)^{k+1}/k^{2/3} = -\sum_{k=0}^{\infty} (-1)^k/\sqrt{k} = -1/(1 + 1/\pi^2) = -\pi^2/(\pi^2 + 1)$.
   c) $\sum_{k=0}^{\infty} (-1)^k/(2k+1) = 35 \sum_{k=0}^{\infty} (4/9)^k = 36(4/9)^2/(1 - 4/9) = 64/5$.
   d) $\sum_{k=0}^{\infty} (3k+1)^2/(7k+2)^2 = (5 \sum_{k=0}^{\infty} (7/9)^k + \sum_{k=0}^{\infty} (3/7)^k)/7^2 = (5/2 + 1/10)/7 = 13/35$.

6.1.2. a) $\sum_{k=1}^{\infty} 1/(k(k+1)) = \sum_{k=1}^{\infty} [1/(k-1) - 1/(k+1)] = 1 - \lim_{k \to \infty} 1/k = 1$.
   b) $\sum_{k=1}^{\infty} 1/(k+2) = -\sum_{k=1}^{\infty} (2k/(k+2) - (2k+2)/(k+3)) = \lim_{k \to \infty} (2k+2)/(k+3) - 3 = 3/2 = 3$. 
   c) $\log((k+2)/(k+1)^2) = \log(k/(k+1)) - \log((k+1)/(k+2))$. Therefore, by telescoping we obtain

   \[
   \sum_{k=2}^{\infty} \log((k+2)/(k+1)^2) = \log(2/3) - \lim_{k \to \infty} \log(k/(k+1)) = \log(2/3).
   \]

   d) Since $2 \sin(a - b) \cos(a + b) = \sin(2a) - \sin(2b)$, we have

   \[
   \sum_{k=1}^{\infty} 2 \sin\left(\frac{1}{k} - \frac{1}{k+1}\right) \cos\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{\infty} \left(\frac{\sin \frac{1}{k} - \sin \frac{1}{k+1}}{k} \right) = \sin 2 - 0 = \sin 2.
   \]

6.1.3. a) $\cos(1/k^2) \to \cos 1$. Hence this series diverges by the Divergence Test.
   b) By L'Hôpital's Rule, $(1 - 1/k)^k \to e^{-1}$. Hence this series diverges by the Divergence Test.
   c) $s_n := \sum_{k=1}^{n} (k+1)/k^2 \geq t_n := \sum_{k=1}^{n} 1/k$. Since $t_n \to \infty$, it follows from the Squeeze Theorem that $s_n \to \infty$ as $n \to \infty$. Therefore, the original series diverges.

6.1.4. Since $s_{k+1} - 2a_k + a_{k+1} = (a_{k+1} - a_k) + (a_{k+1} - a_k)$, this series is the sum of two telescopic series. Hence

   \[
   \sum_{k=1}^{\infty} (a_{k+1} - 2a_k + a_{k-1}) = \sum_{k=1}^{\infty} (a_{k+1} - a_k) + \sum_{k=1}^{\infty} (a_{k+1} - a_k) = L - a_1 + a_0 - L = a_0 - a_1.
   \]

6.1.5. By telescoping,

   \[
   \sum_{k=1}^{\infty} (x^{2k} - x^{2(k+1)}) = (1 - \lim_{k \to \infty} x^{2k}) = \begin{cases} 
   -1 & |x| < 1 \\
   0 & |x| = 1 \\
   \text{diverges} & |x| > 1.
   \end{cases}
   \]

6.1.6. a) Let $s_n := \sum_{k=1}^{n} a_k$. If $\sum_{k=1}^{\infty} a_k$ converges then $s_n \to s$ for some $s \in \mathbb{R}$. By Theorem 2.8, convergent sequences are bounded. Therefore, \{s_n\} is bounded.
   b) The partial sums of $\sum_{k=1}^{\infty} (-1)^k$ assume only the values $-1, 0$, hence are bounded. But the series itself diverges by the Divergence Test.

6.1.7. a) Let $x, y \in I$. By the Mean Value Theorem,

   \[
   F(x) - F(y) = F'(c)(x - y) = \left(1 - \frac{f'(c)}{f'(a)}\right)(x - y).
   \]
Thus by hypothesis, $|F(x) - F(y)| \leq r|x - y|$.

b) By a) and induction, $|x_{n+1} - x_n| = |F(x_n) - F(x_{n-1})| \leq r|x_n - x_{n-1}| \leq r^n|x_1 - x_0|.

c) Since $x_0 \in I$ and $F(I) \subseteq I$, all $x_n$'s belong to $I$. Thus by b) and Geometric series, if $m = n + k$ then

$$|x_m - x_n| \leq \left(\frac{r^n + r^{n+1} + \cdots + r^{n+k-1}}{1 - r}\right)|x_1 - x_0| \leq \frac{r^n + r^{n+1} + \cdots + r^{n+k-1}}{1 - r}|x_1 - x_0|.$$ 

Since $r^n \to 0$ as $n \to \infty$, we see that $x_n$ is Cauchy, hence converges to some $b \in I$, since $I$ is closed. Taking the limit of $x_{n+1} = x_n - f(x_n)/f'(a)$, we obtain $b = f(b)/f'(a)$. We conclude that $f(b) = 0$.

6.1.8. a) Since the $a_k$'s are decreasing, $ka_k = a_2 + \cdots + a_k \leq a_1 + a_2 + \cdots + a_k = \sum_{j=k+1}^{2k} a_j$. Let $\varepsilon > 0$ and choose $N$ so large that $\sum_{j=k+1}^{2k} a_j < \varepsilon$ for $k \geq N$. Then $|ka_k| < \varepsilon$ for $k \geq N$, i.e., $2ka_k \to 0$ as $k \to \infty$. On the other hand, since $0 \leq 2(2k + 1)a_{2k+1} \leq 2ka_{2k} + a_{2k} \to 0$ as $k \to \infty$, it follows from the Squeeze Theorem that $2ka_{2k+1} \to 0$ as $k \to \infty$.

b) Clearly, $s_{2n+2} = s_{2n+1} - 1/(2n+2) > s_{2n}$ and $s_{2n+4} = s_{2n-1} - 1/(2n+2) < s_{2n-1}$. Also, $0 \leq s_{2n+1} - s_{2n} = 1/(2n+1) \to 0$ as $n \to \infty$. Hence by the Squeeze Theorem, $s_{2n+1} - s_{2n} \to 0$ as $n \to \infty$.

c) By part b), $\{s_{2n+1}\}$ is increasing and bounded above by $s_1 = 1$, $\{s_{2n+1}\}$ is decreasing and bounded below by $s_0 = 1/2$, and $s_{2n+3} - s_{2n} \to 0$ as $n \to \infty$. Hence both these sequences converge to the same value, i.e., the series $\sum_{n=1}^{\infty} (-1)^{k+1}/k$ converges. However, $k \cdot (-1)^{k+1}/k = (-1)^{k+1}$ does not converge to 0 as $k \to \infty$.

6.1.9. a) $|\sum_{k=1}^{n} b_k - nb| = |\sum_{k=1}^{n} (b_k - b) - \sum_{k=1}^{N} (b_k - b) + \sum_{k=N+1}^{n} (b_k - b)| \leq \sum_{k=1}^{N} |b_k - b| + \sum_{k=N+1}^{n} |b_k - b| \leq \sum_{k=1}^{N} |b_k - b| + M(n - N).

b) Set $B_n = (b_1 + \cdots + b_n)/n$. Let $\varepsilon > 0$ and choose $N$ so large that $|b_k - b| < \varepsilon$ for $k \geq N$. By part a), $|B_n - b| \leq (N|b_k - b| + \varepsilon(n - N))/n$. Since $N$ is fixed, $\sum_{k=1}^{N} |b_k - b|/n \to 0$ and $(n - N)/n \to 1$ as $n \to \infty$. Consequently, $\limsup_{n \to \infty} |B_n - b| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\limsup_{n \to \infty} |B_n - b| = 0$. Therefore, $B_n - b$ converges to 0 as $n \to \infty$.

c) If $b_k = (-1)^k$ then $B_n = -1/n$ if $n$ is odd and 0 if $n$ is even, so $B_n \to 0$ as $n \to \infty$. However, $B_n$ does not converge as $k \to \infty$.

6.1.10. a) $s_n = \sum_{k=0}^{n-1} (1/k)n/a_k = \sum_{k=0}^{n-1} (1/k)(n-k)/n = (na_0 + (n-1)a_1 + \cdots + a_{n-1})/n = (a_0 + (a_0 + a_1) + \cdots + (a_0 + \cdots + a_{n-1}))/n = (s_1 + \cdots + s_n)/n$.

b) If $\sum_{k=0}^{\infty} a_k = L$ then $s_n \to L$ as $n \to \infty$. Hence by part a) and Exercise 6.1.9b, $s_n \to L$ as $n \to \infty$, i.e., $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to $L$.

c) Since $s_n = \sum_{k=0}^{n-1} (-1)^k$ is 1 when $n$ is odd and 0 when $n$ is even, the corresponding averages are given by

$$\sigma_n = \begin{cases} (n+1)/(2n) & \text{when } n \text{ is odd} \\ 1/2 & \text{when } n \text{ is even}. \end{cases}$$

Therefore, $\sigma_n \to 1/2$ as $n \to \infty$ although $\sum_{k=0}^{\infty} (-1)^k$ diverges.

d) Suppose $\sum_{k=0}^{\infty} a_k$ diverges. Since $a_k \geq 0$, it follows that $s_n \to \infty$ as $n \to \infty$. Hence given $M > 0$, choose $N$ so large that $s_N \geq M$ for $n \geq N$. Then $s_n \geq \sum_{k=N}^{n-1} s_k/n \geq (n - N)/n$. Since $(n - N)/n \to 1$ as $n \to \infty$, it follows that $s_n > M/2$ for $n$ large, i.e., $\sigma_n \to \infty$ as $n \to \infty$.

6.1.11. Let $\varepsilon > 0$ and choose $N$ so large that $\sum_{k=N+1}^{\infty} |a_k|/k < \varepsilon/2$. Since $N$ is fixed, $\sum_{k=1}^{N} a_k/(j+k) \to 0$ as $j \to \infty$. Hence we can choose $J$ so large that $|a_k/(j+k)| < \varepsilon/2$. Consequently, if $j > J$ then (since $k + j > k$)

$$\sum_{k=1}^{\infty} a_k/(j+k) \leq |\sum_{k=1}^{N} a_k/(j+k)| + \sum_{k=N+1}^{\infty} |a_k|/k < \varepsilon.$$

6.1.12. Fix $n \geq 2$. By hypothesis,

$$a_n = \frac{n+1}{n(n+1)(n+2)} = \frac{1}{2} \left( \frac{1}{n(n+1)} - \frac{1}{n(n+1)(n+2)} \right).$$

By telescoping, we have
\[ \sum_{k=2}^{\infty} a_k = \frac{1}{2} \left( \frac{1}{6} - 0 \right) = \frac{1}{12}. \]

Since \( a_1 = 2/3 \), we conclude that
\[ \sum_{k=1}^{\infty} a_k = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}. \]

6.2 Series with nonnegative terms.

6.2.0. a) False. If \( a_k = 1/k^2 \) and \( b_k = 1/k \), then \( a_k/b_k \to 0 \) as \( k \to \infty \) and \( \sum_{k=1}^{\infty} a_k \) converges, but \( \sum_{k=1}^{\infty} b_k \) does not.

b) True. By hypothesis, \( 0 \leq a_k \leq k^k \). Since \( 0 < \alpha < 1 \), the Geometric series \( \sum_{k=1}^{\infty} a_k \) converges, thus by the Comparison Theorem, \( \sum_{k=1}^{\infty} a_k \) converges.

c) True. By hypothesis, \( a_k \leq \alpha^k \) for all \( k \in \mathbb{N} \). Choose \( N \in \mathbb{N} \) such that \(|a_k| \leq 1/2 \) for \( k \geq N \). Then \( a_{N+1} \leq a_N^2 \leq 1/4, a_{N+2} \leq a_{N+1}^2 \leq 1/16, \) and in general, \( a_{N+k} \leq 1/4^k \) for \( k = 1, 2, \ldots \). Since the Geometric series \( \sum_{k=1}^{\infty} 1/4^k \) converges, it follows from the Comparison Theorem that \( \sum_{k=N+1}^{\infty} a_k \) converges.

d) False. Let \( f(k) = 1/2^k \) and \( f^{k+1}(x) dx \geq 1/k \). (Such a function can be constructed by making \( f \) piecewise linear on each \([k, k+1] \), its graph forming a triangle whose peak occurs at the midpoint of \([k, k+1]\) with height \( 2/k \).) Then \( \sum_{k=1}^{\infty} f(k) = 1 \) converges but \( \int_{1}^{\infty} f(x) dx = \infty \).

6.2.1. a) It converges by the Limit Comparison Test, since
\[ \frac{(2k+5)(3k^2 + 2k - 1)}{1/k^2} \to \frac{2}{3} \neq 0 \] as \( k \to \infty \).

b) It converges by the Comparison Test and the Geometric Series Test, since \( 0 \leq (k-1)/(k2^k) \leq 1/2^k \).

c) Since \( p > 1 \), choose \( c > 0 \) such that \( p - \alpha > 1 \). But \( |\log x| \leq Cx^\alpha \), so \( \log k/k^p \leq C/k^{p-\alpha} \). Hence the series converges by the Comparison Test and the \( p \)-Series test.

d) Since \( \log k < \sqrt{k} \) for \( k \) large, \( k^3 \log^2 k/e^k < k^4/e^k \) for \( k \) large. But by six applications of l'Hôpital's Rule,
\[ \lim_{k \to \infty} \frac{k^4/e^k}{1/k^2} = \lim_{k \to \infty} \frac{k^6}{e^k} = \lim_{k \to \infty} \frac{6!}{e^k} = 0. \]

But \( \sum_{k=1}^{\infty} e^{-k} \) is a geometric series which converges, so by Theorem 6.16ii, \( \sum_{k=1}^{\infty} k^4/e^k \) converges. Thus the original series converges by the Comparison Test.

e) It converges by the Limit Comparison Test, since
\[ \frac{(\sqrt{k} + \pi)/(2 + k^{3/5})}{1/k^{11/16}} \to 1 \neq 0 \] as \( k \to \infty \).

f) \( k \geq 3 \) implies \( \log k \geq \log 3 > \log e = 1 \), so \( \log k \geq p := \log 3 \). Thus \( k^{\log k} \geq k^p \) for \( k \geq 3 \), and it follows from the Comparison Test that \( \sum_{k=1}^{\infty} k^{\log k} \) converges.

6.2.2. a) It diverges by the Limit Comparison Test since
\[ \frac{(3k^3 + k - 4)/(5k^4 - k^2 + 1)}{1/k} \to \frac{3}{5} \neq 0 \] as \( k \to \infty \).

b) Since \( (\sqrt{k}/k) \geq (1/k) \), this series diverges by the Comparison Test.

c) \((k+1)/k \geq 1\) so the terms of this series are all \( \geq 1 \). Thus the original series diverges by the Divergence Test.

d) Let \( f(x) = (x(\log x)^p)^{-1} \) for \( x > 0 \). Since
\[ f'(x) = -(x \log^p x)^{-2}(p \log^{p-1} x + \log^p x) \leq 0 \]

for \( x > 1 \), \( f(x) \) is decreasing for \( x > 1 \). Since

\[
\int_1^\infty \frac{dx}{x \log^p x} = \int_1^\infty \frac{du}{u^p} = \infty
\]

for \( p \leq 1 \), this series diverges by the Integral Test.

6.2.3. Let \( M \geq a_k \) and note that \( 1/(k+1)^p \leq 1/k^p \) for all \( k \in \mathbb{N} \). Thus the series \( a_k/(k+1)^p \) has nonnegative terms and is dominated by \( M/k^p \). It follows from the Comparison Test and the \( p \)-Series Test that this series converges for all \( p > 1 \).

6.2.4. Since \( \log^p(k+1) \geq \log^p k \), we have

\[
\sum_{k=2}^{\infty} \frac{1}{k \log^p(k+1)} \geq \sum_{k=2}^{\infty} \frac{1}{k \log^p k}.
\]

But by the Integral Test, this last series converges when \( p > 1 \). Hence by the Comparison Test, the original series converges when \( p > 1 \). Similarly,

\[
\sum_{k=2}^{\infty} \frac{1}{k \log^p(k+1)} = \sum_{k=2}^{\infty} \frac{1}{(k+1) \log^p(k+1)} \leq \sum_{k=3}^{\infty} \frac{1}{k \log^p k}
\]

diverges when \( p \geq 1 \).

6.2.5. When \( p \leq 0 \) use the Comparison Test, since in this case, \( k^p \geq 1 \) for all \( k \in \mathbb{N} \), so the series is dominated by \( \sum_{k=1}^{\infty} |a_k| \). When \( p < 0 \), the result is false, since \( a_k = 1/k^{1-p} \) generates a convergent series by the \( p \)-Series Test \((1-p) \) is GREATER than \( 1 \) in this case), but \(|a_k|/k^p = 1/k \) which generates the harmonic series, which diverges.

6.2.6. a) If \( a_n/b_n \to 0 \) then \( a_n \leq b_n \) for \( n \) large. If \( \sum_{k=1}^{\infty} b_k \) converges, then it follows from the Comparison Test that \( \sum_{k=1}^{\infty} a_k \) converges.

b) If \( a_n/b_n \to \infty \) then \( a_n \geq b_n \) for \( n \) large. If \( \sum_{k=1}^{\infty} b_k \) diverges, then it follows from the Comparison Test that \( \sum_{k=1}^{\infty} a_k \) diverges.

6.2.7. Since \( b_n \to 0 \), it surely is bounded. Thus \( a_k b_k \) is nonnegative and dominated by \( M a_k \). Hence the product converges by the Comparison Test. Notice, we really only need that one of the series is bounded and the other convergent.

6.2.8. Notice that \( ak+b \neq 0 \) for \( k \in \mathbb{N} \), since otherwise, \( b/a = -k \in \mathbb{Z} \). Also notice that \((1/k^q)\) is \([-1/(ak+b)]\) is bounded. Since \( a_k+b \) and \( a \) are both positive or both negative for large \( k \), the terms \( \frac{1}{(ak+b)} \) are eventually all positive or all negative. It follows from the Limit Comparison Test that we need only consider \( \sum_{k=1}^{\infty} (k q)^{-1} \).

If \( 0 < q \leq 1 \) then \( 1/q \geq 1 \) so \( \sum_{k=1}^{\infty} (k q)^{-1} \geq \sum_{k=1}^{\infty} 1/k = \infty \) diverges. If \( q > 1 \) then the geometric series \( \sum_{k=1}^{\infty} (k q)^{-1} \geq \sum_{k=1}^{\infty} 1/q < \infty \). Thus the original series diverges when \( 0 < q \leq 1 \) and converges when \( q > 1 \).

6.2.9. If \( s_n := \sum_{k=1}^{n} a_k \) converges then so does \( s_{2n+1} \). Thus

\[
\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) = \lim_{n \to \infty} (a_2 + a_3) + \ldots + (a_{2n} + a_{2n+1}) = \lim_{n \to \infty} s_{2n+1}
\]

converges. Conversely, if \( L := \sum_{k=1}^{\infty} (a_{2k} + a_{2k+1}) \) converges then

\[
\sum_{k=1}^{2n+1} a_k = \sum_{k=1}^{2n} (a_{2k} + a_{2k+1}) \to L \quad \text{and} \quad \sum_{k=2}^{2n+1} a_k = -a_{2n+1} + \sum_{k=2}^{2n+1} a_k \to L
\]

as \( n \to \infty \). Therefore, \( \sum_{k=1}^{\infty} a_k = a_1 + L \) converges.

6.2.10. If \( p \leq 0 \), then the series diverges by the Divergence Test. If \( p > 0 \), then \( \log(\log(\log k)) > 2/p \) for large \( k \) implies that \( p \log(\log(\log k)) > 2 \) for large \( k \). It follows that

\[
\frac{1}{\log(\log k)^p \log k} = \frac{1}{\epsilon^p \log k \log(\log(\log k))} < \frac{1}{\epsilon^2 \log k} = \frac{1}{k^2}.
\]

Thus the original series converges by the Comparison Test.
6.3 Absolute Convergence.

6.3.0. a) True. Since

\[ \lim_{k \to \infty} \sup |a_k|^{1/k} = \lim_{k \to \infty} |a_k|^\alpha = a_0 \]

by Remark 6.2.1ii and \( a_0 < 1 \), it follows from the Root Test that \( \sum_{k=1}^{\infty} a_k^\alpha \) is absolutely convergent.

b) False. If \( a_k = 1/k^2 \), then \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent, but \( |a_k|^{1/k} \to 1 \) as \( k \to \infty \).

c) False. If \( a_k = -1/k \) and \( b_k = 1/k^2 \), then \( a_k \leq b_k \) for all \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} b_k \) converges absolutely, but \( \sum_{k=1}^{\infty} a_k \) diverges.

d) True. If \( \sum_{k=1}^{\infty} a_k \) converges absolutely, then \( |a_k| \leq 1 \) for large \( k \). But \( |a_k| \leq 1 \) implies \( |a_k|^\alpha \leq |a_k| \). Hence, \( |a_k|^\alpha \leq |a_k| \) for large \( k \), and it follows from the Comparison Theorem that \( \sum_{k=1}^{\infty} a_k^\alpha \) converges absolutely.

6.3.1. a) Since \( [1/(k+1)]/[1/k] = 1/(k+1) \to 0 \) as \( k \to \infty \), this series converges by the Ratio Test.

b) Since \( \sqrt[1]{1/k^k} = 1/k \to 0 \) as \( k \to \infty \), this series converges by the Root Test.

c) Since \( (e^{k+1}/(k+1))/(e^k/k!) = \pi/(k+1) \to 0 \) as \( k \to \infty \), this series converges by the Ratio Test.

d) Since by L'Hôpital's Rule \( \sqrt{(k/(k+1))^e} = (k/(k+1))^k \to e^{-1} \) as \( k \to \infty \), this series converges by the Root Test.

6.3.2. a) The Ratio Test gives 1, but the series converges by the Comparison Test since \( k^e \) implies \( \log k > 5 \) so

\[ \frac{k^3}{(k+1)e^k} < \frac{k^3}{(k+1)^2} \to \frac{1}{k^2} \]

b) It converges by the Ratio Test, since

\[ \frac{(k+1)^{100}/e^{k+1}}{k^{100}/e^k} = \frac{((k+1)/k)^{100}}{e} \to \frac{1}{e} \]

as \( k \to \infty \).

c) It converges by the Root Test, since

\[ \sqrt[k]{a_k} \equiv \frac{k+1}{2k+3} \to \frac{1}{2} < 1. \]

d) It converges by the Ratio Test, since

\[ \frac{|a_{k+1}|}{|a_k|} = \frac{2k+1}{(2k+1)(2k+2)} \to 0 \]

as \( k \to \infty \).

e) It converges by the Root Test, since

\[ |a_k|^{1/k} = \frac{(k-1)!}{k!} + 1 < \frac{(k-1)!}{k!} = \frac{1}{k} \to 0 \]

as \( k \to \infty \).

f) It converges by the Root Test, since

\[ \sqrt[k]{a_k} \equiv \frac{3 + (-1)^k}{5} \]

has a limit supremum of \( 4/5 \).

g) It diverges by the Root Test, since

\[ \sqrt[k]{a_k} \equiv \frac{3 - (-1)^k}{\pi} \]

has a limit supremum of \( 4/\pi \).

6.3.3. a) By the Integral Test (see Exercise 6.2.2d) it converges for all \( p > 1 \) and diverges for \( 0 < p \leq 1 \). It also diverges for \( p \leq 0 \) by the Divergence Test. Therefore, this series converges if and only if \( p > 1 \).

b) It diverges for all \( p > 0 \) since \( \log k \leq Ck^{1/p} \) implies \( 1/\log^p k \geq 1/k \) for \( k \geq 2 \). If \( p \leq 0 \), then the series diverges by the Divergence Test.

c) If \( p = 0 \), the series obviously doesn’t make sense, so we can suppose that \( p \neq 0 \). We shall use the Ratio Test. Since
\[
\frac{a_{k+1}}{a_k} = \frac{1}{|p|} \left( \frac{k+1}{k} \right)^p - \frac{1}{|p|}
\]
as \( k \to \infty \), \( \sum_{k=1}^{\infty} k^p/p^k \) converges absolutely when \( |p| > 1 \) and diverges when \( |p| < 1 \). By inspection, it does not converge absolutely when \( |p| = 1 \). Therefore, the series converges absolutely if and only if \( |p| > 1 \).

d) Since
\[
\frac{1}{\sqrt[k]{k}(k^p - 1)} = \frac{k^p}{k^{p/2}} \to 1
\]
as \( k \to \infty \), it follows from the Limit Comparison Test and the p-Series Test that this series converges if and only if \( p + 1/2 > 1 \), i.e., \( p > 1/2 \).

e) Rationalizing the numerator, the terms of this series look like \( 1/\sqrt[k]{k^p + 1} + k^p \). By the Limit Comparison Test, \( \sum_{k=1}^{\infty} 1/\sqrt[k]{k^p + 1} + k^p \) converges if and only if \( \sum_{k=1}^{\infty} 1/k^p \) converges, i.e., if and only if \( p > 1 \). Since \( 2k^p \leq \sqrt[k]{k^p + 1} + k^p \leq 2\sqrt[k]{k^p + 1} \), it follows from the Limit Comparison Test that the original series converges if and only if \( p > 1 \).

f) We shall use the Ratio Test. Since
\[
\frac{a_{k+1}}{a_k} = 2^p \left( \frac{k+1}{k} \right)^k - \frac{2^p}{e}
\]
as \( k \to \infty \), by L’Hopital’s Rule, \( \sum_{k=1}^{\infty} 2^p k!/k^k \) converges absolutely when \( 2^p < e \), i.e., when \( p < \log_2(e) \), and diverges when \( p > \log_2(e) \). When \( p = \log_2(e) \), we compare the series with \( \sqrt{k} \). Indeed, by Stirling’s Formula,
\[
\frac{e^{\frac{k}{k}}}{\sqrt{k}} = \sqrt{2\pi}
\]
as \( k \to \infty \). Therefore, the original series diverges when \( p = \log_2(e) \).

6.3.4. Notice that \( \frac{1}{a_{k+2}} = \sqrt[k]{k} \mid x \mid \to a \mid x \mid \) as \( k \to \infty \). Hence if \( a \neq 0 \), then it follows from the Root Test that this series converges absolutely when \( a|x| < 1 \), i.e., \( |x| < 1/a \). If \( a = 0 \), then the limit is zero for all \( x \), so by the Root Test the series converges absolutely for all \( x \in \mathbb{R} \).

6.3.5. Notice that all \((-1)^ka_k \)'s are nonnegative. Hence \( \sum_{k=1}^{\infty} a_k \) converges absolutely by the Ratio Test, since
\[
\frac{a_{k+1}}{a_k} = \left( 1 + (k+1) \sin \left( \frac{1}{k+1} \right) \right)^{-1} - \frac{1}{2}
\]
as \( k \to \infty \) by L’Hôpital’s Rule.

6.3.6. a) Since \( a_{kj} \geq 0 \), \( 0 \leq \sum_{j=1}^{N} a_{kj} \leq \sum_{j=1}^{\infty} a_{kj} = A_k \) for all \( N \in \mathbb{N} \). Hence by the Comparison Test,
\[
\sum_{j=1}^{N} \sum_{k=1}^{\infty} a_{kj} = \sum_{k=1}^{\infty} \sum_{j=1}^{N} a_{kj} \leq \sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}.
\]
Taking the limit of this inequality as \( N \to \infty \) we obtain the desired inequality.

b) By part a), \( \sum_{k=1}^{\infty} a_{kj} \) converges. Hence by reversing the roles of \( k \) and \( j \), we obtain the reverse inequality.

c) By inspection, \( \sum_{j=1}^{\infty} a_{kj} = 0 \) for all \( k \in \mathbb{N} \) but \( \sum_{k=1}^{\infty} a_{kj} = 1 \) if \( j = 1 \) and 0 if \( j > 1 \). Therefore,
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = 0 \neq 1 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{kj}.
\]

6.3.7. a) Since \( a_k \to 0 \) as \( k \to \infty \), \( |a_k| < 1 \) for \( k \) large. Hence \( |a_k|^p \leq |a_k| \) for \( k \) large and it follows from the Comparison Test that \( \sum_{k=1}^{\infty} |a_k|^p \) converges.

b) If \( \sum_{k=1}^{\infty} k^p a_k \) converges for some \( p > 1 \), then \( k^p a_k \to 0 \) as \( k \to \infty \), i.e., \( k^p |a_k| < 1 \) for large \( k \). Thus \( |a_k| \leq 1/k^p \) for large \( k \). Since \( p > 1 \), it follows from the Comparison Test that \( \sum_{k=1}^{\infty} |a_k| \) converges, a contradiction.
6.3.8. a) The middle inequality is obvious since the infimum of a set is always less than or equal to its supremum.

To prove the right-most inequality, suppose that \( r = \lim \sup_{k \to \infty} a_{k+1}/a_k \). We may suppose that \( r \neq \infty \). For any \( r_0 > r \), by Remark 6.22i, there is an \( N \in \mathbb{N} \) such that \( k \geq N \) implies \( a_{k+1}/a_k \leq r_0 \). Fix \( j \in \mathbb{N} \). It follows that
\[
a_{N+j} \leq a_{N+j-1}r_0 \leq a_{N+j-2}r_0^2 \leq \cdots \leq a_Nr_0^j,
\]
i.e., \( a_k \leq a_Nr_0^{k-N} \) for all \( k \geq N \). In particular, if \( n > N \), then
\[
\sup_{k > n} \sqrt[n]{a_k} \leq (a_Nr_0^{-N})^{1/k} \cdot r_0.
\]
Taking the limit of this last inequality as \( k \to \infty \), we see that \( \lim \sup_{k \to \infty} \sqrt[n]{a_k} \leq r_0 \). Finally, letting \( r_0 \downarrow r \), we conclude that \( \lim \sup_{k \to \infty} \sqrt[n]{a_k} \leq r \), as required.

To prove the left-most inequality, repeat the steps above, using part a) in place of Remark 6.22i, but with infimum in place of supremum and \( r_1 < r \) in place of \( r_0 > r \), proves part c).

d) If \( |b_{k+1}/b_k| \to r \) as \( k \to \infty \), then by Remark 6.22iii and part b), \( \lim \sup_{k \to \infty} |b_{k+1}/b_k| = r = \lim \inf_{k \to \infty} |b_{k+1}/b_k| \).

We conclude from part c) that \( \lim \sup_{k \to \infty} |b_k| = \lim \inf_{k \to \infty} |b_k| = r \). But if we translate this back into \( \varepsilon, \delta \) language, we conclude that \( |b_k| \to r \) as \( k \to \infty \).

6.3.9. By hypothesis,
\[
\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \cdot \frac{\pi^2}{24}.
\]
Therefore,
\[
\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.
\]

6.3.10. Since each \( a_k^+ \) and \( -a_k^- \) (where \( a_k \neq 0 \)) is either \( a_k \) or 0, it suffices to show there are integers \( 0 < k_1 < k_2 < k_3 < \ldots \) such that \( b_1 = a_1^+ \), \( b_2 = a_2^+ \), \ldots, \( b_{k_1} = a_{k_1}^+ \), \( b_{k_1+1} = -a_{k_1}^- \), \ldots, \( b_{k_2} = -a_{k_2}^- \), \( b_{k_2+1} = a_{k_2+1}^+ \), \ldots, and \( s_n = \sum_{j=1}^{n} b_j \), then \( \lim_{n \to \infty} s_n = x \) and \( \sup_{n \to \infty} s_n = y \). We suppose for simplicity that \( x \) and \( y \) are both finite.

Since \( \sum_{k=1}^{\infty} a_k^+ = \infty \), choose an integer \( k_1 \in \mathbb{N} \) least such that
\[
s_{k_1} := b_1 + b_2 + \cdots + b_{k_1} := a_1^+ + a_2^+ + \cdots + a_{k_1}^+ > y.
\]

Since \( b_1 \) is least, \( s_{k_1-1} \leq y \), hence \( s_{k_1} \leq y + b_1 \). Similarly, since \( \sum_{k=1}^{\infty} a_k^- = \infty \) we can choose an integer \( k_1 \) least such that
\[
s_{k_2} := b_1 + b_2 + \cdots + b_{k_2} := a_1^- - a_2^- - \cdots - a_{k_2}^- < x,
\]
and \( s_{k_2} \leq x + b_1 \). Since the \( -a_k^- \)'s are nonpositive, it is clear that \( s_\ell \leq s_{k_2} \leq y + b_{k_2} \) for \( k_1 < \ell \leq k_2 \). Therefore,
\[
s_{k_2} > y \quad \text{and} \quad x + b_{k_2} \leq s_\ell \leq y + b_{k_2}
\]
for all \( k_1 \leq \ell \leq k_2 \). By a similar argument, if \( k_2 > k_1 \) is least such that \( s_{k_2} > y \), then \( s_{k_1} < x \) and \( x + b_{k_2} \leq s_\ell \leq x + b_{k_2} \) for all \( k_1 \leq \ell \leq k_2 \). In particular,
\[
y < \sup_{k_1 \leq \ell \leq k_2} s_\ell \leq y + \max\{b_{k_1}, b_{k_2}\} \leq y + \sup_{\ell \geq k_1} b_\ell.
\]
In the same way, if \( r_2 > r_3 \) is least such that \( s_{r_2} < x \), then
\[
x + \inf_{\ell \geq r_2} b_\ell \leq \sup_{k_1 \leq \ell \leq k_2} s_\ell < x.
\]

Continuing this process, we generate integers \( k_1 < r_1 < k_2 < r_2 < \ldots \) such that for each \( j \in \mathbb{N} \),
\[
y < \sup_{k_1 \leq \ell \leq k_{j+1}} s_\ell \leq y + \sup_{\ell \geq k_1} b_\ell \quad \text{and} \quad x + \inf_{\ell \geq r_2} b_\ell \leq \inf_{r_2 \leq \ell \leq r_2+1} s_\ell < x.
\]

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The first of these inequalities implies
\[ y < \sup_{x \in A} x \leq y + \sup_{x \in B} x. \]
Taking the limit of this inequality as \( j \to \infty \), bearing in mind that by the Divergence Test \( b_n \to 0 \) as \( n \to \infty \), we conclude that
\[ y \leq \limsup_{n \to \infty} s_n \leq y + \liminf_{n \to \infty} b_n = y. \]
This proves \( s_n \) has limit supremum \( y \). A similar argument establishes that \( s_n \) has limit infimum \( x \).

6.3.11. By Exercise 4.4.4 and the Squeeze Theorem, it suffices to show that \( s_n := \sum_{k=0}^{n} (-1)^k x^{2k+1}/(2k+1)! \) converges as \( n \to \infty \) for all \( x \in \mathbb{R} \). But it does converge by the Ratio Test:
\[
\left| \frac{(-1)^{k+1} x^{2k+3}/(2k+3)!}{(-1)^{k} x^{2k+1}/(2k+1)!} \right| = \frac{|x^2|}{(2k+2)(2k+3)} \to 0
\]
for all \( x \in \mathbb{R} \). A similar argument works for the cosine series.

6.4 Alternating series.

6.4.9. a) True. If \( \sum_{k=1}^{\infty} b_k \) converges, then its partial sums are bounded. Hence apply Dirichlet's Test to \( a_k \downarrow 0 \) and \( \sum_{k=1}^{\infty} b_k \).

b) False. Let \( a_k = (-1)^k / k \). Then \( \sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} 1/k \), which diverges.

c) False. Let \( a_k = 1/k \) if \( k \) is odd and \( a_k = 2/k \) if \( k \) is even. Then
\[
a_{2k} - a_{2k+1} = \frac{2}{2k} - \frac{1}{2k+1} = \frac{k+1}{2k^2 + k}.
\]
The series associated with this last fraction diverges by the Limit Comparison Test (compare it with \( 1/k \)). Therefore, \( \sum_{k=1}^{\infty} (-1)^k a_k \) diverges.

d) False. Let \( a_k = 1/k^2 \) if \( k \) is odd and \( a_k = 2/k^2 \) if \( k \) is even. Since \( 2k+1 < k^2 \) for \( k \geq 3 \) implies \( (k+1)^2 < 2k^2 \), it is easy to check that \( a_k \) is not monotone when \( k > 3 \). On the other hand, \( \sum_{k=1}^{\infty} (-1)^k a_k \) converges absolutely by the Comparison Test since
\[
\sum_{k=1}^{\infty} |a_k| \leq 2 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

6.4.1. a) Clearly, \( 1/k^p \downarrow 0 \) as \( k \to \infty \) for all \( p > 0 \). Therefore, the series converges by the Alternating Series Test.

b) By Example 6.32, \( \sum_{k=1}^{\infty} \sin(kx) \) has bounded partial sums for all \( x \in \mathbb{R} \). Hence the series converges by the Dirichlet Test.

c) Since \( (1 - \cos(1/x))^r = -\sin(1/x)/x^2 < 0 \) for \( x \geq 1 \), \( 1 - \cos(1/k) \) is decreasing. Thus the original series converges by the Alternating Series Test.

d) Since \( (x/3)^r = (3^r - 1 - x \log 3 \cdot 3^r)/3^{2r} = (1 - x \log 3)/3^r < 0 \) for \( x \geq 1, k/3^k \) is decreasing. Thus the original series converges by the Alternating Series Test.

e) Let \( f(x) = \pi/2 - \arctan x \). Since \( f'(x) = -1/(1+x^2) < 0 \) for all \( x \in \mathbb{R}, f(k) \downarrow 0 \) as \( k \to \infty \). Hence this series converges by the Alternating Series Test.

6.4.2. a) By the Ratio Test, this series converges for all \( |x| < 1 \) and diverges for all \( |x| > 1 \). It converges at \( x = 1 \) (the harmonic series) and converges at \( x = -1 \) (an alternating series). Thus it converges if and only if \( x \in [-1,1) \).

b) Since \( x^{2k}/2^k = (x^2/2)^k \), this series is geometric. Hence, it converges if and only if \( |x^2| < 2 \), i.e., if and only if \( x \in (-\sqrt{2}, \sqrt{2}) \).

c) By the Ratio Test, this series converges when \( |x| < 1 \) and diverges when \( |x| > 1 \). When \( x = 1 \) it converges by the Alternating Series Test. When \( x = -1 \) it diverges by the Limit Comparison Test (compare it with \( 1/k \)).

d) The absolute value of the ratio of successive terms of this series is given by
\[
k\sqrt{k+1} |x+2|/((k+1)\sqrt{k+2}).
\]
Thus by the Ratio Test, this series converges when \( |x+2| < 1 \) (i.e., when \(-3 < x < -1 \)) and diverges when \( |x+2| > 1 \). If \( x = -1 \) or \( x = -3 \), this series is \( \sum_{k=1}^{\infty} (1/k)/((k+1)\sqrt{k+1}) \) which converges absolutely by the Limit Comparison Test, since \( \sum_{k=1}^{\infty} k^{-3/2} < \infty \). Therefore, the original series converges if and only if \( x \in [-3,-1] \).
6.4.3. a) Since \( [(k+1)^3/(k+2)!]/[k^3/(k+1)!] = (k+1)^3/(k^3(k+2)) \to 0 \) as \( k \to \infty \), this series converges absolutely by the Ratio Test.

b) Since
\[
\left| \frac{-1(-3) \ldots (1 - 2k)(-1 - 2k)/(1 \cdot 4 \ldots (3k - 2)(3k + 1))}{-1(-3) \ldots (1 - 2k)/(1 \cdot 4 \ldots (3k - 2))} \right| = \frac{-1 - 2k}{3k + 1} = \frac{-1}{3} < 1,
\]
this series converges absolutely by the Ratio Test.

c) Since \( (k+2)^{k+1}/(k^{k+1} + 1)/(1/k) = (k+2)/(k+1)^{k+1} \cdot (1/p) \to 0 \) as \( k \to \infty \) and \( e/p < 1 \), this series converges absolutely by the Ratio Test.

d) Let \( f(x) = \sqrt{x}/(x + 1) \) for \( x > 0 \). Since \( f'(x) = (1-x)/(2\sqrt{x}(x+1)^2) < 0 \) for \( x > 1 \), \( f \) is strictly decreasing on \( (1, \infty) \). Thus \( f(k) \downarrow 0 \) as \( k \to \infty \) and this series converges by the Alternating Series Test. On the other hand, \( (\sqrt{k}/(k+1))/(1/\sqrt{k}) = k/(k+1) \to 1 \) as \( k \to \infty \). Hence it follows from the Limit Comparison Test that \( \sum_{k=1}^{\infty} \sqrt{k}/(k+1) \) diverges. Hence the original series is conditionally convergent.

e) Since \( (\sqrt{k+1}/(k+2))^{1/k} = \sqrt{k+1}/\sqrt{k} \to 1 \) as \( k \to \infty \), this series converges absolutely by the Limit Comparison Test.

6.4.4. If \( b_k \downarrow 0 \) then \( b_k - b \downarrow 0 \) as \( k \to \infty \). Moreover, if \( \sum_{k=1}^{\infty} a_k \) converges, then it surely has bounded partial sums. Hence by Dirichlet’s Test, \( \sum_{k=1}^{\infty} a_k(b_k - b) \) converges, say to \( s \). But \( \sum_{k=1}^{\infty} a_k b_k \) converges, so we can add it to both sides of \( s = \sum_{k=1}^{\infty} a_k(b_k - b) = \sum_{k=1}^{\infty} a_k b_k - \sum_{k=1}^{\infty} a_k b \). We obtain
\[
\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} a_k b + s.
\]

6.4.5. By Abel’s Formula, \( \sum_{k=1}^{n} a_k b_k = b_n s_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k-1}) \). Take the limit of this identity as \( n \to \infty \), bearing in mind that \( s_n \) is bounded and \( b_n \to 0 \) as \( n \to \infty \). We obtain \( \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} a_k b_k \).

6.4.6. By Abel’s Formula, \( \sum_{k=1}^{n} a_k b_k = B_n,m a_n - \sum_{k=m}^{n-1} B_k,m (a_{k+1} - a_k) \) where \( B_n,m := \sum_{k=m}^{n} b_k \). By hypothesis, \( |B_{n,m}| \leq 2M \). Hence
\[
\left| \sum_{k=m}^{n} a_k b_k \right| \leq 2M|a_n| + 2M \sum_{k=m}^{n-1} |a_{k+1} - a_k|.
\]
Since \( a_n \to 0 \) and \( \sum_{k=1}^{\infty} |a_{k+1} - a_k| < \infty \), it follows that \( \sum_{k=1}^{\infty} a_k b_k \) is Cauchy, hence convergent.

6.4.7. Let \( c_n = \sum_{k=1}^{n} a_k b_k \) for \( n \in \mathbb{N} \). Given \( \varepsilon > 0 \) choose \( N \) so large that \( b_k > 0 \) and \( |c_k| < \varepsilon/2 \) for \( k \geq N \). By Abel’s Formula,
\[
\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} a_k b_k / b_k = \sum_{k=m}^{n} c_k / b_k = c_m / b_m + \sum_{k=m}^{n-1} (c_k - c_{k+1}) \left( \frac{1}{b_{k+1}} - \frac{1}{b_k} \right).
\]
Now \( 1/b_k \to 0 \) as \( k \to \infty \) so
\[
\sum_{k=m}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=m}^{n-1} (c_k - c_{k+1}) \left( \frac{1}{b_{k+1}} - \frac{1}{b_k} \right)
\]
and this limit must exist. Let \( m > N \). Since the \( 1/b_k \)'s are decreasing, we have by telescoping that
\[
\left| \sum_{k=m}^{n} a_k \right| \leq 2 \sup_{k \geq m} |c_k| \frac{1}{b_m} - \frac{1}{b_n} \leq 2 \sup_{k \geq m} |c_k| \left( \frac{1}{b_m} - 0 \right) \leq 2 \sup_{k \geq m} |c_k| \left( \frac{1}{b_m} \right) < \varepsilon.
\]
We conclude that \( |b_m \sum_{k=m}^{\infty} a_k| < \varepsilon \) for \( m \geq N \), i.e., \( b_m \sum_{k=m}^{\infty} a_k \to 0 \) as \( m \to \infty \).

6.4.8. By a sum angle formula and telescoping, we see that \( 2 \sin(x/2) \sum_{k=1}^{n} \cos(kx) = \sum_{k=1}^{n} (\sin((k - 1/2)x) - \sin((k + 1/2)x)) = \sin(x/2) - \sin((n + 1/2)x) \). Thus
\[
\left| \sum_{k=1}^{n} \cos(kx) \right| \leq 1/|\sin(x/2)| < \infty
\]

for each fixed \( x \in (0,2\pi) \). Hence by Dirichlet’s Test, \( \sum a_k \cos(kx) \) converges for each \( x \in (0,2\pi) \). When \( x = 0 \), the series converges if and only if \( \sum_{k=1}^{\infty} a_k \) converges.

\[ \left| \sum_{k=1}^{n} \sin(2k+1)x \right| \leq 2/|\sin x| < \infty \]

for each fixed \( x \in (0,\pi) \cup (\pi,2\pi) \). Hence by Dirichlet’s Test, \( \sum a_k \sin(2k+1)x \) converges for each \( x \in (0,\pi) \cup (\pi,2\pi) \). Since the series is identically zero when \( x = 0, \pi, 2\pi \), it converges everywhere on \([0,2\pi]\). But \( \sin(2k+1)x \) is periodic of period \( 2\pi \). Hence this series converges everywhere on \( \mathbb{R} \).

6.5 Estimation of series.

6.5.1. a) Let \( f(x) = \pi/2 - \arctan x \). Since \( f'(x) = -1/(1 + x^2) < 0 \) for all \( x \in \mathbb{R} \), \( f(k) \downarrow 0 \) as \( k \to \infty \). Hence this series converges by the Alternating Series Test. Since \( f(100) = 0.000999, n = 100 \) terms will estimate the value to an accuracy of \( 10^{-2} \).

b) Let \( f(x) = x^{2-k} - x^2 e^{-x/2} \). Since \( f'(x) = x^{2-k}(2 - x/2) < 0 \) for all \( x > 2/\log 2 \), \( f(x) \) is strictly decreasing for \( x \) large. Therefore, the series converges by the Alternating Series Test. Since \( f(15) = 0.0008, n = 15 \) terms will estimate the value to an accuracy of \( 10^{-2} \).

c) Let \( a_k = (2 \cdot 4 \cdot \ldots \cdot 2k)/(1 \cdot 3 \cdot \ldots \cdot (2k-1)k^2) \) and observe that

\[ a_{k+1}/a_k = (2k + 2)(2k+1)^2/(2k+1)(2k+2) = 2k^2/(2k+2)(2k+1) < 1. \]

Thus \( a_{k+1} < a_k \). Moreover, \( a_k = (2/3)(4/5)\ldots((2k-2)/(2k-1)) \cdot (2k/k^2) < 2/k \to 0 \) as \( k \to \infty \). Therefore, this series converges by the Alternating Series Test. Since \( a_0 = \approx .0105 \) and \( a_{10} \approx .0055 \), \( n = 10 \) terms will estimate the value to an accuracy of \( 10^{-2} \).

6.5.2. a) \( p > 1 \) (see Exercise 6.2.4).

b) Let \( f(x) = 1/(x \log^p(x+1)) \). By Theorem 6.35,

\[ -\int_n^\infty f(x) \, dx \leq s_n - s \leq f(n) - \int_n^\infty f(x) \, dx, \]

so

\[ |s - s_n| \leq f(n) + \int_n^\infty f(x) \, dx. \]

Since

\[ \int_n^\infty f(x) \, dx = \int_n^\infty \frac{dx}{x \log^p(x+1)} \leq \int_n^\infty \frac{dx}{x \log^p(x)} = \frac{1}{(p-1) \log^{p-1}(n)}, \]

it follows that

\[ |s - s_n| \leq \frac{1}{n \log^p(n+1)} + \frac{1}{(p-1) \log^{p-1}(n)} \leq \frac{n + p - 1}{n(p-1)} \left( \frac{1}{\log^{p-1}(n)} \right). \]

6.5.3. a) Since \( \lfloor 1/(k+1) \rfloor / \lfloor 1/k \rfloor = 1/(k+1) \to 0 \) as \( k \to \infty \), this series converges by the Ratio Test. The ratio is less than or equal to \( 1/3 \) for \( k > N = 1 \). Hence by Remark 6.40, \( |s_n - s| \) is dominated by \( (1/3)^n/(2/3) = (1/2)(1/3)^{n-1} \). For \( n = 7 \), this last ratio is about \( 0.000069 \) still a little too big, but it’s about \( 0.00023 < 0.0005 \) for \( n = 8 \).

b) Since \( \sqrt[4]{k^4} = 1/k \to 0 \) as \( k \to \infty \), this series converges by the Root Test. The root is less than or equal to \( 1/2 \) for \( k \geq N = 2 \). Hence by Remark 6.40, \( |s_n - s| \) is dominated by \( (1/2)^{n+1}/(1/2) = (1/2)^n \) for \( n \geq 2 \). Since \( 1/2^a \approx 0.000098 \) for \( n = 10 \) and \( \approx 0.000049 \) for \( n = 11 \), choose \( n = 11 \).

c) Since \( (2^{k+1}/(k+1))!/(2^k/k)! = 2/(k+1) \to 0 \) as \( k \to \infty \), this series converges by the Ratio Test. The ratio is less than or equal to \( 1/2 \) for \( k > N = 2 \). Hence by Remark 6.40, \( |s_n - s| \) is dominated by \( (2^2/2)!/(1/2)^{n-1}/(1/2) = (1/2)^{n-1} \). Thus by the calculations in part b), choose \( n = 14 \).

d) Since by L’Hôpital’s Rule \( \sqrt[k]{(k+1)!} = (k/(k+1))^k \to e^{-1} \) as \( k \to \infty \), this series converges by the Root Test. The root is less than or equal to \( 1/2 \) for \( k \geq N = 1 \) by Example 4.30. Hence by Theorem 6.40, \( |s_n - s| \) is dominated by \( (1/2)^{n+1}/(1/2) = (1/2)^n \). Thus by the calculations in part b), choose \( n = 11 \).

6.5.4. Fix \( n \geq N \). If \( |a_{k+1}|/|a_k| \leq x \) for \( k > N \), then \( |a_{N+1}| \leq x |a_N|, |a_{N+2}| \leq x^2 |a_N|, \ldots \), hence \( |a_k| \leq |a_N|x^{k-N} \) for any \( k > N \). Hence given \( n \geq N \),

\[
0 \leq s - s_n = \sum_{k=n+1}^{\infty} |a_k| \leq |a_N| \sum_{k=n+1}^{\infty} x^{k-N} = |a_N| \frac{x^{2n-N+1}}{1-x}.
\]

6.6 Additional tests.

6.6.1. a) The ratio of successive terms of this series is

\[
\frac{2k+3}{2k+2} > 1.
\]

Hence \( a_{k+1} \geq a_k > 0 \), so the series diverges by the Divergence Test.

b) The ratio of successive terms of this series is

\[
\frac{2k+1}{2k+5} = 1 - \frac{4}{2k+5} = 1 - \frac{2}{k+5/2}.
\]

Hence it converges absolutely by Raabe’s Test.

c) Let \( u = \log k \) and note that \( u \to \infty \) as \( k \to \infty \). Now for \( k > e \),

\[
\frac{\log(1/|a_k|)}{\log k} = \frac{\log(\log k \log \log k)}{\log k} = \frac{(\log \log k)^2}{\log k} = \frac{\log^2 u}{u}.
\]

But the limit of this last quotient is (by L'Hôpital’s Rule twice)

\[
\lim_{u \to \infty} \frac{2 \log u \cdot (1/u)}{1} = \lim_{u \to \infty} \frac{2 \log u}{u} = 0.
\]

Hence the series diverges by the Divergence Test.

d) Applying L'Hôpital's Rule twice, we obtain

\[
p := \lim_{k \to \infty} \frac{k \log(\sqrt{k}/(\sqrt{k} - 1))}{\log k} = \lim_{k \to \infty} \frac{\log(\sqrt{k}/(\sqrt{k} - 1))}{\log k/k}
\]

\[
= \lim_{k \to \infty} \frac{\sqrt{k} - 1}{\sqrt{k}} \lim_{k \to \infty} \frac{-k^2/2\sqrt{k}}{(\sqrt{k} - 1)^2(1 - \log k)}
\]

\[
= \lim_{k \to \infty} \frac{k}{(\sqrt{k} - 1)^2} \lim_{k \to \infty} \frac{-\sqrt{k}/2}{1 - \log k}
\]

\[
= \lim_{k \to \infty} \frac{-1/(4\sqrt{k})}{-1/k} = \infty.
\]

Hence the series converges absolutely by the Logarithmic Test.

6.6.2. a) It converges absolutely for all \( p > 0 \) by the Ratio Test, since

\[
\frac{(k+1)/e^{(k+1)p}}{k/e^{kp}} = \frac{k+1}{ke^p} \to \frac{1}{e^p} < 1
\]

for all \( p > 0 \). If \( p \leq 0 \), this series diverges by the Divergence Test.

b) Since \( \log((\log k)^{p \log k})/\log k = p \log \log k \to \infty \) if \( p > 0 \), this series converges absolutely for all \( p > 0 \) by the Logarithmic Test. It diverges for \( p \leq 0 \) by the Divergence Test.

c) It converges absolutely for all \( |p| < 1/e \) by the Ratio Test, since

\[
\left| \frac{(p(k+1))^{k}/(k+1)!}{(pk)^k/k!} \right| = \frac{|p(k+1)^{k+1}}{(k+1)^k} = |p| \left( \frac{k+1}{k} \right)^k \to |p| < 1
\]

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for all $|p| < 1/e$. Similarly, if $|p| > 1/e$, this series diverges by the Ratio Test. If $p = 1/e$, then the terms of the series become

$$\frac{k^k}{e^k \cdot k!} > \frac{1}{e \sqrt{k}}$$

by Stirling’s Formula. By the Comparison Test and the $p$-Series Test, the original series diverges. For $p = -1/e$, the series converges conditionally by the Alternating Series Test and what we just proved.

6.6.3. a) By L'Hôpital's Rule, $\sqrt[3]{1/(\log k)^{\log k}} \to e^0 = 1$ as $k \to \infty$ so the Root Test yields $r = 1$. However, the series converges by the Logarithmic Test since $\log((\log k)^{\log k})/\log k = \log \log k \to \infty$ as $k \to \infty$.

b) The ratio of consecutive terms of this series is $(2k + 1)/(2k + 4)$ which converges to 1 as $k \to \infty$. However, since $(2k + 1)/(2k + 4) = 1 - (3/2)/(k + 2)$, the series converges by Raabe's Test.

6.6.4. Since the range of $f$ is positive, $|f(k)| = f(k)$ for all $k \in \mathbb{N}$. Moreover, by L'Hôpital's Rule,

$$\lim_{k \to \infty} \frac{\log(1/f(k))}{\log k} = -\lim_{k \to \infty} \frac{f'(k)/f(k)}{1/k} \equiv -\alpha.$$

By the Logarithmic Test, if $-\alpha > 1$, then this series converges absolutely. Hence it surely converges.

6.6.5. If $p > 1$ is infinite, let $q = 2$. If $p > 1$ is finite, let $q = \sqrt{p}$. Note that in either case, $q > 1$. By hypothesis, $k(1 - |a_{k+1}/a_k|) > q$ for $k$ large. (Indeed, in either case, $q < p$ so this expression is eventually bigger than $q$.) The inequality implies $|a_{k+1}/a_k| < 1 - q/k$ for $k$ large. Since $q > 1$, it follows from Raabe's Test that $\sum_{k=1}^{\infty} a_k$ converges absolutely.
CHAPTER 7

7.1 Uniform Convergence of Sequences.

7.1.1. a) Given $\varepsilon > 0$ choose $N$ so large that $N > \max\{|a|, |b|\}/\varepsilon$. Then $n \geq N$ and $x \in [a, b]$ imply

$$\frac{|x/n|}{x^3 + nx^{36}} < \frac{1}{n} \leq \max\{|a|, |b|\}/N < \varepsilon.$$ 

Hence $x/n \to 0$ uniformly on $[a, b]$.

b) Given $x \in (0, 1)$, $nx \to 0$, hence $1/(nx) \to 0$ as $n \to \infty$. If $\{1/(nx)\}$ were uniformly convergent, then there is an $N \in \mathbb{N}$ such that $|1/(N^2)| \leq 1$ for all $x \in (0, 1)$. Applying this inequality to $x = 1/(2N)$ we obtain $2 = 1/(N \cdot (1/(2N)) \leq 1$, a contradiction.

7.1.2. a) Since $(3^{36} + 3)/N \to 0$ as $N \to \infty$, given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so that $0 < (3^{36} + 3)/N < \varepsilon$. Since $x \in [1, 3]$ implies $|3 - x^3| \leq 3 + 3^{36}$ and $x^3 + nx^{36} \geq 0 + n = n$, it follows that

$$\left| \frac{nx^{36} + 3}{x^3 + nx^{36}} - x^3 \right| \leq \frac{3 + 3^{36}}{n} \leq \frac{3 + 3^{36}}{N} < \varepsilon$$

for all $x \in [1, 3]$ and $n \geq N$. Hence $(nx^{36} + 3)/(x^3 + nx^{36}) \to x^3$ uniformly on $[1, 3]$, so by Theorem 7.10,

$$\lim_{n \to \infty} \int_1^3 \frac{nx^{36} + 3}{x^3 + nx^{36}} \, dx = \int_1^3 x^3 \, dx = \frac{3^4 - 1}{3^4}.$$ 

b) Since $e > 1$ implies $e^{4/n} > 1$ and $e^{4/n} \to 1$ as $N \to \infty$, given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so that

$$0 < e^{4/n} - 1 < \varepsilon.$$ 

Since $x \in [0, 2]$ implies $e^{x/n} \leq e^{x}$, it follows that

$$|e^{x/n} - 1| = e^{x/n} - 1 \leq e^{x/n} - 1 \leq e^{4/n} - 1 < \varepsilon$$

for all $x \in [0, 1]$ and $n \geq N$. Hence $e^{x/n} \to 1$ uniformly on $[0, 2]$, so by Theorem 7.10,

$$\lim_{n \to \infty} \int_0^2 e^{x/n} \, dx = \int_0^2 \, dx = 2.$$ 

c) Let $x \in [0, 3]$. Since $\sin(x/n) > 0$ for $n \geq 3$, we have $f(x) := \sin(x/n) + x + 1 + \sqrt{x + 1} > \sqrt{1 + 1} = 2$ for $n \geq 3$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $N \geq 3$ and $2/N < \varepsilon$. Since $0 < \sin(x/n) \leq x/n$, it follows by rationalizing the numerator that

$$\left| \sin(x/n) + x + 1 - \sqrt{x + 1} \right| = \frac{\sin(x/n) + x + 1 - (x + 1)}{\sin(x/n) + x + 1 + \sqrt{x + 1}}$$

$$\leq \frac{x/n}{2} \leq \frac{3}{2n} < \varepsilon$$

for all $x \in [0, 3]$ and $n \geq N$. Hence $\sin(x/n) + x + 1 \to \sqrt{x + 1}$ uniformly on $[0, 3]$, so by Theorem 7.10,

$$\lim_{n \to \infty} \int_0^3 \sin(x/n) + x + 1 \, dx = \int_0^3 \sqrt{x + 1} \, dx = \frac{2}{3} (x + 1)^{3/2} \bigg|_0^3 = \frac{14}{3}.$$ 

7.1.3. Choose $N$ so large that $|f(x) - f_n(x)| < 1$ for all $x \in E$ and $n \geq N$. Set $M := \sup_{x \in E} |f_n(x)|$ and observe by the Triangle Inequality that $|f(x) - f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M$ for all $x \in E$. Therefore, $|f_n(x)| \leq |f(x)| + 1 \leq (1 + M) + 1 = 2 + M$ for all $n \geq N$ and $x \in E$, i.e., $\{f_n\}_{n \geq N}$ is uniformly bounded on $E$. In particular,

$$|f_n(x)| \leq M := \max(2 + M, \sup_{x \in [a, b]} |f_1(x)|, \ldots, \sup_{x \in [a, b]} |f_{N-1}(x)|) < \infty$$

for all $n \in \mathbb{N}$ and $x \in E$.

7.1.4. Since $g$ is continuous on $[a, b]$, it is bounded by the Extreme Value Theorem, i.e., there is a $C > 0$ such that $|g(x)| \leq C$ for all $x \in [a, b]$. Since $f$ is bounded and $\{f_n\}$ is uniformly bounded, there is an $M > 0$ such that $\max\{|f_n(x) - f(x)| : x \in [a, b], n \in \mathbb{N}\} \leq M$. Given $\varepsilon > 0$ choose $\delta > 0$ so small that $a < x < a + \delta$ or

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\[ b \geq x > b - \delta \text{ implies } |g(x)| < \varepsilon / M. \] By hypothesis, \( f_n \to f \) uniformly on \([a + \delta, b - \delta]\). Thus choose \( N \) so large that \( x \in [a + \delta, b - \delta] \) and \( n \geq N \) imply \( |f_n(x) - f(x)| < \varepsilon / C \). If \( n \geq N \) and \( x \in [a, b] \), then

\[
|f_n(x)g(x) - f(x)g(x)| = |f_n(x) - f(x)||g(x)| < \begin{cases} 
(\varepsilon / C) : C = \frac{\varepsilon}{\varepsilon / M} = \varepsilon & x \in [a + \delta, b - \delta] \\
\varepsilon / M & x \notin [a + \delta, b - \delta].
\end{cases}
\]

Therefore, \( f_n g \to fg \) uniformly on \([a, b] \).

7.1.5. a) Given \( \varepsilon > 0 \) choose \( N \) so large that \( n \geq N \) and \( x \in E \implies |f_n(x) - f(x)| < \varepsilon / \max(2, |a| + 1) \) and \( |g_n(x) - g(x)| < \varepsilon / \max(2, |a| + 1) \). Then \( n \geq N \) and \( x \in E \) imply

\[
|f(x + g(x)) - (f_n + g_n)(x)| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

and

\[
|(\alpha f)(x) - (\alpha f_n)(x)| = |\alpha||f(x) - f_n(x)| < |\alpha| \frac{\varepsilon}{|a| + 1} < \varepsilon.
\]

b) See Theorem 2.12.

c) Given \( \varepsilon > 0 \) choose \( M > 0 \) so that \( \sup\{|f(x)|, |g(x)| : x \in E\} \leq M \). Choose \( N_1 \) so large that \( n \geq N_1 \) and \( x \in E \implies |f_n(x) - f(x)| < \varepsilon / (2M) \) and \( |g_n(x) - g(x)| < \varepsilon / (2M) \). Since \( g_n \to g \) and \( g \) is bounded by \( M \), choose \( N_2 \) so large that \( |g_n(x)| < 2M \) for all \( n \geq N_2 \) and \( x \in E \). If \( n \geq N := \max\{N_1, N_2\} \) and \( x \in E \) then

\[
|f(x + g(x)) - (f_n + g_n)(x)| \leq |f(x) - f_n(x)||g_n(x)| + |f(x)||g(x) - g_n(x)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

d) Let \( f_n(x) = 1/n \) and \( g_n(x) = 1/x \). Then \( f_n \to 0 \) uniformly on \( \mathbb{R} \) and \( g_n(x) \to 1/\varepsilon \) uniformly on \((0, \infty)\), in particular, on \((0, 1)\), as \( n \to \infty \). However, by Exercise 7.1.1b, \( f_n(x)g_n(x) = 1/(nx) \) does not converges uniformly on \((0, 1)\).

7.1.6. Given \( \varepsilon > 0 \) choose \( \delta > 0 \) so small that \( x, y \in E \) and \( |x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon / 3 \). If \( x, y \in E \) and \( |x - y| < \delta \), then

\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.
\]

Hence \( f \) is uniformly continuous on \( E \).

7.1.7. Let \( \varepsilon > 0 \) and choose \( \delta \) such that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \varepsilon \). Let \( x \in \mathbb{R} \) and choose \( N \) such that \( n \geq N \) implies \( |\alpha| < \delta \). If \( n \geq N \), then \( |x + y_n - x| = |y_n| < \delta \), so \( |f_n(x) - f(x)| = |f(x + y_n) - f(x)| < \varepsilon \).

7.1.8. Choose \( N \) so large that \([a, b] \subset [-N, N] \). Let \( x \in [a, b] \) and \( n \geq N \). Then \( x \geq -N \geq -n, x/n \geq 1 \), and it follows from Bernoulli’s Inequality that \((1 + x/n)^n \geq e^x \) for \( n \geq N \).

Let \( n > N, x > 0 \), and set \( f(x) = e^x - (1 + x/n)^n \). Then

\[
f'(x) = e^x - \left(1 + \frac{x}{n}\right)^{n-1} \geq e^x - \left(1 + \frac{x}{n}\right)^n > 0
\]

since \( 1 + x/n > 1 \). Thus \( f \) takes its maximum on \([a, b] \) at \( x = b \). Therefore,

\[
|e^x - \left(1 + \frac{x}{n}\right)^n| \leq e^b - \left(1 + \frac{b}{n}\right)^n \to 0
\]

as \( n \to \infty \). It follows that \((1 + x/n)^n \to e^x \) uniformly on \([a, b] \). In particular,

\[
\lim_{n \to \infty} \int_a^b \left(1 + \frac{x}{n}\right)^n e^{-x} \, dx = \int_a^b dx = b - a.
\]

7.1.9. a) By the Extreme Value Theorem, \( f \) is bounded on \([a, b] \) and there are positive numbers \( \varepsilon_0 \) and \( M \) such that \( \varepsilon_0 < |g(x)| < M \) for all \( x \in [a, b] \). Hence \( 1/M < 1/|g(x)| < 1/\varepsilon_0 \) for all \( x \in [a, b] \) and it follows that \( 1/g \) is bounded on \([a, b] \) and \( 1/(2M) < 1/|g_n(x)| < \varepsilon_0 \) for large \( n \) and all \( x \in [a, b] \), i.e., \( 1/g_n \) is defined and bounded on \([a, b] \). Hence by Exercise 7.1.5c, \( f_n/g_n = f_n \cdot (1/g_n) \to f \cdot (1/g) = f/g \) uniformly on \([a, b] \) as \( n \to \infty \).
b) Let \( f_n(x) = 1/n \) and \( g_n(x) = x \). Then \( f_n \to 0 \) uniformly on \( \mathbb{R} \), \( |g_n| > 0 \) for \( x \neq 0 \), and \( g_n(x) \to x \) uniformly on \((0, \infty)\), in particular, on \((0, 1)\), as \( n \to \infty \). However, by Exercise 7.1.1b, \( f_n(x)/g_n(x) = 1/(nx) \) does not converge uniformly on \((0, 1)\).

7.1.10. Given \( \epsilon > 0 \) choose \( N_0 \) so large that \( k \geq N_0 \) and \( x \in E \) imply \( |f_n(x) - f(x)| < \epsilon/2 \). Since \( \sum_{k=1}^{N_0} |f_k(x) - f(x)| \) is bounded on \( E \), choose \( N \) such that \((1/n) \sum_{k=1}^{N_0} |f_k(x) - f(x)| < \epsilon/2 \) for all \( n \geq N \) and \( x \in E \). If \( x \in E \) and \( n \geq \max\{N_0, N\} \) then

\[
\left| \frac{1}{n} \sum_{k=1}^{N} f_k(x) - f(x) \right| \leq \frac{1}{n} \sum_{k=1}^{N_0} |f_k(x) - f(x)| + \frac{\epsilon}{2} \left( 1 - \frac{N_0}{n} \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

7.1.11. Since \( f \) is integrable, there is an \( M > 0 \) such that \( |f(x)| \leq M \) for all \( x \in [0, 1] \). Choose \( n_0 \in \mathbb{N} \) so that \( 1 - b_{n_0} \leq \epsilon/(2M) \) and \( N > n_0 \) so large that \( |f_n(x) - f(x)| < \epsilon/2 \) for \( n \geq N \) and \( x \in [0, 1] \). Suppose \( n \geq N \) and \( x \in [0, 1] \). Since the \( b_n \)'s are increasing, \( b_n \leq 1 \) for all \( n \in \mathbb{N} \) and \( n \geq n_0 \) imply that \( 1 - b_n \leq 1 - b_{n_0} \). Therefore,

\[
\left| \int_{0}^{1} f(x) \, dx - \int_{0}^{b_n} f_n(x) \, dx \right| \leq \int_{0}^{b_n} |f(x) - f_n(x)| \, dx + \int_{b_n}^{1} |f(x)| \, dx \\
\leq \frac{\epsilon}{2} b_n + M(1-b_n) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

7.2 Uniform Convergence of Series.

7.2.1. a) Since \( |\sin(x/k)| \leq |x|/k^2 \leq \max\{|a|, |b|\}/k^2 \) for any \( x \in [a, b] \), this series converges uniformly on \([a, b]\) by the Weierstrass M-Test.

b) Let \( I = [a, \infty) \subset (0, \infty) \). Then \( x \in I \) implies \( e^{-ka} \leq e^{-kr} \). Since this last series converges (it’s Geometric with \( r = e^{-a} < 1 \), the original series converges uniformly on \([a, b]\) by the Weierstrass M-Test.

7.2.2. Clearly, \( |x|^r \leq x^k \) for \( x \in [a, b] \) and \( r = \max\{|a|, |b|\} \). Since \([a, b] \subset (-1, 1) \) implies \( r < 1 \) and the geometric series \( \sum_{k=0}^{\infty} x^k \) converges, it follows from the Weierstrass M-Test that the original series converges uniformly on \([a, b]\).

7.2.3. a) Since \( |x|^{k+1}/(k+1)!! \leq |x|^k/k! \) as \( k \to \infty \), this series converges pointwise on \( \mathbb{R} \) by the Ratio Test. Moreover, since \( x \in [a, b] \) implies \( |x|^k/k! \leq e^c/k! \), where \( c := \max\{|a|, |b|\} \), it follows from the Weierstrass M-Test that the original series converges uniformly on \([a, b]\).

b) Integrating term by term, we have

\[
\int_{a}^{b} E(x) \, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} \bigg|_{a}^{b} = b - a = E(b) - E(a).
\]

c) Clearly, \( E(0) = 1 \). Differentiating term by term, we obtain \( E'(x) = \sum_{k=0}^{\infty} x^k/k! = E(x) \). Thus \( y = E(x) \) solves the initial value problem \( y' - y = 0 \), \( y(0) = 1 \).

7.2.4. The series converges uniformly on \( \mathbb{R} \) by the Weierstrass M-Test. Hence integrating term by term, we obtain

\[
\int_{0}^{\pi/2} f(x) \, dx = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{0}^{\pi/2} \cos(kx) \, dx = \sum_{k=1}^{\infty} \frac{1}{k^3} \sin \left( \frac{k \pi}{2} \right)
\]
 Since \( \sin(k \pi/2) = -1 \) when \( k = 3, 7, \ldots \), \( \sin(k \pi/2) = 1 \) when \( k = 1, 5, \ldots \), and \( \sin(k \pi/2) = 0 \) when \( k = 2, 4, \ldots \), it follows that

\[
\int_{0}^{\pi/2} f(x) \, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k - 1)^3} + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(2k + 1)^3}.
\]

7.2.5. Since \( |\sin(x/(k+1))/k| \leq |x|/(k(k+1)) \), the series converges uniformly on any closed bounded interval \([a, b]\) by the Weierstrass M-Test. In fact, for any \( x \in \mathbb{R} \),

\[
|f(x)| \leq \sum_{k=1}^{\infty} \left| \frac{\sin(x/(k+1))}{k} \right| \leq \sum_{k=1}^{\infty} \frac{|x|}{k(k+1)} = |x| \sum_{h=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = |x|.
\]