

RESEARCH TOPICS

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1. INTRODUCTION

The project in §5 is suggested jointly with Serkan Hosten.

1.A. General description. The proposed research problems concern the aspects of lattice point configurations that are relevant to combinatorial commutative algebra, toric geometry and discrete geometry. Many important special cases of these problems are amenable to algorithmic/computer aided research. They do not require a sophisticated background. Moreover, a nontrivial experimental finding may well serve as a basis for a publication in a decent journal. Simultaneously, by working on these problems, a student gets an opportunity to learn related advanced techniques for performing a focused research on a deeper conceptual level.

The project proposed in §6 is different in nature: it invites a student to master a basic techniques in the direction of advanced topics in algebraic geometry that will be used in a research activity at a later stage.

All projects proposed below offer the possibility of working on various subprojects by several students simultaneously, without essential overlaps.

1.B. Point configurations in algebraic combinatorics. For a subset $X \subset \mathbb{R}^d$ its convex hull in \mathbb{R}^d ($d \in \mathbb{N}$) is denoted by $\text{conv}(X)$. All considered polytopes $P \subset \mathbb{R}^d$ are assumed to be *convex*, and a ‘ d -polytope’ means a ‘ d -dimensional polytope’.

A *lattice polytope* is a polytope whose vertices belong to the standard integral lattice \mathbb{Z}^d . A lattice d -simplex of the smallest possible Euclidean volume (equal to $1/d!$) is called *unimodular*. All polytopes below are assumed to be lattice polytopes. For a polytope $P \subset \mathbb{Z}^d$ the set of its lattice points $P \cap \mathbb{Z}^d$ will be denoted by $L(P)$. Subsets of the form $L(P) \subset \mathbb{Z}^d$, $P \subset \mathbb{R}^d$ a polytope, will be called *convex point configurations*.

A *cone* $C \subset \mathbb{R}^d$ is a subset of \mathbb{R}^d for which $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \in \mathbb{R}_+$. A cone is called *unimodular* if it is spanned by a part of a basis of \mathbb{Z}^d . All considered cones are assumed to be *polyhedral*, *rational* and *pointed*. For such a cone C the set $C \cap \mathbb{Z}^d$ has the finite set of additively indecomposable elements which is called the *Hilbert basis* of the cone C . It is denoted by $\text{Hilb}(C)$. Thus a cone $C \subset \mathbb{Z}^d$ is unimodular if and only if $\#\text{Hilb}(C) = \dim C$.

To a field \mathbf{k} and a polytope P (a cone C) one associates *projective (affine) toric varieties* – certain objects on a crossroad of algebraic and discrete convex geometries. Historically, the initial motivation for studying convex point configurations and closely associated objects came from algebro-geometric, homological, and K -theoretical investigation of toric varieties. But in this text we confine ourselves only to the discrete geometry aspect of the theory.

2. CARATHÉODORY RANK AND UNIMODULAR COVERING

A fundamental problem is a characterization of point configurations of the form $\text{Hilb}(C) \subset \mathbb{Z}^d$ for d -cones $C \subset \mathbb{R}^d$.

Two distinguished conjectures in this direction since the 1980s [3, 5, 6, 10] have been:

Unimodular Hilbert Cover (UHC). A cone $C \subset \mathbb{R}^d$ is the union of the unimodular cones in \mathbb{R}^d that are spanned by elements of $\text{Hilb}(C)$.

Integral Carathéodory Property (ICP). An arbitrary lattice point of a cone $C \subset \mathbb{R}^d$ is a non-negative integral linear combination of at most d elements of $\text{Hilb}(C)$.

(The terminology is explained by the observation that (ICP) is a discrete analogue of Carathéodory's corresponding classical result on cones [14, Ch.1].)

For a cone $C \subset \mathbb{R}^d$ its *Carathéodory rank*, denoted by $\text{CR}(C)$, is the smallest natural number k such that every element $x \in C \cap \mathbb{Z}^d$ is a non-negative integral linear combination of k elements of $\text{Hilb}(C)$. Thus (ICP) asserts that $\text{CR}(C) = \dim C$ for a cone C . Sebö has shown [10] that $\text{CR}(C) \leq 2 \dim C - 2$ for arbitrary cone C .

Contrary to the existing expectations, both conjectures (UHC) and (ICP) have been disproved in [1, 2] by a single six-dimensional cone C_6 , found by an extensive computer search that lasted for several months.

Surprisingly, after several years of extensive computer search, using powerful computers, only one more counterexample to (UHC) and (ICP) have been found. Moreover, if one starts by a random choice of initial cones and proceeds by the *shrinking process* (the terminology is introduced in §3), one is usually led to the same counterexample!

It is clear that (UHC) implies (ICP). However, it is still not known whether (ICP) is in general weaker than (UHC). There are further challenges surrounding (UHC) and (ICP). Namely, a counterexample is found in dimension 6 while positive results end in dimension 3. Already the following 3-dimensional problem is a real challenge and even a partial result would be interesting.

Problem 2.1. *Prove or disprove that cones of dimension 4 have unimodular Hilbert cover.*

3. SHRINKING AND TIGHT CONFIGURATIONS

A cone $C \subset \mathbb{R}^d$ is called *tight* if for every element $x \in \text{Hilb}(C)$ the set $\text{Hilb}(C) \setminus \{x\}$ is not the Hilbert basis of a cone in \mathbb{R}^d , or equivalently: $\text{Hilb}(C) \setminus \{x\} \not\subseteq \text{Hilb}(\mathbb{R}_+(\text{Hilb}(C) \setminus \{x\}))$, where $\mathbb{R}_+(\text{Hilb}(C) \setminus \{x\}) \subset \mathbb{R}^d$ is the cone spanned by $\text{Hilb}(C) \setminus \{x\}$.

The tight cones are the smallest among possible candidates for a counterexample to (UHC) and (ICP). This observation suggests the following strategy for finding such counterexamples.

One starts with a randomly generated cone C and then inspects whether there exists an element $x \in \text{Hilb}(C)$ such that $\text{Hilb}(C) \setminus \{x\}$ is the Hilbert basis of the cone $\mathbb{R}_+(\text{Hilb}(C) \setminus \{x\})$. (Currently, there exist many implemented algorithms for computing Hilbert bases/lattice points in convex regions: `4ti2`, `Latte`, `Normaliz`.) It is clear that the equality

$$\text{Hilb}(C) \setminus \{x\} = \text{Hilb}(\mathbb{R}_+(\text{Hilb}(C) \setminus \{x\}))$$

can only hold for *extremal* elements $x \in \text{Hilb}(C)$, i. e. those spanning the edges of the cone. If such x exists, pick it randomly and repeat the process for the cone $C_1 = \mathbb{R}_+(\text{Hilb}(C) \setminus \{x\})$. This process is called *shrinking*. Continue the shrinking process until it halts. There are two possible outcomes: the last term of the produced sequence of cones C, C_1, C_2, \dots, C_k is either (i) a tight cone, or (ii) $C_k = 0$. Clearly, $\dim C_k \leq \dim C$.

Of a particular interest are *homogeneous cones* – the cones C with the property that $\text{Hilb}(C)$ lies on an affine hyperplane. In this situation the whole story can be reformulated in terms of polytopes. A polytope $P \subset \mathbb{R}^d$ is called *normal* if $L(P)$ is affinely equivalent to $\text{Hilb}(C)$ for some cone C of dimension $\dim P + 1$.

In pure algebraic terms, P is normal iff for any natural number t and any lattice point $x \in tP$ there exist lattice points $x_1, \dots, x_t \in P$ (not necessarily distinct) such that $x_1 + \dots + x_t = x$.

Call a normal polytope *tight* if the cone $C(P)$ is tight. One can perform the shrinking process starting with a randomly chosen normal polytope. This is a recipe of finding examples of tight polytopes.

By the same token, one is naturally led to the partially ordered set SNP_d whose elements are normal polytopes of dimension $\leq d$, with $P < Q$ if and only if P can be obtained from Q by a finitely many shrinking steps. The study of SNP_d for $d \geq 2$ is of fundamental importance in geometry of numbers and may well have applications beyond mathematics.

Even for $d = 2$ the poset SNP_2 deserves a better understanding. Contrary to higher dimensions, this poset is completely describable in terms of elementary planar geometry – a consequence of *Pick's theorem* from the 19th century, characterizing empty lattice triangles in \mathbb{R}^2 .

Problem 3.1. *Study SNP_2 from the perspective of partially ordered sets.*

4. HDRH vs. RH?

Call a point $(a, b) \in \mathbb{Z}^2$ in the plane *primitive* if $\gcd(a, b) = 1$. A pre-WWI observation of Littlewood is that the most famous open problem in all of mathematics – *the Riemann Hypothesis* – admits a translation into a fairly elementary statement on the primitive vectors in the rectangular triangles

$$\text{conv}((0, 0), (0, n), (n, n)) \subset \mathbb{R}^2, \quad n \in \mathbb{N}.$$

There is highly developed theory of these systems of primitive vectors – the theory of *Farey series* [4, 8, 13]. In this theory one almost proves the RH, but this ‘almost’ has become the stumbling stone for generations of mathematicians.

Nobody so far has paid attention to the fact Littlewood’s elementary formulation of the HR admits a natural higher dimensional analogue in terms of the primitive vectors in the lattice simplices:

$$\Delta_n^{(d)} = \text{conv}((0, 0, \dots, 0), (n, 0, \dots, 0), (n, n, \dots, 0), (n, n, \dots, n)) \subset \mathbb{R}^d, \quad n \in \mathbb{N}.$$

Problem 4.1. ¹ *Formulate a Higher Dimensional Riemann Hypothesis, based on the approach of primitive directions $\Delta_n^{(d)}$; perform computer experiments to test whether a reasonable part of the theory of Farey series extends to higher dimensions; make the first steps in extending the theory of Farey series to higher dimensions.*

This project may be linked to the area of computer vision (sic!) – the details to be discussed upon request.

A masters thesis defended at SFSU on 2003 presents very lucid account of Littlewood’s observation – it can serve as an introduction to the subject.

5. SMOOTH POINT CONFIGURATIONS

This is a project suggested jointly with Serkan Hosten, and the students involved will work with both faculty.

A convex point configurations $L(P)$, P a polytope, is called *smooth* if the cone at every vertex of P is unimodular, i.e., the primitive vectors on the edges at every vertex form a basis of \mathbb{Z}^d .

These polytopes have attracted attention because of a homological property, called ‘quadratic generation’, they conjecturally have. There have been faulty attempts to prove this conjecture but it remains widely open [11]. Again, despite the original homological formulation, the conjecture admits a pure elementary geometric translation.

In the recently launched joint project Joseph Gubeladze and Serkan Hosten [7] devise a reasonably fast implemented algorithm for constructing many ‘random’ normal smooth point configurations and then testing the quadratic generation conjecture. The project was driven by an inkling that there should be a counterexample. Another motivation was to detect families of smooth polytopes, since currently there is a lack of explicitly described big classes of such polytopes. We invite students

¹Anyone who signs up for this project has to comply with the condition that if, all of sudden, a proof of the RH emerges as a result of the project then 47% of the \$ 1 000 000 prize, established by the Clay Mathematical Institute, will go to the supervisor, aka J. G.

with strong coding skills and the desire to study an exciting field, where one easily runs into outstanding open mathematical problems, to further develop our software and refine our insight into smooth point configurations.²

A closely related project is as follows. The quadratic generation condition for a convex point configuration $L(P)$ can be formulated in terms of the existence of lattice walks in P of special type. There is another kind of lattice walk in polytopes which remotely resembles the lattice walk related to the quadratic generation condition.

An oriented broken line with lattice vertices within a lattice polytope $P \subset \mathbb{R}^d$ is called a *Hilbert walk* if its steps belong to $\bigcup_{v \in \text{vert}(P)} \text{Hilb}(C_v)$, where $\text{Hilb}(C_v)$ refers to the Hilbert basis of the corner cone spanned by P at vertex v ³. The convex point configuration $L(P)$ is called *Hilb-connected* if for any two points $x, y \in L(P)$ there is a Hilbert walk from x to y and backwards.

Problem 5.1. *Are there big explicitly described classes of convex point configurations that are Hilb-connected?*

We conjecture that smooth lattice point configurations are Hilb-connected. But if this conjecture fails on some smooth configuration then a very natural candidate for a counterexample to quadratic generation condition is born!

Another related problem which heads us in the direction of classification of smooth polytopes is the following:

Problem 5.2. *Classify all smooth 3-dimensional polytopes with at most k lattice points? How far can k be pushed?*

6. PROJECTIVE MODULES AND VECTOR BUNDLES

This is a project based on two well known pieces of fine mathematical writing [9, 12].

²That the existing software really needs further refinement is exhibited by the fact that when Serkan and Joseph were running the cluster computers in TH919 the room was heating up and a lot of noise was made, forcing to halt the computations. A new upgraded software should at least produce less heat and noise!

³According to our terminology on cones, one first needs to consider the parallel translate of C_v by $-v$ so that the apex coincides with the origin

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