Math 890: Algebra, Geometry, Combinatorics
San Francisco State University
Spring 2005

RESEARCH TOPICS

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Abstract. This is an overview of five research directions, corelated by the fact that they deal with various combinatorial, algebraic, and arithmetic aspects of the same objects – *convex polytopes* and their discrete analogs. Each of this directions is itself a sort of network of inter-related problems, techniques and concepts, organized around a central motivating goal/conjecture – potentially a basis for a long term focused research. But we also highlight very concrete working problems. Despite their pure mathematical nature, these problems are very open to a computer experimentation. Even a substantial computational evidence supporting this or another conjecture can well serve as a basis for a publishable work.

1. Hom-polytopes

The basic notion we deal with in all these projects is that of a *(convex) polytope*. This includes such formations as flat polygons, tetrahedra, pyramids, prisms, Platonic solids etc. These objects fascinated philosophers and artists alike since ancient times.

Formally, a *polytope* (we only consider convex polytopes) is a subset of an Euclidean space $\mathbb{R}^n$ that can be represented as a bounded intersection of finitely many closed half-spaces. Equivalently, a polytope is the *convex hull* of a finite set of points in $\mathbb{R}^n$. Thus a polytope is a certain piece of the ambient real vector space. It is natural to ask whether the standard linear algebra concepts admit similar 'piecewise' analogs for polytopes. The importance of *the category of polytopes* is hinted at in the concluding pages of the well known modern treatise on polytopes [Z]. Great deal of work has been done in this direction, but at present we do not even know what a natural polyhedral version of some of very basic linear algebra concepts should be.

Linear algebra over the real numbers $\mathbb{R}$ studies *the category of real vector spaces* – all possible vector spaces and all linear maps between them. Here one encounters such constructions as *the hom-space* (the vector space of all linear maps between two vector spaces), *tensor product*, *kernel*, *cokernel* etc.

The category of polytopes consists of all polytopes and all posible *affine maps* between them. Here an *affine map* is a map that preserves *barycentric coordinates*. Equivalently, an affine map between two polytopes is a map that can be obtained by a linear map between the ambient Euclidean space, composed with a parallel translation. It is easily shown that the set of all affine maps between two polytopes

2000 *Mathematics Subject Classification.* 11P21, 13P10, 14M25, 52B11, 52B20.

is itself a polytope in a natural way – *the hom-polytope*. Already here can we pose the following challenging problem:

Problem 1.1. Characterize (or determine the f-vector/the combinatorial type of) the hom-polytope for special classes of polytopes: cubes, platonic solids, permutahedra etc.

Problem 1.2. Write an implemented algorithm for computing the hom-polytopes.

2. Quotients of polytopes

The real story, however, starts when we try to extend the linear algebra concept of a *quotient space* to the category of polytopes. The polytopal analogue of the dual object – *the kernel of a linear map* – is provided by *fibered polytope*, a remarkable concept invented by Billera and Sturmfels [BiSt]. The fibered polytopes play a crucial role in the theory of *polyhedral subdivisions* and *triangulations*, a cornerstone of the contemporary computational geometry. Roughly, the fiber polytope, associated to an affine map between two polytopes, is the 'average' of the preimage polytopes over the points in the target polytope.

However, nobody knows what the right definition of the *cokernel polytope* should be. Recall, *the cokernel of a linear map* is the quotient of the target space by the image of the map. Cokernel has certain *universal property*, which is also its defining property.

Quite recently, in the totally different context of a geometric approach to the notion of conditional independence in statistics, Mond, Smith and van Straten introduced *the space of sandwiched triangles* between two polygons[1] [MSmvS]. No longer is this space a polytope. Rather, it is a *compact real algebraic variety* in a high-dimensional Euclidean space. In other words, it is the solution set to a system of real multivariate non-strict polynomial inequalities which, in addition, is bounded.

I conjecture that this approach is the key to the hypothetical theory of quotients of polytopes. The notion of the space of sandwiched triangles easily extends to higher dimensions, admits *relativization*[2], is amenable to an algorithmic study and, most importantly, resembles the aforementioned universal property of a quotient vector space. Let us call the Mond-Smith-van Straten construction the *polytopal pre-quotient*. (We add 'pre' because the notion of the actual quotient still remains elusive and needs to be crystallized.)

The work [MSmvS] also shows how *Morse theory* – a high point of the 20th century mathematics, also called *global analysis* – can be put in good use to study pre-quotient polyhedra.

In the talk I will briefly discuss the construction and explain the relevance of the Morse theory.

Problem 2.1. Characterize polytopal pre-quotients for simple polytopes: cubes, simplices etc.

[1] i. e. 2-dimensional polytopes.

[2] i. e. instead of a pair of polytopes $P \subset Q$, one considers the more general situation of an affine maps $P \to Q$.

Problem 2.2. The work [MSmvS] lists several open questions on the maximal number of connected components for polytopal pre-quotients in dimension 2. Try two answer these questions by some ingenious constructions in planar geometry.

Problem 2.3. Write an implemented algorithm for estimating the number of connected components of polytopal pre-quotients. This can be done by first writing up the defining polynomial inequalities and then studying the sign behavior of the values of these inequalities on certain sampling finite set of points.

3. Hom-varieties for polytopal algebras

This direction of research was initiated recently in the series of publications [BrG1, BrG2, BrG3]. Here one studies *the category of polytopal algebras*. It simultaneously contains the category of *lattice polytopes* – discrete versions of polytopes – and the category of vector spaces. This is the next natural step in the direction of enriching polytopes with an algebraic structure. Motivation comes from applications to the geometry of toric varieties and K-theory, not discussed here.

One starts with the following basic constructions.

First we interpret the lattice of integer points $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$ as the set of *monomials* in $n+1$ variables $X_1, \dots, X_n, X_{n+1}$. (The reason why we choose $n+1$ and not n will become clear below.) In particular, we also allow negative powers of variables.

Let $\mathbf{k}$ be a field, say $\mathbb{R}$, or $\mathbb{C}$, or a finite field. The set of all $\mathbf{k}$-linear combinations of such monomials is called the *Laurent polynomial ring* in the variables $X_1, \dots, X_{n+1}$. This is a commutative ring. It contains the usual ring of polynomials $\mathbf{k}[X_1, \dots, X_{n+1}]$ as a subring.

So we identify Laurent monomials with integer (lattice) points by looking at the corresponding *exponent vectors*. Then the product of monomials corresponds to the usual sum of integer points. In particular, the identity element 1 of the Laurent polynomial ring is the same as the origin 0 in $\mathbb{R}^{n+1}$.

Now consider any subset $S \subset \mathbb{Z}^{n+1}$ satisfying the conditions:

(1) $0 \in S$,
(2) if $s_1, s_2 \in S$ then $s_1 + s_2 \in S$.

In other words, S is a *subsemigroup* of $\mathbb{Z}^{n+1}$. A typical example is given by the set of all lattice points in a *cone* $C \subset \mathbb{R}^{n+1}$. Recall, a *cone* in an Euclidean space is the intersection of a finite family of closed halfspaces whose boundary hyperplanes are n-dimensional linear subspaces. (A cone cannot be empty – it always contains $0 \in \mathbb{R}^{n+1}$ and the only bounded cone is $\{0\}$.)

If S is a semigroup as above then we can consider the set of those Laurent polynomials whose *support monomials* belong to S. This will be a subring of the Laurent polynomial ring. For instance, the polynomial ring $\mathbf{k}[X_1, \dots, X_n]$ corresponds to the semigroup of the integer points in $\mathbb{R}^{n+1}$ with nonnegative coordinates. For a semigroup S the resulting ring will be denoted by $\mathbf{k}[S]$ and called *the semigroup ring of S* (over $\mathbf{k}$).

Now the *polytopal rings* are defined as follows. Let P be a *lattice polytope* in $\mathbb{R}^n$, that is all vertices of P belong to $\mathbb{Z}^n$. We can embed P into $\mathbb{R}^{n+1}$ by assigning $(x, 1)$ to every point $x \in P$. That is, we put a copy of P on height 1 in $\mathbb{R}^{n+1}$ above the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ (viewed as the subspace of the first n coordinates). Let $(P, 1)$ denote the shifted polytope and consider the cone $C(P) \subset \mathbb{R}^{n+1}$, spanned over 0 by $(P, 1)$. The polytopal ring $\mathbf{k}[P]$ is by definition the semigroup ring $\mathbf{k}[S(P)]$ of the semigroup $S(P) = C(P) \cap \mathbb{Z}^{n+1}$ of all lattice points in the cone $C(P)$. (The actual definition of a polytopal ring, as given in [BrG1, BrG2, BrG3] is slightly more general, but here we ignore such subtleties.)

After we have defined the polytopal rings (over $\mathbf{k}$) we need to define *homomorphisms* between them. The first observation is that $\mathbf{k}[P]$ is a *graded ring* in a natural way: for a monomial $m \in S(P)$ its last coordinate – the height of m over $\mathbb{R}^n$ – is defined to be *the degree of* m. Let d be a natural number. Then the elements in $\mathbf{k}[P]$ of degree d are exactly the $\mathbf{k}$-linear combinations of the lattice points in $C(P)$, living on height d over $\mathbb{R}^n$.

The homomorphisms between two polytopal algebras $\mathbf{k}[P]$ and $\mathbf{k}[Q]$ are simply $\mathbf{k}$-*algebra homomorphisms* (i. e. ring homomorphisms which are also $\mathbf{k}$-vector space homomorphisms), respecting the graded structures: degree d elements map to degree d elements for all $d \in \mathbb{N}$.

The last condition is equivalent to the condition that degree one monomials in $\mathbf{k}[P]$ map to $\mathbf{k}$-linear combinations of degree one monomials in $\mathbf{k}[Q]$. This remark makes the following two things possible (under a mild condition on how P sits inside $\mathbb{R}^n$ which is generically satisfied, so we do not emphasize it here):

(1) we can identify a homomorphism $f : \mathbf{k}[P] \to \mathbf{k}[Q]$ with a matrix over $\mathbf{k}$ of size $\#\,\mathrm{L}(P) \times \#\,\mathrm{L}(Q)$, where $\mathrm{L}(P)$ is the set of lattice points in P and $\mathrm{L}(Q)$ is the set of lattice point in Q; namely – the row, indexed by a lattice point $x \in P$, is given by the sequence of the coefficients in the linear combination

$$f\big((x, 1)\big) = a_1(y_1, 1) + a_2(y_2, 2) + \ldots$$

where y runs over $\mathrm{L}(Q)$,

(2) we can write an algorithm for finding polynomials in $\#\,\mathrm{L}(P) \times \#\,\mathrm{L}(Q)$ variables whose zero set in the space of matrices $\mathbf{k}^{\#\,\mathrm{L}(P)\times\#\,\mathrm{L}(Q)}$ is exactly the set of $\big(\#\,\mathrm{L}(P) \times \#\,\mathrm{L}(Q)\big)$-matrices that correspond to homomorphisms $f : \mathbf{k}[P] \to \mathbf{k}[Q]$ in the sense of (1). (Not all matrices correspond to homomorphisms!)

In conclusion, we can realize the set of all homomorphisms between two polytopal algebras as the zero set in $\mathbf{k}^{\#\,\mathrm{L}(P)\times\#\,\mathrm{L}(Q)}$ of certain system of polynomials. In other words, we have an *algebraic variety* of homomorphisms – *the hom-variety.*

In [BrG1, BrG2, BrG3] we have described several crucial subvarieties of the hom-varieties – those corresponding to *automorphisms* (invertible homomorphisms) and *codimension 1 idempotent homomorphisms* $f : \mathbf{k}[P] \to \mathbf{k}[P]$ (i. e. $f^2 = f$, $\dim(\operatorname{Im} f) = \dim \mathbf{k}[P] - 1$). We have a very concrete *geometric conjecture* [BrG3] about the nature of arbitrary homomorphisms between arbitrary polytopal algebras. It is phrased in terms of combinatorics of the underlying polytopes. The verification of this conjecture – very interesting result on its own – would shed much light on the structure of the hom-varieties.

Problem 3.1. Characterize further special subvarieties of the hom-varieties (e. g. those corresponding to higher codimension idempotent homomorphisms).

Problem 3.2. Verify the mentioned geometric conjecture for special classes of polytopes.

Problem 3.3. Write an implemented algorithm that finds the defining equations of hom-varieties.

Time permitting, I will explain the results in [BrG2] on codimension 1 idempotent endomorphisms.

4. Counting lattice polytopes by lattice points

The idea here is that the classical theory of counting lattice points in lattice polytopes admits a very interesting counterpart, obtained by swapping the rôles of what is counted and what is given.

More precisely, in the classical theory, beautifully exposed in [BeSi], one is interested in the number of lattice points in various lattice polytopes. For instance, the family of all integral dilations of a given lattice polytope leads to *Ehrhart theory*.

However, here we are interested in the reverse problem: counting lattice polytopes (in a fixed dimension) that contain a given number of lattice points. Of course, for this problem to make sense we have to identify *equivalent polytopes*: two lattice polytopes are considered equivalent if there is a bijective affine mapping between them that restricts to a bijection on the sets of lattice points they contain. It is not difficult to show that the numbers we are interested in are in fact finite. But their actual computation is a big challenge.

Already in dimension 2 this is a nontrivial question. Arnold has initiated this research some 25 years ago [A]. He proved that the number $H(A)$ of lattice polygons in $\mathbb{R}^2$ (up to equivalence) that have area A satisfies the inequalities

$$c_1 A^{1/3} < \log H(A) < c_2 A^{1/3} \log A$$

(Here c_1 and c_2 are some absolute constants.) Arnold's formula has been extended to all dimensions by Barany and Vershik [BaV].

On the other hand there is a very close relationship between the number of lattice points in a lattice polytope and the volume of the polytope. In fact, in dimension 2 we have a precise 19th century formula (*Pick's Theorem*) and in all dimensions the two numbers are asymptotically same for the series of the integral dilations of the polytope.

Problem 4.1. Based on the formulas of Arnold and Barany-Vershik, give explicit estimates for the number of lattice n-dimensional polytopes (up to equivalence) that contain N lattice points.

Problem 4.2. Using Pick's Theorem, derive *sharp* inequalities as in the previous problem when the dimension is 2.

5. Infinitesimal nonlinear extension of GL_n

Most likely, there will be no time left to discuss this research direction. But I would like to make it available to the interested students. Hence this section. In some sense (which can be made more precise), this project is complementary to the one in Section 3.

One starts with a formal power series ring $\mathbf{k}[[X_1, \dots, X_n]]$ over a field $\mathbf{k}$. Unlike polynomial rings, this ring admits infinite sums (but not infinite products!). The main object of our investigation is the group $GA_n(\mathbf{k})$ of automorphisms of $\mathbf{k}[[X_1, \dots, X_n]]$. This group can be thought of as an infinitesimally nonlinear extension of the group $GL_n(\mathbf{k})$ of all invertible $n \times n$-matrices over $\mathbf{k}$. In the work [GM] we show that the *abelianization* – the maximal commutative quotient – of the huge group $GA_n(\mathbf{k})$ coincides with the abelianization of $GL_n(\mathbf{k})$ when the characteristic of $\mathbf{k}$ is not 2 or 3.

Problem 5.1. Prove or disprove the similar claim on the abelianizations when the characteristic of $\mathbf{k}$ is 2 or 3, or when instead of a field the underlying power series has coefficients in the ring of integers $\mathbb{Z}$.

Problem 5.2. Show that *the second integral homology group* of the *commutator subgroups* of $GL_n(\mathbf{k})$ and $GA_n(\mathbf{k})$ are the same, at least for n large. This would be the second level homological analogue of the results of [GM]. The same question can be posed for all higher integral homology groups. The reason we single out the second homology group is that here the problem can be easily expressed in group theoretical language, while for the higher groups the sophisticated machinery of homological algebra and algebraic topology becomes unavoidable.

Problem 5.3. Study the similar questions for truncated polynomial rings. This amounts to relaxing the previous problem by ignoring all higher coefficients, starting from certain degree.

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