The Elementary Action on Unimodular Rows over a Monoid Ring, II

JOSEPH GUBELADZE

A. Razmadze Mathematical Institute of the Academy of Sciences of Georgia, Rukhadze 1, 380093 Tbilisi, Republic of Georgia

Communicated by Wilberd van der Kallen
Received January 9, 1991

In \[G1\] we proved that for an arbitrary submonoid \(M \subset \mathbb{Q}'_+\) (\(r\) is a natural number and \(\mathbb{Q}'_+\) denotes the additive monoid of nonnegative rational numbers) such that \(M \subset \mathbb{Q}'_+\) is an integral extension (i.e., for any \(a \in \mathbb{Q}'_+\), there exists natural \(c\) for which \(ca \in M\)) and for a commutative noetherian ring \(R\) with Krull \(\dim R = d < \infty\) the group of elementary matrices \(E_n(R[M])\) acts transitively on the set of unimodular \(n\)-rows \(Um_n(R[M])\) whenever \(n \geq \max(d+2, 3)\). It should be noted that “classical” results imply the mentioned transitivity only for \(n \geq d + r + 2\) \([B, Sw, V]\). In terms of our \(\Phi\)-correspondence \([G1–G3]\) the integrality condition on \(M \subset \mathbb{Q}'_+\) exactly coincides with the condition on \(\Phi(M)\) to be an \((r-1)\)-dimensional simplex. In order to involve all commutative, cancellative, and torsion free monoids we have to consider the case of arbitrary finite convex polyhedra \(\Phi(M)\). In the present paper we extend the mentioned result on transitivity of the action of \(E_n(R[M])\) on \(Um_n(R[M])\) when \(n \geq \max(d+2, 3)\) (notation as above) for the class of monoids of \(\Phi\)-simplicial growth, i.e., for those monoids \(M\) which admit the sequence of finite convex polyhedra \(P_1 \subset P_2 \subset \cdots \subset P_k = \Phi(M)\) where \(P_1\) and the closures (in Euclidean metric) of \(P_{i+1}\setminus P_i\) are simplices (of arbitrary finite dimensions) for all \(i \in [1, k-1]\).

A word on notation: all the considered rings are assumed to be commutative, \(Um_n(A)\) denotes the set of all unimodular \(n\)-rows over \(A\); for \(a, b \in Um_n(A)\) we will write \(a \sim b\) iff there exists \(e \in E_n(A)\) (the group of elementary \(n \times n\) matrices) such that \(ae = b\); in the case when \(e\) can be chosen from \(E_n(A')\) for some overriding \(A \subset A'\) we will write “\(a \sim b\) over \(A'\).” \(\mathbb{N}\) will denote the natural numbers and, as usual, \(GL_n\) the general linear group.
1. The Monoid Terminology

For the reader's convenience we shall recall the basic definitions and notation from [Gi]. All the considered monoids $M$ are assumed to be commutative, cancellative, and torsion free, i.e., the natural homomorphisms $M \rightarrow K(M) \rightarrow \mathbb{Q} \otimes K(M)$ are injective where $K(M)$ is the fraction group and $\mathbb{Q}$ denotes the rational numbers. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_+$, and $\mathbb{Q}_+$ denote integers, reals, nonnegative integers, and nonnegative rational numbers, respectively (when they are regarded as monoids we mean the additive monoids). $\text{rank}(M)$ denotes $\text{rank}(K(M)) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes K(M)$. For a monoid $M$ we fix an embedding $M \hookrightarrow \mathbb{R}^r$ coming from the composition

$$M \hookrightarrow K(M) \hookrightarrow \mathbb{Q} \otimes K(M) \xrightarrow{\sim} \oplus \mathbb{Q} \rightarrow \oplus \mathbb{R}.$$

$M$ will be identified with its image in $\mathbb{R}^r$. In case $M$ is finitely generated such that $U(M)$ (the group of invertible elements) is trivial the cone $C(M) \subset \mathbb{R}^r$ spanned by $M$ will be spanned by $0 \in \mathbb{R}^r$ and the intersection $H \cap C(M)$ for some hyperplane $H \subset \mathbb{R}^r$ (of dimension $r - 1$) not passing through the origin $O$. In this situation the polyhedron $\Phi(M) = H \cap C(M)$ will be finite and convex. The various choices of $H$ determine the polyhedra $\Phi(M)$ which are equivalent to each other under projective transformations. The notions and properties we are going to deal with endure these projective transformations and hence there are no problems with correctness. For a finitely generated monoid $M$ for which $U(M) = 0$ int$(M)$ will refer to the submonoid of $M$ determined by the interior of $\Phi(M)$ (note that we always mean that the submonoid has the same zero element as the containing one), i.e.,

$$\text{int}(M) = \{ \text{the elements from } M \setminus \{0\} \text{ passing through} \}
\text{the interior of } \Phi(M) \cup \{0\}.$$

In this section the monoid operation is written additively.

**Definition 1.1.** A monoid $M$ is called normal if $x \in K(M)$ and $nx \in M$ for some natural $n$ imply $x \in M$.

Assume $M$ is a finitely generated monoid with $U(M) = 0$. For any non-zero $m \in M$ by $\Phi(m)$ will be denoted the point of intersection of the radial direction in $\mathbb{R}^r$ ($r = \text{rank}(M)$) determined by $m$ with polyhedron $\Phi(M)$. Of course, $\Phi(M)$ is the convex hull of $\{ \Phi(m) | m \in M \text{ and } m \neq 0 \}$; we do not expel the case $M = 0$ when $\Phi(M) = \emptyset$. Actually $\Phi$-correspondence will be considered in a more general situation, namely, for those submonoids $N \subset \mathbb{Q}^r$ ($r \in \mathbb{N}$) for which there exist hyperplanes $H \subset \mathbb{R}^r \setminus \{0\}$...
(dim $H = r - 1$) satisfying the condition: for each $n \in N \setminus \{0\}$ the radial direction of $n$ intersects $H$. $\Phi(N)$ again will refer to the convex hull of $\{\Phi(n) \mid n \in N, n \neq 0\}$, the set which is again dense in this hull. We will freely use Gordan’s classical lemma: a submonoid $N \subset \mathbb{Z}^r$ for some natural $r$ with $U(N) = 0$ is finitely generated iff $N$ admits the existence of the aforementioned hyperplane (in $\mathbb{R}^r$) and $\Phi(N)$ is a finite (we always mean closed) convex polyhedron.

A triple $(P, A_1, A_2)$ is called a truncated triple [G1] if $A_1 \subset A_2$ are simplices of the same dimension and $P$ is their common vertex such that the two cones spanned by $P$ and the bases of $A_1$ and $A_2$, situated oppositely to $P$, coincide. A nonzero monoid $M$ is called truncated [G1] if it is finitely generated, normal, $U(M) = 0$, and there exists a triple $(t, M, F)$ such that $F \approx \mathbb{Z}_+^r$ ($r = \text{rank}(M)$), $t \in M \setminus \{0\}$, $(\Phi(t), \Phi(M), \Phi(F))$ is a truncated triple, $M$ is integrally closed in $F$, and $t$ is the free generator of $\{m \in M \mid m \neq 0, \Phi(m) = \Phi(t)\} \cup \{0\} \approx \mathbb{Z}_+$. The point $P = \Phi(t)$ will be called the vertex of the truncated monoid $M$.

**Theorem 1.2 [G1]**. Let $N$ be a finitely generated normal monoid with $U(N) = 0$ such that $\Phi(N)$ is a simplex and $Q$ an arbitrary vertex of $\Phi(N)$. Then $\text{int}(N)$ is a filtered union of truncated monoids the vertices of which converge to $Q$.

**Remark.** Actually in [G1] Theorem 1.2 is stated in a form not involving the condition on vertices, but the proof given in [G1] exactly implies the present form.

**Definition 1.3.** A monoid $M \neq 0$ will be called a monoid of $\Phi$-simplicial growth if it is finitely generated, $U(M) = 0$, and there exists a sequence of finite convex polyhedra

$$P_1 \subset P_2 \subset \cdots \subset P_k = \Phi(M)$$

such that $P_1$ and the closures (in the sense of Euclidean metric) of $P_{i+1} \setminus P_i$ are simplices for all $i \in [1, k - 1]$.

It can be shown that $M$ is of $\Phi$-simplicial growth iff the convex polyhedron $\Phi(M)$ can be constructed step by step beginning with some simplex (or even a point, it does not matter), adding at each step a new simplex not violating convexity so that the previously obtained polyhedron does not intersect the interior of this new simplex.

**Examples 1.4.** (a) The simplest representatives of the monoids of $\Phi$-simplicial growth for which the corresponding $\Phi$-images are not simplices are the finitely generated monoids with $U(M) = 0$ for which $\Phi(M)$ are the unions of two simplices having common bases;
(b) all finitely generated nonzero monoids \( M \) with \( U(M) = 0 \) of rank \( \leq 3 \) are monoids of \( \Phi \)-simplicial growth: indeed, if rank(\( M \)) \( \leq 2 \) then \( \Phi(M) \) is necessarily a simplex and for rank(\( M \)) = 3 the desired decomposition of \( \Phi(M) \) (dim \( \Phi(M) = 2 \)) into triangles can be obtained by the diagonals emerging from an arbitrarily chosen vertex;

(c) a finitely generated monoid \( M \) with \( U(M) = 0 \) is also a monoid of \( \Phi \)-simplicial growth when rank(\( M \)) = 4 and \( \Phi(M) \) is a cube (or, more generally, a 3-dimensional polyhedron combinatorially equivalent to a cube): \( \Phi(M) \) decomposes into five simplices one of which is situated between the others and has with them single common faces; the case when \( \dim(M) = 3 \) and \( \Phi(M) \) is a truncated tetrahedron (in usual sense) can be regarded analogously.

Remark. In terms of Swan (from his notes on [G1]) a finitely generated normal monoid \( M \) with \( U(M) = 0 \) is a monoid of \( \Phi \)-simplicial growth iff there exists a sequence of pyramidal extensions of normal monoids \( \mathbb{Z}_+ = M_1 \subset M_2 \subset \cdots \subset M_k = M \).

A finite convex polyhedron \( \Phi \) of type \( \Phi(M) \) where \( M \) is a monoid of \( \Phi \)-simplicial growth will be called a polyhedron of simplicial growth (Sect. 8).

Remark 1.5. In the following we will use the elementary observations:

(a) For a finitely generated monoid \( M \) with \( U(M) = 0 \) all the vertices of the polyhedron \( \Phi(M) \) are \( \Phi \)-images of some elements from \( M \);

(b) the set \( \{ \Phi(m) \mid m \in M \setminus \{0\} \} \) is a dense subset in \( \Phi(M) \).

2. Monoids with \( \Phi \)-Simplicial Vertices

Let \((P, A_1, A_2)\) be a truncated triple (Sect. 1) and let \( I \) be a finite convex polyhedron for which \( \dim I = \dim A_1 = \dim A_2 \), \( I \subset A_2 \), and \( I \cap A_1 \) is the base of \( A_1 \) situated oppositely to the vertex \( P \). Put \( V = A_1 \cup I \). A pair of type \((P, V)\) will be called a convex set with simplicial vertex and \( P \) will be called its vertex (we mean that the objects \( A_1, A_2 \), and \( I \) will be clear from the context). Sometimes we will use \( V \) instead of \((P, V)\). Now, let \( r \) be some natural number, \( H \) an \((r - 1)\)-dimensional hyperplane in \( \mathbb{R}' \) not passing through the origin \( 0 \in \mathbb{R}' \), \( V \) a convex set with simplicial vertex for which \( V \subset H \) and \( \dim V = \dim H \). Denote by \( C(V) \) the cone in \( \mathbb{R}' \) spanned by \( 0 \) and \( V \). We denote by \( H^+ \) the half space of \( \mathbb{R}' \) bounded by \( H \) and not containing \( 0 \). Put \( C(V)^+ = C(V) \cap H^+ \).

Definition 2.1. A monoid \( N \neq 0 \) will be called a monoid with \( \Phi \)-simplicial vertex if \( N \) is finitely generated, \( U(N) = 0 \), and there exist a truncated
triple \((P, \Delta_1, \Delta_2)\), a convex polyhedron \(\Gamma\) of the aforementioned type, and a free monoid \(F\) such that the following conditions are satisfied:

(a) \(\Phi(N) = \Delta_1 \cup \Gamma\),

(b) \(N \subset F\),

(c) \(\Phi(F) = \Delta_2\),

(d) \(C(V)^+ \cap F = C(V)^+ \cap N\) where \(V = \Delta_1 \cup \Gamma\)

(here \(C(V)^+\) is considered with respect to the affine space \(H\) spanned by \(\Phi(N)\); the vertex \(P\) also will be called the vertex of \(N\).

**Proposition 2.2.** Let \(L\) be a finitely generated monoid with \(U(L) = 0\) such that \(\Phi(L)\) is a convex set with simplicial vertex. Then \(\text{int}(L)\) is a filtered union of monoids \(N_i\) with \(\Phi\)-simplicial vertices the vertices of which converge to the vertex of \(\Phi(L)\). If \(\Phi(L) = \Delta \cup \Gamma\) is the representation mentioned in the definition of a convex set with simplicial vertex then the filtered union can be chosen so that the intersections \(\Phi(N_i) \cap \Delta\) will be simplices.

We need the following lemmas (notations are again additive).

**Lemma 2.3 [G1].** Let \(L\) be a finitely generated monoid with \(U(L) = 0\) and \(L'\) its normalization, i.e., \(L' = \{x \in K(L) \mid \text{some positive multiple of } x \text{ belongs to } L\}\). Then \(L'\) is a finitely generated normal monoid with \(U(L') = 0\) and the conductor ideal \(c_{L', L} = \{l \in L' \mid l + L' \subset L\}\) contains an element from \(\text{int}(L) \setminus \{0\}\).

**Lemma 2.4 [G1].** Let \(L_1 \subset L_2\) be an arbitrary extension of finitely generated monoids without nontrivial invertible elements. Assume \(\dim \Phi(L_1) = \dim \Phi(L_2)\) (i.e., \(\text{rank}(L_1) = \text{rank}(L_2)\)) and \(L_1 = \{l \in L_2 \mid l \neq 0\}\) and \(\Phi(l) \in \Phi(L_1)\cup \{0\}\) ("\(L_1\) is integrally closed in \(L_2\)"). Then \(K(L_1) = K(L_2)\).

**Proof of Proposition 2.2.** Let \(P\) be the vertex of the convex set \(\Phi(L)\) with simplicial vertex. Of course \(P\) is a rational point in the sense that it is the \(\Phi\)-image of some element from \(K(L)\). It is also obvious that the faces of the simplex \(\Delta\) mentioned in Proposition 2.2 span rational affine spaces (= these affine spaces are spanned by \(\Phi\)-images of some elements from \(K(L)\)). From these observations we conclude that there exists a simplex \(\Delta'\) for which \((P, \Delta, \Delta')\) is a truncated triple, \(\Phi(L) \subset \Delta'\), and all vertices of \(\Delta'\) are \(\Phi\)-images of some elements from \(K(L)\). Note that here we used the convexity of \(\Phi(L)\). Now consider the normal monoid

\[ M = \{m \in K(L) \mid m \neq 0, \Phi(m) \in \Delta'\} \cup \{0\}. \]

By Gordan's lemma \(M\) will be finitely generated. Denote by \(\Gamma\) the convex polyhedron which intersects \(\Delta\) in its base and for which \(\Phi(L) = \Delta \cup \Gamma\).
By Theorem 1.2 int(M) is a filtered union of truncated monoids: say int(M) = \bigcup_i M_i. We can also assume that the vertices of M_i converge to the vertex P. Then the interior of \( A' \) will be a filtered union of the simplices \( \Phi(M_i) \). Therefore (taking in account the special position of \( A \) and \( A' \)) this filtered union can be chosen so that the intersections \( \Phi(M_i) \cap \Delta \) (where we assume \( \dim \Phi(M_i) = \dim \Delta \)) will be simplices. It can be also achieved that \( \dim(A \cap \Phi(M_i)) = \dim A \). Consider the simplices \( A_i = \Phi(M_i) \cap \Delta \) and the convex polyhedra \( \Gamma_i = \Phi(M_i) \cap \Gamma \). Now for each index choose a subpolyhedra \( \Gamma_i' \subset \Gamma_i \), for which the intersections \( A_i \cap \Gamma_i' \) are one of the faces and the vertices not belonging to these faces are included in the interiors of \( \Gamma_i' \)’s (resp.) so that the polyhedra \( V_i = A_i \cap \Gamma_i' \) are convex and rational in the sense that they are spanned by \( \Phi \) images (note that the convexity of \( V_i \) holds automatically). We can also achieve that \( V_i \)'s will approximate the interior of the polyhedron \( \Phi(L) \). The existence of such geometric objects is obvious from the elementary observations. Put

\[ N_i = \{ n \in L \mid n \neq 0 \text{ and } \Phi(n) \in V_i \} \cup \{0\}. \]

We have the filtered union \( \text{int}(L) = \bigcup_i N_i \). It remains to show that \( N_i \)'s are monoids with \( \Phi \)-simplicial vertices. First of all note that (by virtue of the rationality of \( V_i \)'s) \( \Phi(N_i) = V_i \). Let \( x \in c_{L:L} \cap \text{int}(L), \ x \neq 0 \) (Lemma 2.3). Denote by \( C(L) \) and \( C(N_i) \) the cones in \( \mathbb{R} \otimes K(L) = \mathbb{R}^r \) \((r = \text{rank}(L))\) spanned by \( L \) and \( N_i \), resp. Then they will be convex closed polyhedral cones and since \( C(N_i) \) is contained in the interior of \( C(L) \) by elementary reasonings \( C(N_i) \setminus (C(L) + x) \) will be a bounded set in \( \mathbb{R}^r \) for each \( i \). Therefore there exist hyperplanes \( H_i \subset \mathbb{R}^r \setminus \{0\} \) (for all \( i \)) of dimension \( r - 1 \) which are parallel to the hyperplane spanned by \( \Phi(L) \) and

\[ C(N_i)^+ := C(N_i) \cap H_i^+ \subset (C(L) + x) \cap C(N_i), \]

where \( H_i^+ \) are the half spaces bounded by \( H_i \) and not containing 0. Since \( C(L) \) and \( C(L') \) (the cone spanned by \( L' \)) coincide and \( x \in c_{L:L} \), we obtain that \( L' \cap C(N_i)^+ = N_i \cap (C(N_i)^+ = K(L') \cap C(N_i)^+ \). Since the monoids \( M_i \) are truncated (Sect. 1) there exist triples \((t_i, M_i, F_i)\) of the type mentioned in the definition of truncated monoids (in particular \( F_i \) are free monoids). By Lemma 2.4 \( K(M) = K(M_i) = K(F_i) \). We also have the obvious identity \( K(M) = K(L) \). Taking all these in account we conclude that the objects \((\Phi(t_i), A_i, \Phi(F_i)), \Gamma_i', V_i, F_i, H_i, \) and \( N_i \) satisfy the conditions of Definition 2.1 (here the equalities \( K(F_i) \cap C(N_i)^+ = F_i \cap C(N_i)^+ \) are used). Therefore the monoids \( N_i \) are of the desired type.
3. AUTOMORPHISMS OF MONOID RINGS WITH $\Phi$-SIMPLICIAL VERTICES

From now on the monoid operation will be written multiplicatively. A monoid ring $R[N]$ will be called a monoid ring with $\Phi$-simplicial vertex if $N$ is a monoid with $\Phi$-simplicial vertex.

Let $N$ be a monoid with $\Phi$-simplicial vertex embedded in the corresponding free monoid $F$ mentioned in Definition 2.1 (we will freely use the notation from this definition). Denote by $t_1, \ldots, t_r$ the free basis of $F$ so that $\Phi(t_r) = P$ (here $r = \text{rank}(N)$).

**Lemma 3.1.** Let $R[N]$ be a monoid ring with $\Phi$-simplicial vertex (notation as above). Then there exists a natural number $c$ such that for any natural numbers $c_i > c$, $i \in [1, r - 1]$, the $R$-automorphism $\tau$ of $R[F]$, determined by $t_i \mapsto t_i + t_i^r$ for $i < r$ and $t_r \mapsto t_r$, induces the restriction on $R[N]$ an $R$-automorphism of $R[N]$.

**Proof.** Let $n_1, \ldots, n_k$ be a finite generating set of $N$. Let $\tau$ be as in Lemma 3.1. We want to show that for all sufficiently large $c_i$'s $\tau(n_i) \in R[N]$. Without loss of generality we can restrict ourselves to the consideration of such generators $n_i$ for which in the representations $n_i = t_{i_1}^{a_{i_1}} \cdots t_{i_r}^{a_{i_r}} t_r^{a_r}$ there necessarily occurs a strictly positive number $a_{j_k}$ for some $j_k < r$. In this situation we have $\tau(n_i) = n_i + m_i$ where $m_i$ is an $R$-linear form of monomials of type

$$t_1^{x_1} t_2^{x_2} \cdots t_r^{x_r} t_r^{a_r + c_j} + \cdots + t_r^{a_r + c_{j_k}}$$

for which $x_i + y_i = a_{j_i}$ ($i \in [1, r - 1]$) and at least one of the $y_i$'s is strictly positive. From elementary geometric reasonings we conclude that for sufficiently large natural numbers $c_i$ $\Phi$-images of the mentioned monomials (involved in the elements $m_i \in R[F]$) are sufficiently close to the vertex $P$ and (hence) belong to $A_1$. It should be noted that the finite generation of $N$ is essentially used here. Now observe that at the same time the mentioned monomials are divisible (for large $c_i$'s) in $R[F]$ by $t_r^d$ with sufficiently large $d$. By the definition of a monoid with $\Phi$-simplicial vertex all these conditions imply $m_i \in R[N]$. Hence $\tau(R[N]) \subseteq R[N]$. Since the same arguments can be applied to the $R$-automorphism $\tau^{-1}$ of $R[F]$ (determined by $t_i \mapsto t_i - t_i^r$ for $i < r$ and $t_r \mapsto t_r$) we conclude that $\tau^{-1}(R[N]) \subseteq R[N]$ as well. The lemma is proved.

**Remark.** The method of constructing the automorphisms of monoid rings with $\Phi$-simplicial vertices, used in Lemma 3.1, is different from the method of constructing the automorphisms of truncated (and quasi-truncated) monoid rings suggested in [G1].
In what follows we fix the lexicographical ordering on $R[F] = R[t_1, ..., t_r]$ under which $t_i$ is lower than $t_j$ iff $i > j$. The highest member of an element $f \in R[F]$ with respect to this ordering will be denoted by $H(f)$. $H(f)$ will refer to the highest homogeneous member of $f$ with respect to the grading $R[F] = R_0 \oplus R_1 \oplus \cdots$ relative to the powers of $t_r$; in particular $R_0 = R[t_1, ..., t_{r-1}]$.

**Definition 3.2.** An element $f \in R[F]$ will be called monic if $H(f) = ut_i^a$ for some $u \in U(R)$ (group of units) and $a \geq 0$; $f \in R[F]$ will be called quasimonic if $H(f) = ut_1^{a_1} \cdots t_r^{a_r}$ for some $u \in U(R)$ and $a_1, ..., a_r \geq 0$. An element from a monoid ring $R[N]$ with $\Phi$-simplicial vertex will be called monic (quasimonic) if it is so in $R[F]$ (notation as above).

In the following when we write $a \preccurlyeq b$ we mean that $b$ is sufficiently greater than $a$.

**Proposition 3.3.** Let $R[N]$ be a monoid ring with $\Phi$-simplicial vertex and let $f \in R[N]$ be quasimonic element. Then for any natural numbers $c_1, ..., c_r$, satisfying the condition $0 \preccurlyeq c_{r-1} \preccurlyeq \cdots \preccurlyeq c_1$, the element $\tau(f) \in R[N]$ is monic where $\tau$ is the $R$-automorphism of $R[N]$ induced by $t_i \mapsto t_i + t_r^i$ for $i < r$ and $t_r \mapsto t_r$.

**Proof.** The fact that $\tau(f)$ is monic in $R[t_1, ..., t_r]$ for appropriate natural numbers $c_i$ is well known (see, for example, [L, p. 83, 84 and 92, 93]) and Lemma 3.1 enables us to choose the numbers $c_i$ so that the restriction of $\tau$ on $R[N]$ will be an $R$-automorphism of $R[N]$.

4. The Equivalence of Unimodular Rows with Isomorphic Last Components

Notation is the same as in the previous sections.

**Proposition 4.1.** Let $R$ be a (commutative) noetherian ring with Krull dimension $d < \infty$, $n \geq \max(d + 2, 3)$, $N$ a monoid with $\Phi$-simplicial vertex, $f = (f_1, ..., f_n) \in Um_n(R[N])$, and $\tau$ an $R$-automorphism of $R[F]$ of type $t_i \mapsto t_i + t_r^i$ for $i < r$ and $t_r \mapsto t_r$. Then for all sufficiently large natural numbers $c_i$ ($i \in [1, r - 1]$) there exist rows $g = (g_1, ..., g_n) \in Um_n(R[N])$ such that $g_n = \tau(f_n)$ and $f \sim g$ over $R[N]$.

We need the main result of [G1] which equivalently can be formulated as follows.

**Theorem 4.2 [G1].** Let $R$, $d$, and $n$ be the same as in Proposition 4.1 and $M$ a finitely generated monoid with $U(M) = 1$ for which $\Phi(M)$ is a
simplex. Then the group of elementary matrices $E_n(R[M])$ acts transitively on $Um_n(R[M])$.

Let us introduce two more auxiliary notations:

(a) $A' = A_1 \setminus (A_1 \cap \Gamma)$,

(b) $N' = \{ n \in N \mid \text{either } n = 1 \text{ or } \Phi(n) \in A' \}$.

**Lemma 4.3.** $N'$ is a filtered union of monoids of the type mentioned in Theorem 4.2.

**Proof.** Since the edges (= one-dimensional faces) of the simplex $A_1$ containing the vertex $P$ span (in the affine space of $\Phi(N)$) the rational lines (i.e., these lines are spanned by $\Phi$-images of some elements from $N$) we see that there exist $(r-2)$-dimensional rational simplices $D_j (j \in \mathbb{N})$ the vertices of which belong to the interiors of the aforementioned edges so that the vertices of $D_j$ converge to the corresponding vertices of the $(r-2)$-dimensional simplex $A_1 \cap \Gamma$ (which need not be rational!) when $j \to \infty$ (here we used Remark 1.5). Let $S_j$ denote the convex hull of $P$ and $D_j$. Then $S_j$ will be rational simplices. Hence by Gordan's lemma the monoids $N_j = \{ n \in N \mid \text{either } n = 1 \text{ or } \Phi(n) \in S_j \}$ will be finitely generated ($\Phi(N_j) = S_j$). It remains to note that the monoids $N_j$ are of the type mentioned in Therem 4.2 and $N' = \bigcup N_j$.

**Lemma 4.4.** Let $R[N]$ be a monoid ring with $\Phi$-simplicial vertex and $n$ an arbitrary natural number. Assume $\tau$ is an $R$-automorphism of $R[F]$ ($F$ corresponds to $N$) of type $t_i \mapsto t_i + t_i'$ for $i < r$ and $t_r \mapsto t_r$ and $A \in GL_n(R[F])$. Then for all sufficiently large natural numbers $c_i (i < r)$ the matrix $A^{-1}\tau(A)$ belongs to $GL_n(R[N'])$.

**Proof.** Consider the matrix $A' = \tau(A) - A$. It is obvious that when the numbers $c_i$ are sufficiently large the monomials involved in the components of $A'$ have sufficiently high degrees (we mean the total degrees relative to all variables $t_i$) and (using the reasonings used in Lemma 3.1) $\Phi$-images of these monomials are sufficiently close to the vertex $P$. Since the constant terms of the components of $A'$ are equal to 0 we conclude that for appropriate $c_i$'s all the monomials involved in the components of $\tau^{-1}A'$ will belong to $R[N']$. Indeed the mentioned monomials have a form $mm'$ where $m$ is a monomial from $A^{-1}$ and $m'$ that from $A'$. Now the aforementioned claim follows from the observation that the multiplication by $m$ slightly affects the radial direction in $\mathbb{R} \otimes K(N)$ of the monomial $m'$ and at the same time the degree of $mm'$ remains sufficiently high (here we essentially used the special position of $A'$ in $\Phi(N)$). Therefore $A^{-1}A'$ is a matrix over the ring $R[N']$. Finally, $A^{-1}\tau(A) = I_n + A^{-1}A' \in GL_n(R[N'])$ where $I_n$ denotes the identity matrix.
Proof of Proposition 4.1. By the "classical" case of Theorem 4.2 when $M$ (from this theorem) is a free monoid [Su] the row $f$ is completable to some matrix $A \in E_n(R[F])$. Let the natural numbers $c_i$ mentioned in the proposition be sufficiently large. Then by Lemma 3.1 $\tau(f) \in Um_n(R[N])$. Denote by $B$ the matrix $A^{\tau(A)}$. By Lemma 4.4 we can assume $B \in GL_n(R[N'])$. We have $fB = \tau(f)$. By Lemma 4.3 and Theorem 4.2 $E_n(R[N'])$ acts transitively on $Um_n(R[N'])$. Using the fact of normality of $E_n$ in $GL_n$ (for arbitrary commutative rings) when $n \geq 3$ [Su] we see that there exists $e \in E_n(R[N'])$ for which

$$B = e \begin{pmatrix} B' & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B' \in GL_{n-1}(R[N'])$. In this situation the row $fe$ obviously satisfies the desired condition. The proposition is proved.

5. Preliminary Transformations of Unimodular rows

Proposition 5.1. Let $R$ be a local noetherian ring with Krull dimension $d < \infty$ and $N$ a monoid with $\Phi$-simplicial vertex. Assume $n \geq \max(d + 2, 3)$. Then for any $f = (f_1, ..., f_n) \in Um_n(R[\text{int}(N)])$ there exists $g = (g_1, ..., g_n) \in Um_n(R[N])$ for which $g_n$ is monic and $f \sim g$ over $R[N]$.

We need the following lemma which was proved by R. G. Swan in his notes on [G2].

Lemma 5.2 (Swan). Let $L$ be a finitely generated monoid with $U(L) = 1$ and $\Phi_1 \subset \Phi_2$ two convex subsets of $\Phi(L)$ having the dimension $\dim(\Phi(L))$. Assume $\Phi_1$ is an open subset of $\Phi(L)$ and put

$$L_j = \{l \in L \mid \text{either } l = 1 \text{ or } \Phi(l) \in \Phi_j\}, \quad j = 1, 2.$$

Then for any local ring $A$ and any $h \in A[L_1]$ with invertible constant term (i.e., $h \notin \mu A[L_1] + (L_1 \setminus \{1\}) A[L_1]$ where $\mu$ is the maximal ideal of $A$) the natural homomorphism


is an isomorphism.

Proof. The claim can be checked locally on $A[L_1]$. Let $v \in \max(A[L_1])$ (the maximal spectrum). If $L_1 \setminus \{1\} \subset v$ then both sides of the considered map vanish when we localize them relative to $v$. If $L_1 \setminus \{1\} \notin v$ then from
the openness of $\Phi_1$ we easily conclude that $L_1 \subset A[L_1] \setminus v$ and therefore the localization at $v$ passes through the following localization:

$$L_1^{-1}A[L_1]/h(L_1^{-1}A[L_1]) \rightarrow L_2^{-1}A[L_2]/h(L_1^{-1}A[L_2]).$$

It remains to note that

$$A[K(L)] = L_1^{-1}A[L_1] = L_1^{-1}A[L_2]$$

(Lemma 2.4).

Recall that the height $ht(I)$ of an ideal $I$ in some commutative ring is defined as follows: $\inf\{ht(p), I \subset p \text{ and } p \text{ is a prime ideal}\}$. For arbitrary ideal $I \subset R[t_1, ..., t_r]$ by $\gamma(I)$ will be denoted the ideal of $R$ consisting of the leading coefficients (with respect to our lexicographical ordering of $R[t_1, ..., t_r]$ introduced in Sect. 1) of elements from $I$ (here $R$ is any commutative ring). Now assume $L$ is a submonoid of the free monoid $F$ of monomials in the $t_i$'s. Let $J$ be an ideal of $R[L]$. Analogously denote by $\gamma(J)$ the set of the leading coefficients (with respect to the aforementioned ordering of $R[F]$) of elements from $J$. Then $\gamma(J)$ again will be an ideal of $R$. Indeed, since the multiplicative closedness of $\gamma(J)$ is obvious, we only have to show that $\gamma(J)$ is an additive subgroup of $R$: let $a, b \in \gamma(J)$ and $a + b \neq 0$. Then there exists $x, y \in J$ for which $H(x) = am$ and $H(y) = bn$ for some $m, n \in L$ (for $H$ see Sect. 3); therefore $H(xn + ym) = a + b$ and (hence) $a + b \in \gamma(J)$.

**Lemma 5.3** [G1, Lemma 6.5]. Let $R$ be a noetherian (commutative) ring and $F$ the same as above. Then for any submonoid $M \subset F$ such that $\Phi(M) = \Phi(F)$ and any ideal $J \subset R[M]$ we have $ht(J) \leq ht(\gamma(J))$.

**Lemma 5.4** [L, p. 93]. Let $A$ be a commutative noetherian ring and $(a_1, ..., a_n) \in Um_n(A)$ for some $n \in \mathbb{N}$. Then there exists $(a'_1, ..., a'_n) \in Um_n(A)$ such that $(a_1, ..., a_n) \sim (a'_1, ..., a'_n)$ and for any $s \leq n$ $ht(\{(a'_1, ..., a'_s)\}) \leq s$.

**Theorem 5.5** (Krull, [A–M]). Let $A$ be a commutative noetherian ring and $h \in A \setminus U(A)$ a nonzerodivisor element. Then $ht(hA) = 1$.

As usual, for an extension of monoids $M \subset L$ we will say that $M$ is integrally closed in $L$ if $al \in M$ for some $a \in \mathbb{N}$ and $l \in L$ implies $l \in M$.

**Lemma 5.6.** Let $N$ be a monoid with $\Phi$-simplicial vertex and $F$ the corresponding free monoid (notation as above). Then there exists a natural number $k$ for which $N$ is an integrally closed submonoid in the monoid $M$ generated by $\{t_1^k, ..., t_r^k\} \cup N$. 
Proof. By Definition 2.1 all monomials from $F$ having sufficiently high degrees fall into $N$ whenever their radial directions in $\mathbb{R}^r$ pass through $\Phi(N)$. Now, if the natural number $k$ is sufficiently large all monomials from $M$ passing through $\Phi(N)$ either a priori belong to $N$ or have sufficiently high degrees (and, by the aforementioned remark, again belong to $N$).

**Lemma 5.7.** Let $L$ be an arbitrary submonoid of $F$ ($F$ is the same as above). Then for any commutative ring $R$ an element $h \in R[L]$ is invertible in $R[L]$ iff it is invertible in $R[F]$.

**Proof.** Easily follows from the well-known observation that an element $h \in R[F]$ is invertible iff the constant term of $h$ is invertible and the coefficients at monomials of positive degrees are nilpotent (actually the same holds in all positively graded algebras).

**Proof of Proposition 5.1.** Without loss of generality it can be assumed that the constant term of $f_n$ is invertible in $R$ (since $R$ is local this can be achieved by multiplying $f$ by appropriate $e \in E_n(R)$). We will also assume that $f_n \notin U(R[\text{Int}(N)])$, because otherwise $f_n \in U(R[N])$ and $f \sim (1, 0, ..., 0) \sim (1, 0, ..., 0, t_i)$ for such natural $c$ that $t_i \in N$ (the existence of such $c$ is obvious) and there is nothing to prove. Let $k$ be a sufficiently large natural number and $M$ the monoid mentioned in Lemma 5.6. Lemma 5.2 implies that the bottom row of the commutative square

$$
\begin{array}{ccc}
R[N] & \longrightarrow & R[M] \\
\downarrow & & \downarrow \\
R([N])/(f_n) & \longrightarrow & R[M]/(f_n)
\end{array}
$$

is an isomorphism: indeed, it just suffices to consider the commutative triangle with $R$-isomorphisms $\alpha, \beta$

$$
\begin{array}{ccc}
R[N]/(f_n) & \longrightarrow & R[M]/(f_n) \\
\downarrow & \alpha & \downarrow \\
R[\text{Int}(N)]/(f_n) & \longrightarrow & R[M]/(f_n)
\end{array}
$$

Let $\tilde{f}_1, ..., \tilde{f}_{n-2}, \tilde{f}_{n-1}$ denote the images of $f_1, ..., f_{n-2}, f_{n-1}$ in $R[M]/(f_n)$, respectively. Then $(\tilde{f}_1, ..., \tilde{f}_{n-1}) \in \text{Um}_{n-1}(R[M]/(f_n))$. By Lemma 5.4 there exists $e \in E_{n-1}(R[M]/(f_n))$ such that

$$\text{ht}((\tilde{h}_1, ..., \tilde{h}_{n-3}, \tilde{h}_{n-2})) 
\geq n - 2 \geq d,$$

where $(\tilde{h}_1, ..., \tilde{h}_{n-3}, \tilde{h}_{n-2}, \tilde{h}_{n-1}) = (\tilde{f}_1, ..., \tilde{f}_{n-3}, \tilde{f}_{n-2}, \tilde{f}_{n-1})e$ (note that here we used that $R[M]/(f_n)$ is noetherian). Now the aforemen-
tioned commutative square enables us to find \( E \in E_{n-1}(R[N]) \) for which \((f_1, \ldots, f_{n-2}, f_{n-1})E\) maps (by the composite map) into \((\bar{h}_1, \ldots, \bar{h}_{n-2}, \bar{h}_n-1)\). Let \( J \) denote the ideal

\[
l_1 R[M] + \cdots + l_n R[M] + l_{n-2} R[M] + f_n R[M] \subset R[M],
\]

where \((l_1, \ldots, l_{n-3}, l_{n-2}, l_{n-1}) = (f_1, \ldots, f_{n-3}, f_{n-2}, f_{n-1})E\). Since \( R[M] \) is noetherian, \( f_n \) is not a zerodivisor (because the constant term of \( f_n \) is so) and \( f_n \notin U(R[M]) \) (by Lemma 5.7) Theorem 5.5 implies \( \text{ht}(J) \geq d+1 \). By Lemma 5.3 \( \text{ht}(\gamma(J)) \geq d+1 > \dim R \). Hence \( \gamma(J) = R \). In other words \( J \) contains a quasimonic element \( q \). Say \( q = l_1 q_1 + \cdots + l_{n-2} q_{n-2} + f_n q_n \) for some \( q_1, \ldots, q_{n-2}, q_n \in R[M] \). Then for a sufficiently large natural \( c \) we have

\[
t_1 q_1, \ldots, t_{n-2} q_{n-2}, t_n q_n \in R[N]
\]

(since for such \( c \) all monomials involved in the \( t_i q_i \)'s have sufficiently high degrees and at the same time their radial directions in \( \mathbb{R}^\prime \) pass sufficiently closely to \( P \)). Therefore \( t_i q_i \) is a quasimonic element in the ideal \( I = l_1 R[N] + \cdots + l_{n-2} R[N] + f_n R[N] \subset R[N] \) (recall that the \( l_i \)'s belong to \( R[N] \)). Let \( T \) be an arbitrary element from \( \text{int}(N) \), \( T \neq 1 \) (i.e., all \( t_i \)'s are involved in \( T \)). Then for any natural \( k \) we have \( f \sim (l_1, \ldots, l_{n-2}, l_{n-1} + T^k t_n, q, f_n) \sim (l_1, \ldots, l_{n-2}, l_{n-1} + T^k t_n, q, f_n + l_{n-1} + T^k t_n, q) \) over \( R[N] \). It is obvious that if \( k \) is sufficiently large then \( f_n + l_{n-1} + T^k t_n, q \) is quasimonic. Now Propositions 3.3 and 4.1 apply. Proposition 5.1 is proved.

6. Reductions Modulo Monic Elements

We keep the notation of the previous sections. In particular for a monoid \( N \) with \( \Phi \)-simplicial vertex \( \Gamma \) will denote the same as in Section 2.

**Proposition 6.1.** Let \( R[N] \) be an arbitrary monoid ring with \( \Phi \)-simplicial vertex and \( N(\Gamma) = \{ n \in N | \text{either } n = 1 \text{ or } \Phi(n) \in \Gamma \} \). Then for any monic element \( f \in R[N] \) the homomorphism \( R[N(\Gamma)] \to R[N]/(f) \) is integral (i.e., \( R[N]/(f) \) is integral over the image of the mentioned homomorphism).

**Proof.** Step 1. We know that the vertices of the edges of \( \Phi(N) \) emerging from \( P \) are of type \( \Phi(n) \) for some \( n \in N \) (Remark 1.5). Of course, all these vertices except \( P \) belong to \( \Gamma \). Let \( \Gamma_0 \) be the rational polyhedron spanned by the mentioned vertices and the vertices of \( \Gamma \) not belonging to \( \Delta \). Since the natural map \( R[N(\Gamma_0)] \to R[N]/(f) \), where
$N(\Gamma_0) = \{ n \in \mathbb{N} \mid \text{either } n = 1 \text{ or } \Phi(n) \in \Gamma_0 \}$, passes through the homomorphism mentioned in Proposition 6.1 we see that it suffices to show the integrality of the homomorphism $R[N(\Gamma_0)] \to R[N](f)$.

Step 2. For arbitrary monoid $L$ let $\mathcal{L}$ denote the divisible envelope of $L$. More precisely $\mathcal{L} = \lim_{\rightarrow, c_\ell} (L \rightarrow L)$ where $L \rightarrow L$ denotes the homomorphism $l \mapsto cl$. In what follows we will naturally identify $\mathcal{L}$ with the monoid of all rational (in the usual sense) points of $R_{\text{rank}(L)}$ which fall into the cone $C(L)$ spanned by $L$ whenever $L$ is finitely generated with $U(L) = 1$. Assume $\mathcal{L}$ is a finitely generated monoid with $U(L) = 1$ for which $\Phi(L)$ is a simplex. Then it can be easily shown that $\mathcal{L}$ is isomorphic to $\mathbb{Q}^\times\text{rank}(L)$ [G1]. Denote by $\mathcal{A}_0$ the closure (with respect to the Euclidean metric) of $\Phi(N) \setminus \Gamma_0$. Then, of course, $\mathcal{A}_0$ will be a rational simplex for which $(P, \mathcal{A}_1, \mathcal{A}_0)$ will be a truncated triple. Put $\mathcal{N}(\mathcal{A}_0) = \{ n \in \mathbb{N} \mid \text{either } n = 1 \text{ or } \Phi(n) \in \mathcal{A}_0 \}$. By the aforementioned remark $\mathcal{N}(\mathcal{A}_0)$ will be isomorphic to $\mathbb{Q}^\times_+$ where $r = \text{rank}(N)$. Furthermore, the elements of $\mathcal{N}(\mathcal{A}_0)$ will be thought as “monomials” $s_1^{a_1} \cdots s_r^{a_r} \mid s_r^{a_r}$ such that $a_1, \ldots, a_r \in \s$, $\Phi(s_i)$ belong to the segments $[P, \Phi(t_i)]$ (respectively) for all $i \in [1, r-1]$, and $\Phi(s_r) = P$. In particular $\Phi(s_i), i \in [1, r)$, determine the set of all vertices of $\mathcal{A}_0$. For convenience we will assume in addition $s_r = t_r$, (this can be achieved by replacing $s_r$ by $s_r^c$ for suitable positive rational $c$). Analogously the elements of $\mathcal{F}$ will be thought of as monomials of type $t_1^{a_1} \cdots t_r^{a_r} \mid t_r^{a_r}$ where $a_1, \ldots, a_r \in \s$. Put $\text{ord}_s(s_1^{a_1} \cdots s_r^{a_r} \mid s_r^{a_r}) = a_r$ and $\text{ord}_s(t_1^{a_1} \cdots t_r^{a_r} \mid t_r^{a_r}) = a_r$.

Step 3. We will say that an element $x \in R[\mathcal{N}]$ $(x \in R[\mathcal{N}(\mathcal{A}_0)])$ is $t$-monic ($s$-monic) if $x = un + r_1n_1 + \cdots + r_kn_k$ where $u \in U(R)$, $n = t_r^a$ (equivalently, $n = s_r^a$) for some rational $a > 0$, $n_1, \ldots, n_k \in \mathcal{N}(\mathcal{A}_0)$, $r_1, \ldots, r_k \in R$, and $\text{ord}_s(n_j) < a$ ($\text{ord}_s(n_j) < a$) for all $j \in [1, k]$. In particular $x \in R[N] \setminus U(R)$ is monic iff $x$ is $t$-monic. We claim that $x \in R[\mathcal{N}(\mathcal{A}_0)]$ is $s$-monic if it is $t$-monic. The proof of this statement easily follows from the observations that for any $n \in \mathcal{N}(\mathcal{A}_0)$ $\text{ord}_s(n) \leq \text{ord}_s(n)$ and $\text{ord}_s(s_r^a) = \text{ord}_s(s_r^a)$ (for all $a \in \s$).

Step 4. The natural homomorphism $R[\mathcal{N}(\Gamma_0)] \rightarrow R[\mathcal{N}](f)$ is integral.

Proof. Of course, we can assume $f \notin U(R)$. Fix arbitrarily the representation $f = g + h$ where $g \in R[\mathcal{N}(\Gamma_0)]$ and $h \in R[\mathcal{N}(\mathcal{A}_0)]$. In addition we can assume $h$ is monic (such a representation obviously exists and it is not uniquely determined). By the previous step $h$ will be $s$-monic. Assume $h = un + r_1n_1 + \cdots + r_kn_k$ where $u \in U(R)$, $n = s_r^a$ for some rational $a > 0$, $n_1, \ldots, n_k \in \mathcal{N}(\mathcal{A}_0)$, $r_1, \ldots, r_k \in R$, and $\text{ord}_s(n_j) < a$ for all $j \in [1, k]$. Let $c$ be such a positive rational number that all the numbers $a = \text{ord}_s(s_r^a)$ and $\text{ord}_s(n_j), j \in [1, k]$, are integral multiples of $c$. Say $\text{ord}_s(n) = dc$ and
\text{ord}_z(n_j) = d_j c. \text{ Then } d_j \in \mathbb{Z}_+ \text{ and } d \in \mathbb{N} \quad (j \in [1, k]). \text{ Denote by } z \text{ the element } x_j^{(s)}. \text{ Then for each } j \in [1, k] \text{ either } n_j = z^{d_j} \text{ or } n_j = m_j z^{d_j} \text{ for certain } m_j \in N(A_0) \setminus \{ 1 \} \text{ satisfying the condition } \Phi(m_j) \in A_0 \cap I_0 \subset I_0. \text{ In this situation } h = uz^{d_1} + g_1 z^{d_2} + \cdots + g_k z^{d_k} \text{ for some } g_1, \ldots, g_k \in R[N(I_0)]. \text{ Since } d > d_1, \ldots, d_k \text{ we come to the conclusion that the image of } z \text{ in } R[N]/(f) \text{ is integral over the image of the considered homomorphism. Since for any } l \in \tilde{N} \text{ there exists natural } v \text{ for which } l^v \text{ belongs to the submonoid of } \tilde{N} \text{ generated by } N(I_0) \cup \{ z \} \text{ we see that all elements from } \tilde{N} \text{ have the integral images (over the image of our map) as well. This completes the proof.} \\

\textbf{Step 5.} \text{ Let } N' \text{ denote the normalization of } N \text{ (see Lemma 2.3). Then the natural map } R[N(I_0)] \to R[N']/(f) \text{ is integral.} \\

\textbf{Proof.} \text{ Since the inclusion } R[N(I_0)] \subseteq R[N(I_0)] \text{ is an integral extension of rings we see that it suffices to show the injectivity of the natural map } R[N']/(f) \to R[\tilde{N}]/(f). \text{ Hence, we have to show } fR[\tilde{N}] \cap R[N'] = fR[N']. \text{ Let } g \in fR[\tilde{N}] \cap R[N']. \text{ Then all the monomials involved in } g \text{ have a type } nm \text{ where } n \text{ is a monomial involved in } f \text{ and } m \text{ that from } R[\tilde{N}]. \text{ Since } nm \text{ is also a monomial from } R[N'] \text{ we must have } m \in R[K(N')]. \text{ From the normality of } N' \text{ we have } N' = \tilde{N} \cap K(N'). \text{ In other words } g = fh \text{ for some } h \in R[N']. \\

\textbf{Step 6.} \text{ Let } I \text{ denote the image in } R[N]/(f) \text{ of the ideal } (N \setminus \{ 1 \}) R[N] \subseteq R[N]. \text{ Then } I \cap J \text{ is a nilpotent ideal of } R[N]/(f) \text{ where } J \text{ denotes the kernel of the natural homomorphism } R[N]/(f) \to R[N']/(f). \\

\textbf{Proof.} \text{ By the definition of the monoid with } \Phi \text{-simplicial vertex for any } n \in \mathbb{N} \setminus \{ 1 \} \text{ and } g \in R[N'] \text{ there exists } c \in \mathbb{N} \text{ for which } n'g \in R[N]. \text{ Now let } x \in (\mathbb{N} \setminus \{ 1 \}) R[N] \text{ and } x = fy \text{ for some } y \in R[N']. \text{ By our remark } (fy)^{c+1} \in R[N] \text{ for some } c \in \mathbb{N}. \text{ Consequently, } (fy)^{c+1} \text{ maps into } 0 \text{ in } R[N]/(f). \\

\textbf{Step 7.} \text{ To prove Proposition 6.1 it suffices to show that any element from } I \text{ (Step 6) is integral over the considered image of } R[N(I_0)] \text{ (we assume that } f \not\in U(R)). \text{ Let } x \in I \text{ and } x' \text{ denote the natural image of } x \text{ in } R[N']/(f). \text{ By Step 5 there exists a monic polynomial } F \text{ in } A[X] \text{ such that } F(x') = 0 \text{ where } A \text{ refers to the image of the map considered in Step 5. We can assume that the leading coefficient of } F \text{ is equal to } 1. \text{ Considering the polynomial } XF \text{ we can also assume that the mentioned monic polynomial from } A[X] \text{ (of course, here } X \text{ is a variable) does not have a nonzero constant term (we merely replace } F \text{ by } XF). \text{ In this situation there exists a monic polynomial } G \in B[X] \text{ with the leading coefficient } 1 \text{ and without a nonzero constant term, where } B \text{ refers to the image of the homomorphism from Proposition 6.1, which maps into } F \text{ under the natural epimorphism } B[X] \to A[X]. \text{ In conclusion we obtain that } G(x) \in I \text{ (since } G(0) = 0 \text{) and}
\( G(x) \in J \) (\( J \) is from Step 6) since \( F(x') = 0 \). By Step 6 \( G(x') = 0 \) for some \( c \in \mathbb{N} \). Since \( G' \) is again monic we see that \( x \) is integral over \( B \). The proof of Proposition 6.1 is finished.

**Corollary 6.2.** Let \( R[N] \) be an arbitrary monoid ring with \( \Phi \)-simplicial vertex, \( f \in R[N] \) a monic element, and \( \mu \) an arbitrary prime ideal of \( R[N(I')] \) (notation as in Proposition 6.1). Then \( R[N]_\mu/(f) \) is semilocal where \( R[N]_\mu = (R[N(I')]_\mu)^{-1}R[N] \).

**Proof.** By Proposition 6.1 \( R[N]_\mu/(f) \) is integral over the local image of the local ring \( R[N(I')]_\mu \). Since \( N \) is a finitely generated monoid \( R[N] \) must be a finitely generated \( R \)-algebra. Therefore \( R[N]_\mu/(f) \) is finitely generated as an algebra over the mentioned image. Then it is a finitely generated module over the same ring. But any algebra over a semilocal ring, which is finitely generated as module, is itself semilocal (a standard fact of commutative algebra).

7. **The Triviality of Localized Unimodular Rows**

We will use the following notation. Let \( L \) be a finitely generated monoid with \( U(L) = 1 \). Assume \( \Phi(L) \) is a convex set with simplicial vertex (Sect. 2). Thus we have fixed the objects \( A_1, A_2, I \), and \( P \) mentioned in Section 2. By \( L_0 \) will be denoted the submonoid \( \{ 1 \in L \mid \text{either } l = 1 \text{ or } \Phi(l) \text{ belongs to the interior of } I \} \). Furthermore, for an extension of commutative rings \( A \subset B \) and a prime ideal \( p \in \text{spec}(A) \) by \( B_p \) (as it was in Corollary 6.2) will be denoted \((A \setminus p)^{-1}B \).

**Proposition 7.1.** Let \( L \) be a finitely generated monoid with \( U(L) = 1 \). Assume \( \Phi(L) \) is a convex set with simplicial vertex, \( R \) is a noetherian local ring with Krull dimension \( d < \infty \), \( n \geq \max(d+2, 3) \), \( f \in Um_n(R[\text{int}(L)]) \), and \( \mu \) is a prime ideal of \( R[L_0] \). Then \( f_\mu \sim (1, 0, ..., 0) \) over \( R[\text{int}(L)]_\mu \) where \( f_\mu \) is the image of \( f \) in \( Um_n(R[\text{int}(L)]_\mu) \).

We need some more auxiliary notation. For any (finite) convex polyhedron \( \Phi \) of arbitrary dimension by \( \text{int}(\Phi) \) will be denoted the (relative) interior of \( \Phi \). Let \( L \) be a finitely generated monoid with \( U(L) = 1 \) for which \( \Phi(L) \) is a convex set with simplicial vertex. Denote by \( \Phi_1 \) the convex set \( \text{int}(I) \cup \text{int}(A_1 \cap I) \) and put \( L_1 = \{ l \in L \mid \text{either } l = 1 \text{ or } \Phi(l) \in \Phi_1 \} \).

**Lemma 7.2.** Let \( L, R, d, n, \) and \( f \) be as in Proposition 7.1 and let \( \nu \) be a prime ideal of \( R[L_1] \). Then \( f_\nu \sim (1, 0, ..., 0) \) over \( R[\text{int}(L)] \), where \( f_\nu \) is the image of \( f \) in \( Um_n(R[\text{int}(L)] \).
Proof. By Proposition 2.2 \( \text{int}(L) \) can be represented as a filtered union of monoids \( N_i \) with \( \Phi \)-simplicial vertices so that the vertices of \( N_i \) converge to the vertex \( P \) of \( \Phi(L) \) \((i \in \mathbb{N})\) and the intersections \( \Phi(N_i) \cap A_1 \) are simplices. Let \((P_i, A_{i1}, A_{i2}), \Gamma_i, \) and \( F_i \) be the objects corresponding \( N_i \) mentioned in Definition 2.1. Let \( \delta_i \) be the simplices \( \Phi(N_i) \cap A_1 \) and \( \gamma_i \), the Euclidean closures of the convex sets \( \Phi(N_i \setminus \delta_i) \). Now we are going to define a new system of monoids \( M_i \) (for all \( i \)) with \( \Phi \)-simplicial vertices. As monoids \( M_i \) merely coincide with \( N_i \) (respectively), but we assign to them new objects listed in Definition 2.1: the truncated triples \((P_i, \delta_i, A_{i2}), \) the convex polyhedra \( \gamma_i \), and the same free monoids \( F_i \). Let \( v_i \) denote the intersections \( v \cap R[M(\gamma_i)] \) where \( M(\gamma_i) \) are defined analogously to \( N(\Gamma) \) from Proposition 6.1. Then \( v_i \in \text{spec}(R[M(\gamma_i)]) \) and \( R[\text{int}(L)]_v \) is a filtered union of \( R[M_j]_v \)'s. Since \( \text{int}(L) \) is a filtered union of \( M_j \)'s there exists an index \( j \) for which \( f \in Um_n(R[M_j]) \). By Proposition 5.1 there exists \( g = (g_1, ..., g_n) \in Um_n(R[M_j]) \) such that \( g_n \) is monic in the monoid ring \( R[M_j] \) and \( f \sim g \) over \( R[M_j] \) (and, hence, over \( R[\text{int}(L)] \)). By Corollary 6.2 \( R[M_j]_v/(g_n) \) is semilocal. Let \( \bar{g} \) denote reduction modulo \( g_n \). We have \( (\bar{g}_1, ..., \bar{g}_{n-1}) \in Um_{n-1}(R[M_j]) \). Since \( n-1 \geq 2 \) and over any semilocal ring \( A \) the group of elementary matrices \( E_k(A) \) acts transitively on the set of unimodular rows \( Um_k(A) \) whenever \( k \geq 2 \) we come to the conclusion that the natural image of \( (\bar{g}_1, ..., \bar{g}_{n-1}) \) in \( Um_{n-1}(R[M_j]_v) \) and the standard row \((1, 0, ..., 0)\) are in the same \( E_{n-1}(R[M_j]_v) \)-orbit. But then \( g' \sim (1 + h_1 g'_n, h_2 g'_n, ..., h_{n-1} g'_n, g'_n) \sim (1, 0, ..., 0, g'_n) \sim (1, 0, ..., 0) \) over \( R[M_j]_v \), for some \( h_1, ..., h_{n-1} \in R[M_j]_v \), where \( g' \) and \( g'_n \) denote the corresponding images of \( g \) and \( g_n \) in \( R[M_j]_v \). Hence \( f \sim (1, 0, ..., 0) \) over \( R[\text{int}(L)]_v \).

Proof of Proposition 7.1. Let \( z \) be any internal rational point of \( \Gamma \) (i.e., \( z = \Phi(l) \) for some \( l \in L_0 \setminus \{1\} \)). Let \( k \) be any rational number satisfying the condition \( 0 \leq k \leq 1 \). It is obvious that from the very beginning we could choose the affine subspace \( H \) in \( \mathbb{R}^{\text{rank}(L)} \), where we consider our \( \Phi \)-correspondence, to the rational (in the usual sense), i.e., to be spanned by the points from \( \mathbb{Q}^{\text{rank}(L)} \). In this situation the homothetic image of \( \Phi(L) \) with respect to the centre \( z \) and the coefficient \( k \) again will be rational (the vertices of it will be of type \( \Phi(l) \) for appropriate elements from \( \text{int}(L) \setminus \{1\} \). Put \( L_k = \{l \in L \mid \text{either } l = 1 \text{ or } \Phi(l) \text{ belongs to the homothetic image of } \Phi(L) \text{ with respect to the centre } z \text{ and the coefficient } k \} \). Then \( \text{int}(L) \) will be a filtered union of \( L_k \)'s where \( k \) varies over the set of all rational numbers from the open unit interval \((0, 1)\). Let \( L_{1.1} \) be the same for \( L_k \) for \( L \) in Lemma 7.2, where we assume that the geometric objects corresponding to \( L_k \) are the homothetic images of those corresponding to \( L \). From the construction it is clear that \( L_0 \) is a filtered union of \( L_{1.1} \)'s (it should be noted that \( L_0 \) need not be monoids with \( \Phi \)-simplicial vertices).
Now Proposition 7.1 follows from Lemma 7.2 and the observation that $R[\text{int}(L_k)]_{\mu_k}$ is a filtered union of $R[\text{int}(L_k)]_{\mu_k}$'s and the objects $L_k$ and $\mu_k$ satisfy the conditions form Lemma 7.2 where $\mu_k = \mu \cap R[L_k] \subset \text{spec}(R[L_k])$.

**Corollary 7.3.** Let $G$ be a monoid with $U(G) = 1$ for which $K(G)$ is finitely generated. Assume there exists an $(r - 1)$-dimensional hyperplane $H$ in $\mathbb{R} \otimes K(G)$ not passing through the origin where $r = \text{rank}(G)$ and a convex union $\Delta \cup \Gamma \subset H$ of a simplex $\Delta$ and a polyhedron $\Gamma$ having common face $\Delta \cap \Gamma$, $\dim \Delta = \dim \Gamma = \dim H$, such that $\Phi(G) = \text{int}(\Delta \cup \Gamma)$. Put $G_0 = \{ g \in G \mid \text{ either } g = 1 \text{ or } \Phi(g) \in \text{int}(\Gamma) \}$. Then for any noetherian local ring $R$ with Krull dimension $d < \infty$, $n \geq \max(d + 2, 3)$, $f \in \text{Um}_n(R[G])$, and a prime ideal $\mu \subset R[G_{00}]$ we have $f_\mu \sim (1, 0, ..., 0)$ where $f_\mu$ denotes the image of $f$ in $\text{Um}_n(R[G])_{\mu}$.

**Proof.** Actually it can be shown that at the rationality of $\Delta \cup \Gamma$ this corollary merely coincides with Proposition 7.1 (we mean the case when the vertices of $\Delta \cup \Gamma$ are $\Phi$-images of some elements from $K(G)$). The desired claim follows from the observation that $R[G]_{\mu}$ is a filtered union of $R[G]_{\mu_k}$'s where: $G_i = \{ g \in G \mid \text{ either } g = 1 \text{ or } \Phi(g) \in \text{int}(\Delta_i \cup \Gamma_i), \Delta_i \text{ and } \Gamma_i \text{ are chosen so that } \Delta_i \cup \Gamma_i \text{ are rational unions of type } \Delta \cup \Gamma \text{ approximating } \text{int}(\Delta \cup \Gamma), \text{ and the vertices of the } \Gamma_i \text{'s belong to } \text{int}(\Gamma) \}$ and $\mu_i = \mu \cap R[G_{i0}]$ for $G_{i0} = \{ g \in G_i \mid \text{ either } g = 1 \text{ or } \Phi(g) \in \text{int}(\Gamma_i) \}$.

**Remark 7.4.** When we say that some object is a filtered union of those of certain type we merely mean the inductive limit of a directed diagram (the homomorphism of which need not be injective).

8. **Further Remarks on Polyhedra of Simplicial Growth**

Having established Corollary 7.3 in order to use the method developed in [G2, G3] to complete the proof of the main result we have to collect some preliminary results on polyhedra of simplicial growth and Milnor patching technique for unimodular rows. The exposition occupies Sections 8 and 9.

Recall that a finite convex polyhedron $P$ is called of simplicial growth (Sect. 1) if there exists a finite sequence of finite convex polyhedra $P_1 \subset \cdots \subset P_{k-1} \subset P_k = P$ for which $P_i$ and the closures (in Euclidean metric) of the convex sets $P_{i+1} \setminus P_i$ are simplices for all $i \in [1, k - 1]$. Such sequences will be called nondegenerate if all the simplices mentioned above have the same dimensions.

**Lemma 8.1.** A finite convex polyhedron $P$ is of simplicial growth iff it admits a nondegenerate sequence of the aforementioned type.
Proof. Let \( P_1 \subseteq \cdots \subseteq P_{k-1} \subseteq P_k = P \) be a sequence of the type mentioned in the definition of a polyhedron of simplicial growth. By elementary geometric reasonings for arbitrary finite convex polyhedra \( Q \subseteq S \), such that the (topological) closure of \( S \setminus Q \) is a simplex, the condition \( \dim Q < \dim S \) implies that \( S \) itself is a simplex (the mentioned closure merely coincides with \( S \)). Therefore, the restriction of our sequence to its maximal fragment consisting of polyhedra having dimension \( \dim P \) gives us the desired sequence.

Lemma 8.2. Let \( P \) be a polyhedron of simplicial growth. Then all faces of \( P \) are of simplicial growth as well.

Proof. For simplicity we will consider \((\dim P - 1)\)-dimensional faces. Let \( Q \) be such face. The claim easily follows from the observation that if \( T \) is a finite convex subpolyhedron of \( P \) for which the closure of \( P \setminus T \) is a simplex then the closure of \( Q \setminus (T \cap Q) \) is a simplex as well.

Remarks. (a) Somehow unexpectedly it turns out that a polyhedron of simplicial growth does not remain of the same type at small moves of its vertices (this can be seen in the example of a cube).

(b) Let \( P \) and \( Q \) be two arbitrary polyhedra of simplicial growth (of arbitrary dimensions) contained in some real affine spaces \( X \) and \( Y \) (respectively). Embed \( X \) and \( Y \) in a new real affine space \( Z \) so that \( \bar{X} \cap \bar{Y} = \emptyset \) where the bar refers to the projective closure and the intersection is considered in \( \bar{Z} \) (obviously, we can choose \( \dim Z = \dim X + \dim Y + 1 \)). Is then the convex hull (in \( Z \)) of \( P \) and \( Q \) of simplicial growth? Equivalently: is \( M \oplus N \) of \( \Phi \)-simplicial growth whenever \( M \) and \( N \) are monoids of \( \Phi \)-simplicial growth?

9. Milnor Patching for Unimodular Rows

In this section we will collect some properties of unimodular rows over commutative rings involved in special commutative squares. But at the beginning I would like to make some remarks. R. G. Swan observed that in [G2] we actually had the situation of analytic isomorphisms of commutative rings along certain multiplicatively closed subsets [W]. This observation enabled him to apply a well-developed approach to the study of projective modules and essentially simplify the exposition. Thereafter the analogous observation for \( K_1 \)-topics treated in [G3] was made by G. Schabühser [Sc]. It should be noted that the main result of the mentioned work answers in a positive way the general question, raised in [G1, Sect. 8], in the special case of \( c \)-divisible monoids \((c \geq 2)\). Actually this section recalls the needed results from Schabühser’s preprint.
Let us recall that a ring homomorphism $i: A \rightarrow B$ is called an analytic isomorphism along the multiplicatively closed subset $S \subset A$ [W] if $S$ is regular on $A$, $i(S)$ is regular on $B$, and the natural homomorphisms $A/sA \rightarrow B/i(s)B$ are isomorphisms for all $s \in S$. The first observation on analytic isomorphisms is that the square

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
S^{-1}A & \rightarrow & i(S)^{-1}B
\end{array}
\]

is cartesian. The proof is left to the reader.

We will say that a commutative square of rings

\[
\begin{array}{ccc}
A & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A'
\end{array}
\]

has a Milnor patching property (for unimodular rows) if for arbitrary natural $n \geq 3$ and $f \in Um_n(A_1)$ the equivalence $f' \sim (1, 0, \ldots, 0)$ over $A'$, where $f' \in Um_n(A')$ denotes the image of $f$, implies the existence of $g \in Um_n(A)$ having in $Um_n(A_1)$ the image $g_1$ for which $g_1 \sim f$.

**Proposition 9.1.** The commutative squares of rings of the following two types have a Milnor patching property (for unimodular rows):

(a) $\begin{array}{ccc}
A & \rightarrow & A_1 \\
\downarrow & & \downarrow z \\
A_2 & \rightarrow & A'
\end{array}$ a cartesian square with $\pi$ surjective.

(b) $\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
S^{-1}A & \rightarrow & i(S)^{-1}B
\end{array}$ a commutative square where $i$ is an analytic isomorphism along $S \subset A$.

To prove the proposition we need

**Lemma 9.2.** Let

\[
\begin{array}{ccc}
A & \rightarrow & A_1 \\
\downarrow & & \downarrow \psi_1 \\
A_2 & \rightarrow & A'
\end{array}
\]
be a commutative square of the form (a) or (b) and \( \delta' \in E_n(A') \), \( n \geq 3 \). Then there exist \( \delta_1 \in E_n(A_1) \) and \( \delta_2 \in E_n(A_2) \) such that \( \delta' = \varphi_1(\delta_1) \varphi_2(\delta_2) \).

Proof. (a) is trivial and (b) is just Lemma 2.4 from [Vo].

Proof of Proposition 9.1. Let

\[
\begin{array}{ccc}
A & \longrightarrow & A_1 \\
\downarrow & & \downarrow \varphi_1 \\
A_2 & \longrightarrow & A'
\end{array}
\]

be a commutative square of type (a) or (b) and \( f \in Um_n(A_1) \) for some \( n \geq 3 \). Assume \( f' \delta' = (1, 0, ..., 0) \) for some \( \delta' \in E_n(A') \) where \( f' \) is the image of \( f \). By Lemma 9.2 \( \delta' = \varphi_1(\delta_1) \varphi_2(\delta_2) \) for some \( \delta_1 \in E_n(A_1) \) and \( \delta_2 \in E_n(A_2) \). Now consider the rows \( g_1 = f' \delta_1 \in Um_n(A_1) \) and \( g_2 = (1, 0, ..., 0) \delta_2 \). Since \( g_1 \) and \( g_2 \) have the same images in \( Um_n(A') \) we conclude that there exists \( g \in A'' \) with images \( g_i \) in \( A''_i \), \( i = 1, 2 \). We only have to check that \( g \in Um_n(A) \). The verifications are straightforward. Let us verify the mentioned inclusion, for example, for the squares of type (b). We have \( a_1a_1' + \cdots + a_na_n' = s \) for some \( a_1', \ldots, a_n' \in A \) and \( s \in S \) where \( (a_1, \ldots, a_n) = g \) (since \( g_2 \in Um_n(S^{-1}A) = Um_n(A_2) \)). Since \( A/sA \) is naturally isomorphic to \( B/sB \) (and \( g_1 \in Um_n(B) = Um_n(A_1) \)) there exist \( a_1'', \ldots, a_n'' \in A \) for which \( a_1a_1'' + \cdots + a_na_n'' = 1 \) (modulo \( sA \)). In other words \( s \) and \( 1 + sb \) belong to \( a_1A + \cdots + a_nA \) for some \( b \in A \). Hence \( g \in Um_n(A) \).

10. Main Theorem

**Main Theorem 10.1.** Let \( R \) be a commutative noetherian ring with Krull dimension \( d < \infty \) and \( M \) a monoid of \( \Phi \)-simplicial growth. Then the group of elementary matrices \( E_n(R[M]) \) acts transitively on the set of unimodular \( n \)-rows \( Um_n(R[M]) \) whenever \( n \geq \max(d + 2, 3) \).

**Corollary 10.2.** Let \( R, M, \) and \( d \) be as in the Main Theorem. Then

(a) All projective \( R[M] \)-modules of rank > \( d \) which are stably extended from \( R \) are actually extended from \( R \);

(b) the natural map \( GL_{\max(d+1,2)}(R[M]) \rightarrow K_1(R[M]) \) is surjective;

(c) all the aforementioned assertions hold for the monoids \( L \) which either span the same cones in \( \mathbb{R} \otimes K(L) \) as the monoids of \( \Phi \)-simplicial growth do or are of rank \( (L) \leq 3 \).
Proof. (a) is equivalent to the condition that $GL_n(R'[M])$ acts transitively on $Um_n(R'[M])$ whenever $n \geq d + 2$ and $R'$ denotes any localization of $R(\langle L \rangle)$. But this is the case by the Main Theorem and the elementary observation that $GL_2$ acts transitively on $Um_2$.

(b) is obvious.

(c) If $L$ spans the cone of the desired type then $L$ can be represented as a filtered union of monoids of $\Phi$-simplicial growth. Indeed, represent $K(L)$ as a filtered union $\bigcup_i H_i$ of finitely generated groups $H_i$ so that $\text{rank}(H_i) = \text{rank}(L)$. Then $L = \bigcup_i N_i$ where $N_i = H_i \cap L$; the finite generation of $N_i$ follows from Gordan's lemma and $N_i$ are of $\Phi$-simplicial growth since $\Phi(N_i) = \Phi(L)$. Now the Main Theorem applies. It remains to consider the case $\text{rank}(L) \leq 3$. Proposition 9.1, applied to the cartesian square

$$\begin{array}{ccc}
R[L'] & \rightarrow & R[L] \\
\downarrow & & \downarrow \pi \\
R & \rightarrow & R[U(L)],
\end{array}$$

where $L' = (L \setminus U(L)) \cup \{1\}$ and $\pi(l) = l$ if $l \in U(L)$ and 0 otherwise, shows that it would suffice to consider the case $U(L) = 1$ if we would have the transitivity of the action of $E_n(R[U(L)])$ on $Um_n(R[U(L)])$. But this actually is the case by [Su] ($U(L)$ is a filtered union of finitely generated free abelian groups); hence without loss of generality $U(L) = 1$. In this situation $L$ is a filtered union of finitely generated monoids having rank $\leq 3$ and without nontrivial invertible elements. Since (by Example 1.4) such monoids are of $\Phi$-simplicial growth the Main Theorem applies.

We will use the following results:

**Theorem 10.3** [G1]. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring and $f \in Um_n(A)$ for some $n \geq 3$, the natural image of which in $Um_n(A_0)$ is $(1, 0, ..., 0)$. Then $f \sim (1, 0, ..., 0)$ over $A$ iff $f_\mu \sim (1, 0, ..., 0)$ over $A_\mu$, for each maximal ideal $\mu \subset A_0$, where $f_\mu$ denotes the image of $f$ in $Um_n(A_\mu)$.

**Lemma 10.4** [G1 G3]. Let $M$ be a finitely generated monoid with $U(M) = 1$. Then for any (commutative) ring $R$ the monoid ring $R[M]$ admits a grading $R[M] = R_0 \oplus R_1 \oplus \cdots$, where $R_0 = R$.

**Proof of the Main Theorem.** By Theorem 10.3 and Lemma 10.4 we can assume $R$ is local; in this situation we shall carry out the proof in two steps (here we use that $E_n(R)$ acts transitively on $Um_n(R)[B, Sw, V]$).

**Step 1.** The group of elementary matrices $E_n(R[int(M)])$ acts transitively on $Um_n(R[int(M)])$. 

Proof. Let $P_1 \subset \cdots \subset P_{k-1} \subset P_k = \Phi(M)$ be a nondegenerate sequence as mentioned in the definition of a polyhedron of simplicial growth (Lemma 8.1). For each $j \in [1, k]$ put $G_j = \{g \in M \mid \text{either } g = 1 \text{ or } \Phi(g) \in \text{int}(P_i)\}$. Then by Lemma 5.2 and Proposition 9.1(b) the following commutative square of rings has a Milnor patching property,

$$
\begin{array}{ccc}
R[G_j] & \longrightarrow & R[G_{j+1}] \\
\downarrow & & \downarrow \\
R[G_j]_{\mu_j} & \longrightarrow & R[G_{j+1}]_{\mu_{j+1}}
\end{array}
$$

where $j \in [1, k-1]$ and $\mu_j$ is the maximal ideal of $R[G_j]$ generated by that of $R$ and $G_j \setminus \{1\}$. Therefore by Corollary 7.3 for any $h \in Um_n(R[G_k]) = Um_n(R[\text{int}(M)])$ there exists $E \in E_n(R[\text{int}(M)])$ for which $hE \in Um_n(R[G_1])$. Since by Gordan's lemma $G_1$ is a filtered union of finitely generated monoids $\Phi$-images of which are simplices (int($P_1$) must be represented as a filtered union of rational simplices) Theorem 4.2 shows that $E_n(R[G_1])$ acts transitively on $Um_n(R[G_1])$. Hence $h \sim (1, 0, \ldots, 0)$ over $R[\text{int}(M)]$.

Step 2. Consider the following cartesian square of rings

$$
\begin{array}{ccc}
R[\text{int}(M)] & \longrightarrow & R[M] \\
\downarrow & & \downarrow \pi \\
R & \longrightarrow & R[M]/(\text{int}(M)\setminus\{1\})\ R[M]
\end{array}
$$

with $\pi$ surjective. By Proposition 9.1(a) and the previous step, to complete the proof of the Main Theorem it suffices to show that $E_n(B_1)$ acts transitively on $Um_n(B_1)$ where $B_1 = R[M]/(\text{int}(M)\setminus\{1\})\ R[M]$. Let $F$ be an arbitrary $(\dim \Phi(M) - 1)$-dimensional face of $\Phi(M)$ and $M(F) = \{m \in M \mid \text{either } m = 1 \text{ or } \Phi(m) \in F\}$. We have the natural embedding $R[M(F)] \to B_1$ under which the ideal $(\text{int}(M(F))\setminus\{1\})\ R[M(F)] \subset R[M(F)]$ will turn out an ideal in $B_1$ as well. Denote this ideal by $I$ and consider the following cartesian square:

$$
\begin{array}{ccc}
R[\text{int}(M(F))] & \longrightarrow & B_1 \\
\downarrow & & \downarrow \\
R & \longrightarrow & B_1/I.
\end{array}
$$

By Lemma 8.2 $M(F)$ is a monoid of $\Phi$-simplicial growth. Thus by Step 1 $E_n(R[\text{int}(M(F))]])$ acts transitively on $Um_n(R[\text{int}(M(F))])$ and we see that it suffices to show the transitivity of the action of $E_n(B_2)$ on $Um_n(B_2)$.
where $B_2 = B_1/I$. $B_1$ can be thought of as $R[M]$ with elements from $M$ passing through $\text{int}(P_k)$ annihilated, while in $B_2$ in addition are annihilated those elements from $M$ which pass through $\text{int}(F)$. Thereafter we "annul" the interiors of other (dim $\Phi(M) - 1$)-dimensional faces of $\Phi(M)$. When all these faces are done with we turn to the interiors of (dim $\Phi(M) - 2$)-dimensional faces of $\Phi(M)$, etc. Finally we descend to the coefficient ring over which the transitivity of the action of $E_n$ on $Um_n$ is obvious.

**Remarks.** It can be shown that already in dimension 3 there exist finite convex polyhedra which are not of simplicial growth. The typical representatives of such polyhedra are the ones the shape of which approximates a sphere with sufficient precision.

It turns out that any finitely generated abelian group can be realized as a divisor class group of some normal monoid of $\Phi$-simplicial growth (while in [G1] this could be done only for torsion groups).

**Acknowledgment**

I would like to thank the referee for pointing out an error in the first version of Section 4.

**References**


