The Elementary Action on Unimodular Rows
over a Monoid Ring

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In this paper we prove that for a commutative noetherian ring \( R \) with
Krull dim \( R = d < \infty \) and a submonoid \( M \) of \( \mathbb{Z}^r \) for some \( r \in \mathbb{N} \) such that
\( M \subset \mathbb{Q}^r_+ \) is an integral extension, the group of elementary matrices
\( E_n(R[M]) \) acts transitively on \( \mathit{Um}_n(R[M]) \) for all \( n \geq \max(d+2, 3) \).
Hitherto the special case \( M = \mathbb{Z}^r_+ \) was obtained in [Su]. Bass-
Vasershtein's classical results imply the transitivity only for \( n \geq d + 2 + r \).
The key step is the consideration of finitely generated normal monoids. The proof we give here uses some density properties of integer lattices.

0. PRELIMINARIES

All the considered monoids are assumed to be commutative, cancellative,
torsion free (in the group of fractions) and with unit element. For a
monoid \( M \) its group of fractions will be denoted by \( K(M) \); let \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \)
denote the integers, rational and real numbers, respectively; by \( \mathbb{Z}_+, \mathbb{Q}_+ \)
will be denoted the additive monoids of the corresponding nonnegative
numbers. \( \mathbb{N} = \{1, 2, 3, \ldots \} \). For a monoid \( M \) its rank means the rank
of \( K(M) \) (= dim_\mathbb{Q} \mathbb{Q} \otimes K(M))
. By \( \mathbb{Q}_+ \otimes M \) we denote the universal
divisible enveloping monoid for \( M \): i.e., (writing additively)
\( \mathbb{Q}_+ \otimes M = \lim (M \rightarrow M), \ c \in \mathbb{N} \). As usual, an extension of monoids
\( M \subset N \) is called integral if for any \( n \in N \) there exists \( a \in \mathbb{N} \) such that \( an \in M \).
\( M \) is called normal (or integrally closed) if there does not exist an element
in \( K(M) \setminus M \) some positive multiple of which belongs to \( M \) (recall that a
monoid \( M \) is called seminormal if: \( x \in K(M), 2x \in M, 3x \in M \Rightarrow x \in M \).

Our conditions on monoids imply the natural embeddings
\( M \subset K(M) \subset \mathbb{Q} \otimes K(M) \). Consequently, without loss of generality we will
assume that the considered monoids are submonoids of \( \mathbb{Q} \)-spaces. Let
\( \text{rank}(M) < \infty \). Denote by \( C(M) \) the convex cone in the real space
\( \mathbb{R}^n \)
$\mathbb{R} \otimes K(M)$ spanned by $M$. By elementary geometric observation we obtain that in the case when $M$ is finitely generated and the group of invertible elements $U(M)$ is trivial there exists an affine hyperplane $H \subset \mathbb{R} \otimes K(M)$ \hfill (dim $H = \text{rank}(M) - 1$), $0 \notin H$, such that $C(M)$ is spanned by $C(M) \cap H$ and 0. In this situation $C(M) \cap H$ is a convex closed polytope and we denote it by $\Phi(M)$ [G1, G2]. Of course, the shape of $\Phi(M)$ depends on the choice of $H$, but its “combinatoric complicatedness” is invariant. Let $M$ be as above (finitely generated and $U(M) = 0$). Then by $\text{int}(M)$ will be denoted the submonoid of $M$ consisting of all those elements from $M$ the radial directions of which pass through the interior of $\Phi(M)$. In purely algebraic terms $\text{int}(M) = \{ m \in M | m \neq 0, \forall n \in M, \exists a \in \mathbb{N}, am - n \in M \} \cup \{0\}$ (this algebraic definition was suggested by R. G. Swan in his notes on [G1]; in [G1, G2] $\text{int}(M)$ was denoted by $M_\ast$). Note that, usually, $\text{int}(M)$ is not finitely generated.

Recall that a domain is called seminormal if its multiplicative monoid of nonzero elements is so.

**Theorem 0** ([A-A, G1, G2]). A monoid domain $R[M]$ is integrally closed (seminormal) if and only if $R$ and $M$ are integrally closed (seminormal).

1. **$\Phi$-Simplicial Monoids**

In this paragraph the monoid operation will be written additively. We assume that the considered monoids have finite ranks.

**Proposition 1.1.** Let $M$ be a finitely generated monoid with $U(M) = 0$. Then the following conditions are equivalent:

(a) $\Phi(M)$ is a simplex,

(b) $\mathbb{Q}_+ \otimes M \cong \mathbb{Q}_+^r$, where $r = \text{rank}(M)$,

(c) $M$ can be embedded in $\mathbb{Q}_+^r$, $r = \text{rank}(M)$, so that the extension $\mathcal{M} \subset \mathbb{Q}_+^r$ will be integral,

(d) $\mathcal{M}$ can be embedded in $\mathbb{Z}_+^r$, $r = \text{rank}(M)$, so that the extension $M \subset \mathbb{Z}_+^r$ will be integral.

**Proof:** Let $\Phi(M)$ be simplex. Then there exists a system of elements $m_1, \ldots, m_r \in M$ for which the points of intersection of the radial directions in $\mathbb{R} \otimes K(M)$ obtained by $m_i$'s with the hyperplane $H$ are just the vertices of $\Phi(M)$. $m_1, \ldots, m_r$ obviously are linearly independent in $K(M)$ and for any $m \in M$ there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}_+$ for which $m = \lambda_1 m_1 + \cdots + \lambda_r m_r$ (in $\mathbb{Q} \otimes K(M)$). Thus (a) $\Rightarrow$ (b). Now let $\mathbb{Q}_+ \otimes M \cong \mathbb{Q}_+^r$. Then for each $i \in \{1, r\}$ there exist $\mu_i \in \mathbb{N}$ for which $\mu_i e_i \rightarrow n_i \in M$, where $e_1, \ldots, e_r$ is the
standard basis in $\mathbb{Q}'$. Consider the composition $M \subset \mathbb{Q}_+ \otimes M \simeq \mathbb{Q}_+'$. We see that the image of $M$ contains the elements $(0, \ldots, \mu_i, \ldots, 0)$. It remains to note that $\mathbb{Q}_+'$ is integral over the submonoid generated by the elements $(0, \ldots, \mu_i, \ldots, 0)$; (b) $\Rightarrow$ (c). Since $\mu$ is finitely generated there exists $a \in \mathbb{N}$ such that the image of the composition $M' \to M \subset \mathbb{Q}_+'$ is a submonoid of $\mathbb{Z}_+'$. The condition of integrality of $\mathbb{Z}_+$ over this image is obviously satisfied. Hence, (c) $\Rightarrow$ (d). We have the following implications: $M' \simeq M''$ and $\Phi(M')$ is simplex $\Rightarrow \Phi(M'')$ is simplex, and extension $M' \subset M''$ is integral $\Rightarrow \Phi(M') = \Phi(M'')$. It just remains to note that $\Phi(\mathbb{Z}_+)$ is the standard simplex.

**Definition 1.2.** A monoid $M$ will be called $\Phi$-simplicial if $M$ is finitely generated with trivial group of invertible elements $U(M)$ such that the conditions from the previous propositions are satisfied for $M$ (see [G2]).

**Remark.** Since a 1-dimensional closed convex polytope is just a segment (= 1-dimensional simplex) we see that every finitely generated monoid $M$ of rank($M$) $\leq 2$ with $U(M) = 0$ is $\Phi$-simplicial. On the other hand in the case rank($M$) $> 2$ to the monoids correspond arbitrarily complicated polytopes $\Phi(M)$.

Note that $M' \oplus M''$ is $\Phi$-simplicial iff $M''$ and $M''$ are simultaneously.

**Lemma 1.3.** $M \subset \mathbb{R}_+$ be an integral extension and $M$ be normal. Then there exists elements from $\mathbb{Z}_+$ of the type

$$
a_1 = (a_{11}, a_{12}, \ldots, a_{1r}),
$$

$$
\vdots
$$

$$
a_r = (0, 0, \ldots, a_{rr}),
$$

$a_{ii} > 0$, for which $M = \langle a_1, \ldots, a_r \rangle \cap \mathbb{Z}_+$, where $\langle a_1, \ldots, a_r \rangle$ denotes the subgroup in $\mathbb{Z}_+$ generated by $a_i$ and the intersection is considered in $\mathbb{Z}_+$.

**Proof.** Since the extension $M \subset \mathbb{Z}_+$ is integral we obtain that for any $i \in [1, r]$ there exists an element of the type $(0, \ldots, \mu_i, \ldots, 0)$ in $M$. Hence, for all $i \in [1, r]$ the subsets $X_i = \{(0, \ldots, 0, b_{ii}, b_{ii+1}, \ldots, b_{ir}) \in M | b_{ii} > 0\}$ are nonempty. In each $X_i$ choose and element with minimal $b_{ii}$. We obtain a system $a_1, \ldots, a_r \in M$. Let us prove that $M = \langle a_1, \ldots, a_r \rangle \cap \mathbb{Z}_+$. First, note that the normality of $M$ and the integrality of the extension $M \subset \mathbb{Z}_+$ imply $M - K(M) \cap \mathbb{Z}_+$ (the intersection is considered in $\mathbb{Z}_+$). Thus $\langle a_1, \ldots, a_r \rangle \cap \mathbb{Z}_+ \subset M$. Assume $m = (b_1, \ldots, b_r) \in M$. Write $b_i = q_{i1} + s$ for some $q \in \mathbb{Z}_+$ and $0 \leq s < 1$; since for some $c_2, \ldots, c_r \in \mathbb{N}$ we have $m - q_x + c_2x_2 + \cdots + c_rx_r \in K(M) \cap \mathbb{Z}_+$ = $M$ the minimality of $a_{i1}$ implies
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Let $M$ be a finitely generated normal monoid. Then according to [Ch1] we can define the group of divisorial ideals $\text{Div}(M)$. Namely, the elements of $\text{Div}(M)$ are nonempty intersections of the principal divisorial ideals of $M$,

$$\text{Div}(M) = \left\{ \bigcap (x_j + M) \neq \emptyset \mid x_j \in K(M) \right\}$$

(the intersections are considered in $K(M)$; for $D, D' \in \text{Div}(M)$ put $DD' = \bigcap (x_j + M)$, where the intersection is considered for all $x_j \in K(M)$ satisfying the condition $\{d + d' \mid d \in D, \ d' \in D'\} \subseteq x_j + M$. Then $\text{Div}(M)$ will become a group containing the subgroup of principal divisorial ideals $\text{pr}(M) = \{(x + M)/x \in K(M)\}$, which in its turn is isomorphic to $K(M)/U(M)$. The quotient group $\text{Cl}(M) = \text{Div}(M)/\text{pr}(M)$ is called divisor class group. It just measures the deviation of $M$ from monoids of the type $Z^*_+ \oplus Z^*$ ($r, s \geq 0$).

**Proposition 1.4 ([Ch1, Ho]).** Let $M$ be a finitely generated normal monoid with $U(M) = 0$. Then there exist $s \in \mathbb{N}$ and a submonoid $N \subset \mathbb{Z}^+$ such that $N = K(N) \cap \mathbb{Z}^+_+$ (the intersection is considered in $\mathbb{Z}^+$) with $N$ isomorphic to $M$.

**Remark.** In terms of our $\Phi$-correspondence this statement means that an arbitrary convex closed polytope $\Phi$ (of arbitrary dimension) can be obtained as an intersection of some simplex $\Delta$ and some affine subspace $X$ in some real space $\mathbb{R}^s$ (of appropriate dimensions).

**Proposition 1.5 ([Ch1]).** Let $M$ be a submonoid in $\mathbb{Z}^*_+$ for which $M = K(M) \cap \mathbb{Z}^*_+$ (in $\mathbb{Z}^*$). If for any $i \in [1, s]$ the set

$$T_i = \{e_j + K(\mu) \mid j \neq i, j \in [1, s]\}$$

generates $\mathbb{Z}^*/K(M)$ as a monoid then $\text{Cl}(M) \approx \mathbb{Z}^*/K(M)$.

Now assume $M$ is a normal and the extension $M' \subset \mathbb{Z}^*_+$ is integral. Let $a_1, \ldots, a_s$ be the elements mentioned in Lemma 1.3. Denote by $c_i, i \in [1, r]$, the greatest common divisor (g.c.d.) of the numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$. Then $M \simeq M' = \langle x'_1, \ldots, x'_s \rangle \cap \mathbb{Z}^*_+$, $x'_j = (0, \ldots, c_j^{-1} a_{i_1}, \ldots, c_j^{-1} a_{i_r}); \ M'$ is a submonoid in $\mathbb{Z}^*_+$ of the type mentioned in Proposition 1.5. Consequently $\text{Cl}(M) \approx \text{Cl}(M') \approx \mathbb{Z}^*_+/\langle x'_1, \ldots, x'_s \rangle$ and, hence, $\text{Cl}(M)$ is a torsion group. Conversely, if $M$ is a finitely generated normal monoid with $\text{Cl}(M)$ torsion then according to Proposition 1.1 and results from [Ch1, Ch2] $M$ must be $\Phi$-simplicial. In [Ch1] it is also proved that any abelian group can be realized as $\text{Cl}(M)$ for some normal $M$. Thus we obtain
PROPOSITION 1.6. Let $M$ be a finitely generated normal monoid with $U(M) = 0$. Then $M$ is $\Phi$-simplicial iff $\text{Cl}(M)$ is a torsion group; for any finite $G$ there exists a $\Phi$-simplicial normal monoid $M$ for which $\text{Cl}(M) \approx G$.

2. TRUNCATED MONOIDS

We again use the additive notations for monoid structure. Let $\Delta_1$ and $\Delta_2$ be two simplices in some (finite dimensional) real space. A triple $(P, \Delta_1, \Delta_2)$ will be called truncated if the following conditions are satisfied:

(a) $\Delta_1 \subset \Delta_2$ and $P$ is a common vertex of $\Delta_1$ and $\Delta_2$,
(b) $\dim \Delta_1 = \dim \Delta_2$
(c) there exists a hyperplane $X$ in our real space such that $\Delta_1$ is spanned by $P$ and $\Delta_2 \cap X$.

Let $M$ be a $\Phi$-simplicial normal monoid. Then for arbitrary vertex $P$ of the simplex $\Phi(M)$ the maximal submonoid $N \subset M$ for which $\Phi(N) = P$ is isomorphic to $\mathbb{Z}^r$. This follows from the observation that $N$ is normal, $U(N) = 0$, $\text{rank}(N) = 1$ and $N$ is a submonoid in $K(M) \approx \mathbb{Z}^r$. The monoid $N$ will be denoted by $M(P)$.

DEFINITION 2.1. A triple $(t, M_1, M_2)$ will be called truncated if $M_1$ and $M_2$ are $\Phi$-simplicial normal monoids of the same rank and the following conditions are satisfied:

(a) $M_1 \subset M_2$ and $M_1$ is integrally closed in $M_2$ (i.e., there does not exist an element in $M_2 \setminus M_1$ integral over $M_1$),
(b) $(P, \Phi(M_1), \Phi(M_2))$ is a truncated triple for certain $P$ (in the aforementioned sense),
(c) $t$ is free generator of $M_1(P)$.

DEFINITION 2.2. We will say that monoid $M$ is truncated if there exists a truncated triple $(t, M, M')$ with $M' \approx \mathbb{Z}_+^r$ (of course, $r = \text{rank}(M)$).

LEMMA 2.3. Let $(t, M_1, M_2)$ be a truncated triple and $P$ be as above. Then

(a) $M_1(P) = M_2(P) = \mathbb{Z}_+^r t$,
(b) $K(M_1) = K(M_2)$.

Proof. (a) is trivial. Let $x \in K(M_2)$. Choose arbitrary $m \in \text{int}(M_1) \subset \text{int}(M_2)$. Then for sufficiently large $a \in \mathbb{N}$ the element $am + x$ belongs to $C(M_1) \cap M_2$ (this is obvious from elementary geometric reasonings).
Consequently, for some $b \in \mathbb{N}$ we have $b(am + x) \in M_1$. Thus $am + x$ is an element from $M_2$, integral over $M_1$, by the definition of a truncated triple $am + x \in M_1$. Hence, $x \in K(M_1)$.

**Lemma 2.4.** Let $M$ be a truncated monoid and $(t, M', M)$ be a truncated triple. Then $M'$ is a truncated monoid.

**Proof.** The lemma follows from the general observation that if $(t, M_1, M_2)$ and $(t, M_3, M_4)$ are truncated then $(t, M_1, M_3)$ is so.

Let us introduce the following notation $[G_1, G_2]$: For a monoid (of an appropriate type) and any nonzero $m \in M$ denote by $\Phi(m)$ the intersection of the radial ray from $\mathbb{R} \otimes K(M)$ determined by $m$ with the polytope $\Phi(M)$.

**Lemma 2.5.** Let $M$ be a truncated monoid. Then $\text{Cl}(M)$ is a finite cyclic group.

**Proof.** Let $(t, M, M')$ be a truncated triple where $M' \cong \mathbb{Z}'_+$, $r = \text{rank}(M)$. It is obvious that in a free monoid the free generators are uniquely determined. Consequently, there elements $t_2, \ldots, t_r \in M'$ for which $\{t, t_2, \ldots, t_r\}$ is a basis in $M'$ and $\{\Phi(t), \Phi(t_2), \ldots, \Phi(t_r)\}$ is the set of all vertices of $\Phi(M')$. For all natural $a \in \mathbb{N}$ the systems $B_a = \{t, t_2 + at, \ldots, t_r + at\}$ are bases for $K(M') \cong \mathbb{Z}'$. It is obvious that when $a \to \infty$ the points $\Phi(t_2 + at), \ldots, \Phi(t_r + at)$ converge to the point $\Phi(t)$ (in the sense of euclidean metric in $\Phi(M')$). By the definition of a truncated triple we obtain that for a sufficiently large $a \in \mathbb{N}$ the points $\Phi(t_i + at), i \in [2, r]$, belong to $\Phi(M)$. Since $K(M) = K(M')$ (by Lemma 2.3) we obtain $t_i + at \in M, i \in [2, r]$. Consider the free submonoid $M'' \subset M$ generated by $\{t, t_2 + at, \ldots, t_r + at\}$. Since $K(M'') = K(M)$ and $M''$ is normal we come to the conclusion that $M''$ is integrally closed in $M$. In addition we have that the points $\Phi(t_i + at), i \in [2, r]$, belong to the edges $[\Phi(t), \Phi(t)]$ (respectively) of the simplex $\Phi(M'')$. In other words, the triple $(t, M'', M)$ is truncated. By Proposition 1.1 we can identify $M$ with a submonoid in $\mathbb{Z}'_+$ so that the extension $M \subset \mathbb{Z}'_+$ will be integral. In this situation the elements $t, t_2 + at, \ldots, t_r + at$ must have the forms (for appropriate enumeration of the basis in $\mathbb{Z}'_+$),

\[
\begin{align*}
t_2 + at &= (a_1, 0, \ldots, 0, b_1), \\
t_3 + at &= (0, a_2, \ldots, 0, b_2), \\
&\vdots \\
t_r + at &= (0, 0, \ldots, a_{r-1}, b_{r-1}), \\
t &= (0, 0, \ldots, 0, a_r),
\end{align*}
\]
\( a_i > 0, \ i \in \{1, r\} \). Thus \( M = \langle t_2 + at, ..., t_r + at, t \rangle \cap \mathbb{Z}'_+ \approx \langle x_1, x_2, ..., x_r \rangle \cap \mathbb{Z}'_+ \) for

\[
x_1 = (1, 0, ..., 0, b'_1),
\]
\[
x_2 = (0, 1, ..., 0, b'_2),
\]
\[
\vdots
\]
\[
x_{r-1} = (0, 0, ..., 1, b'_{r-1}),
\]
\[
x_r = (0, 0, ..., 0, a'_r),
\]

where \( b'_i = c^{-1}b_i, i \in \{1, r\} \), \( a'_i = c^{-1}a_i \) and \( c = \gcd(b_1, ..., b_{r-1}, a_r) \). By Proposition 1.5, \( \text{Cl}(M) \approx \mathbb{Z}'/\langle x_1, ..., x_r \rangle \approx \mathbb{Z}/(a'_r) \) is a cyclic group.

As a result of Lemma 2.5 and Proposition 1.6 we obtain that a "majority" of \( \Phi \)-simplicial normal monoids are not truncated.

**Lemma 2.6.** Let \( M \) be a \( \Phi \)-simplicial monoid. Then \( M \) is truncated iff there exists a truncated triple \((t, M'', M)\) with \( M'' \) free.

**Proof.** The part "only if" was already considered in the proof of the previous lemma. Now assume \((t, M'', M)\) is a truncated triple and \( M'' \) is a free monoid. Let \( \{t, t_2, ..., t_r\} \) be the basis of \( M'' \). For arbitrary \( a \in \mathbb{N} \) consider the free monoid \( M'_a \) generated by \( \{t, t_2 - at, ..., t_r - at\} \). From elementary geometric reasonings we conclude that for a sufficiently large \( a \in \mathbb{N} \) the cone \( C(M) \) is contained in \( C(M'_a) \) and these cones satisfy the condition: the triple \((\Phi(t), \Phi(M), \Phi(M'_a))\) is truncated (here the hyperplane \( H \) mentioned in Section 0 is chosen corresponding to the cone \( C(M'_a) \), or equivalently, \((t, M, M'_a)\) is truncated.

**Remark.** The condition of \( \text{Cl}(M) \) being cyclic is a necessary one for a \( \Phi \)-simplicial normal monoid \( M \) to be truncated, but it is not a sufficient condition.

### 3. Approximations by Truncated Monoids

We keep to the additive notations for a monoid structure.

**Theorem 3.1.** Let \( M \) be a finitely generated normal \( \Phi \)-simplicial monoid. Then \( \text{int}(M) \) is filtered union of truncated monoids.

The proof is based on some lemmas.

**Lemma 3.2 [G1].** Let \( M \) be a finitely generated normal monoid with \( U(M) = 0 \). Then \( \text{int}(M) \) is normal.
This lemma actually holds for arbitrary monoids which satisfy only the 
seminormality condition [G1].

**Gordan's Lemma 3.3** [Da, G2]. Let $M$ be a submonoid in $\mathbb{Z}^s$ (for some $s \in \mathbb{N}$) satisfying the condition of existence of a hyperplane $H \subset \mathbb{R} \otimes K(M)$ $(0 \notin H)$ for which $C(M)$ is spanned by $0$ and $C(M) \cap H$. Then $M$ is finitely generated iff $\Phi(M) = C(M) \cap H$ is a finite closed (of course, convex) polytope.

**Lemma 3.4.** Let $M$ be a finitely generated monoid with $U(M) = 0$. Then for an arbitrary convex $W \subset \Phi(M)$ for which $\dim W = \dim \Phi(M)$ the submonoid

$$M(W) = \{m \in M \mid m \neq 0, \Phi(m) \in W\} \cup (0) \subset M$$

has the same group of fractions as $M$, i.e., $K(M) = K(M(W))$.

**Proof.** Let $x \in K(M)$. Because of the condition $\dim W = \dim \Phi(M)$ we have $\exists m \in M$, $m \neq 0$, $\Phi(m) \in \text{int}(W)$, or equivalently $\text{int}(M) \cap \text{int}(M(W)) \neq 0$. Let $x = y - z$, $y \in M$, $z \in M$. Using again the equality $\dim W = \dim \Phi(M)$ we obtain that for some $a \in \mathbb{N}$, $am \in \text{int}(M(W))$, $am + z \in \text{int}(M(W))$, where $m$ is of the aforementioned type (this follows from the observation that $\Phi(am + y)$, $\Phi(am + z) \to \Phi(m)$ when $a \to \infty$). In conclusion $x = (am + y) - (am + z) \in K(\text{int}(M(W))) \subset K(M(W))$. Thus $K(M) = K(M(W))$.

**Corollary 3.5.** Let $M$ be a finitely generated monoid with $(M) = 0$. Then $K(M) = K(\text{int}(M))$.

By $S^{r-1}$ denote the standard unit sphere in $\mathbb{R}^r$ ($r > 1$). For any $1 \leq k \leq r$ define a map

$$\theta_k : GL_r(\mathbb{Z}) \to S^{r-1} \times \cdots \times S^{r-1},$$

where on the right hand side the product is taken over $r - k + 1$ copies of $S^{r-1}$, as follows: let $x = (a_r) \in GL_r(\mathbb{Z})$, then for any $i \in [1, r]$ the row $a_i = (a_{i1}, \ldots, a_{ir})$ defines the radial direction in $\mathbb{Z}^r$ is identified with its standard image in $\mathbb{R}^r$, denote by $x_i$ the intersection of this direction $S^{r-1}$ and put $\theta_k : (a_r) \mapsto (x_{k}, x_{k+1}, \ldots, x_r)$.

**Proposition 3.6.** For any $r > 1$ the image of $\theta_k$ is dense in

$$S^{r-1} \times \cdots \times S^{r-1}$$

in the sense of euclidean metric when $k \geq 2$ and it is not so when $k = 1$. 
This proposition for $k \geq 2$ will be proved in the next paragraph and the case $k = 1$ will be considered in Section 10.

Let us define a map $T: GL_r(\mathbb{Z}) \to S^{r-1} \times \cdots \times S^{r-1}$, where on the right hand side the product of $r-1$ copies of $S^{r-1}$ is considered. Consider any $\alpha = (\alpha_{ij}) \in GL_r(\mathbb{Z})$. Let $(A_{ij})$ be the inverse matrix $(\alpha_{ij})^{-1} \in GL_r(\mathbb{Z})$. Let $y_i$ be the point of intersection of the radial direction in $\mathbb{R}^r$ determined by $(A_{i1}, \ldots, A_{ir})$ and put $T(\alpha) = (y_2, \ldots, y_r)$.

**Corollary 3.1.** The image of $T$ is dense in $S^{r-1} \times \cdots \times S^{r-1}$ (in the sense of euclidean metric).

**Proof.** The map $T$ can be obtained as the composition

$$GL_r(\mathbb{Z}) \xrightarrow{\cdot^{-1}} GL_r(\mathbb{Z}) \xrightarrow{\theta_1} S^{r-1} \times \cdots \times S^{r-1}.$$ 

Thus the previous proposition implies the desired density.

In what follows we will keep to the standard terminology. Let $X_1, \ldots, X_r$ be variables ($r > 1$) and $b_1, \ldots, b_r \in \mathbb{R}$. Then to the closed semispace $b_1 X_1 + \cdots + b_r X_r \geq 0$ ($\exists i_0 b_{i_0} \neq 0$) naturally corresponds a point on $S^{r-1}$, namely the point of intersection of the radial direction of $(b_1, \ldots, b_r)$ with $S^{r-1}$. Thus we obtain the natural bijection $S^{r-1} \approx$ (the set of all semispaces in $\mathbb{R}^r$ passing through 0). Consequently, the mentioned set of semispaces can be equipped with a metric structure ($S^{r-1}$ inherits its metric from the enveloping $\mathbb{R}^r$). We will say that a cone $C \subset \mathbb{R}^r$ is simplicial if the following conditions are satisfied:

(a) 0 is a vertex of $C$,

(b) dim $C = r$,

(c) there exists a hyperplane $H \subset \mathbb{R}^r$, for which $0 \notin H$, $C \cap H$ is a simplex and $C$ is spanned by 0 and $C \cap H$.

Every simplicial cone can be represented as an intersection $E_1 \cap \cdots \cap E_r$ of some semispaces $E_i = (b_{i1} X_1 + \cdots + b_{ir} X_r \geq 0)$ with $\det(b_{ij}) \neq 0$. Of course, such a representation is uniquely determined. Consider any group basis $\alpha_i = (\alpha_{ij})$ of $\mathbb{Z}^r$, $i \in [1, r]$. By $\varepsilon_i$, $i \in [1, r]$ denote the semispace in $\mathbb{R}^r$ determined as follows:

(a) the boundary of $\varepsilon_i$ is a hyperplane of dimension $r - 1$, spanned by $\{0, \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$,

(b) $\alpha_i \in \varepsilon_i$.

The existence and uniqueness of $\varepsilon_i$ is obvious.
Let $C, E_i, \varepsilon_i$ be as above.

**Lemma 3.8.** Let $C = E_1 \cap \cdots \cap E_r$ be a simplicial cone. Then there exists a basis \{\(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\)\} of the free group $\mathbb{Z}^r$ for which the semispaces $E_i$ are sufficiently close (in the sense of the aforementioned metric) to $E$, for all $i = 2, \ldots, r$.

**Proof.** Let $x_2, \ldots, x_r$ be the points from $S^{r-1}$ which correspond to $E_2, \ldots, E_r$. By $x$ denote the point

$$(x_2, \ldots, x_r) = \left( x_{r-1}, x_r \right).$$

According to Corollary 3.7 there exists $\beta \in GL_r(\mathbb{Z})$ for which $T(\beta)$ is sufficiently close to $x$. Let $\beta = (b_{ij})$. In the case $\det(b_{ij}) = 1$ put $\varepsilon_i = (b_{ij})$, $\varepsilon_i \in [1, r]$, $j \in [1, r]$. In this situation we have

$$\varepsilon_i = \det \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \cdots & \vdots \\ b_{i-1,1} & \cdots & b_{i-1,r} \\ X_1 & \cdots & X_r \\ b_{i+1,1} & \cdots & b_{i+1,r} \\ \vdots & \cdots & \vdots \\ b_r & \cdots & b_r \end{pmatrix} \geq 0,$$

or equivalently $\varepsilon_i = (B_{11}X_1 + \cdots + B_{rr}X_r \geq 0)$, where $(B_{ij}) = (b_{ij})^{-1} \in GL_r(\mathbb{Z})$ $(i, j \in [1, r])$. Of course, the system \{\(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\)\} possess the desired property. It can be analogously checked that in the case $\det(b_{ij}) = -1$ the system \{-\(\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_r\)\} possess the desired property.

**Proof of Theorem 3.1.** The interior of $\Phi(M)$ can be represented as a filtered union of simplices $\bigcup \mathcal{A}_i$, $i \in \mathbb{N}$, where $\dim \mathcal{A}_i = \dim \Phi(M)$. Moreover, without loss of generality we can assume that the vertices of $\mathcal{A}_i$ are the $\Phi$-images of some elements from $\text{int}(M)$. Here we use the following obvious observation: the set $\{\Phi(m)\mid m \in \text{int}(M), \ m \neq 0\}$ is dense (in the sense of the euclidean metric) in $\Phi(M)$. Each $\mathcal{A}_i$ determines a simplicial cone $C_i$ with vertex in 0. Let $C_i = E_{i1} \cap \cdots \cap E_{ir}$ $(r = \text{rank}(M))$ for certain semispaces $E_{ij}$ in $\mathbb{R}^r$ ($0 \in E_{ij}$). According to Lemma 3.8 there exist bases $\{\varepsilon_{i1}, \ldots, \varepsilon_{ir}\}$ $(i \in \mathbb{N})$ of $K(M) - \mathbb{Z}^r$, for which $\varepsilon_{i2}, \ldots, \varepsilon_{ir}$ are sufficiently close to $E_{i2}, \ldots, E_{ir}$, respectively $(i \in \mathbb{N})$. We keep to the notations introduced above. In this situation the cones $E_{i1} \cap E_{i2} \cap \cdots \cap E_{ir}$ will be simplicial and they will be contained in the interior of the cone $C$ spanned by $M$. Let $H$
be a hyperplane in $\mathbb{R}^r = \mathbb{R} \otimes \mathbb{K}(M)$ for which $0 \not\in H$ and $C$ is spanned by $0$
and $C \cap H$. Consider the simplices $\delta_i = H \cap (E_{i1} \cap E_{i2} \cap \cdots \cap E_{ir})$. These simplices almost coincide with the simplices $\Delta_i$ (in the sense of the euclidean metric of $H$). Since the boundary hyperplanes of the semispaces $E_i$, $e_i$ are determined by rational linear forms in the standard coordinates of $\mathbb{K}(M) = \mathbb{Z}'$, we come to the conclusion that the vertices of the simplices $\delta_i$ are the $\Phi$-images of some elements from $\text{int}(M) \setminus \{0\}$. Thus $\text{int}(\mathcal{M}) = \bigcup M(\delta_i)$ (the filtered union) and, according to Lemmas 3.2 and 3.3, $\mathcal{M}(\delta_i)$ are finitely generated, normal and $\Phi$-simplicial. Moreover, by Lemma 3.4, $\mathbb{K}(M(\delta_i)) = \mathbb{K}(\text{int}(M)) = \mathbb{K}(M)$. It remains to prove that $M(\delta_i)$ are truncated. Denote by $P_i$ the vertex $H \cap \partial E_{i1} \cap \cdots \cap \partial E_{ir}$ ($i \in \mathbb{N}$) of $\delta_i$, where $\partial e_i$ denotes the boundary of the semispaces $e_i$. Then $x_{hi}$ is the free generator of $\mathcal{M}(\delta_i)(P_i) \approx \mathbb{Z}_+$. We have $\Phi(x_{hi} + ax_{hi}) \rightarrow P_i = \Phi(x_{hi})$ when $a \rightarrow \infty$ (here $a \in \mathbb{N}$ and $j \in \{2, r\}$). Thus for all sufficiently large $a$ ($x_{hi}$, $M_i$, $M(\delta_i)$) is truncated, where $M_i$ is the free monoid $\mathbb{Z}_+ x_{i1} + \mathbb{Z}_+ (ax_{i2} + ax_{i1}) + \cdots + \mathbb{Z}_+ (ax_r + ax_{i1})$. Indeed, to ensure that $M_i$ is integrally closed in $M(\delta_i)$ is just suffices to not that $\mathbb{K}(M_i) = \mathbb{K}(M(\delta_i)) = \mathbb{K}(M)$. Finally, by Lemma 2.6, $M(\delta_i)$ is truncated. Q.E.D.

4. The Density of $\text{Im}(\theta_k)$ for $k \geq 2$

First of all note that the density of the image of $\theta_2$ implies the density of $\text{Im}(\theta_k)$ for all $k \in \{3, r\}$. Thus we will assume $k = 2$.

**Lemma 4.1.** Let $a_1, \ldots, a_r$ be a basis of $\mathbb{Z}'$. Let $v \in \mathbb{Z}'$, $v = \sum_{i=1}^r v_i a_i$. Then for each $m \in \mathbb{Z}$ there exists a basis $a'_1, a'_2, a'_3, \ldots, a'_r$ of $\mathbb{Z}'$ with $a'_2 = a_2 + mv_1 v$.

**Proof.** Considering $\mathbb{Z}'^2 = \mathbb{Z}' / \langle a_3, \ldots, a_r \rangle$ we see that without loss of generality it can be assumed $r = 2$. We have to show that $a'_2$ is unimodular in $\mathbb{Z}'^2$. We can also assume that $a_1 = (1, 0)$ and $a_2 = (0, 1)$. In this situation $a'_2 = (mv_1^2, 1 + mv_1 v_2)$ is actually unimodular since $mv_1^2 \cdot mv_2^2 + (1 + mv_1 v_2)(1 - mv_1 v_2) = 1$.

**Proof of Proposition 3.6.** We have to show the existence of a basis of $\mathbb{Z}'$ for which the radial directions of its elements, except for the first one, approximate arbitrarily fixed directions (from $\mathbb{R}'$). Let $a_1, \ldots, a_r$ be an arbitrary basis of $\mathbb{Z}'$. Choose $v \in \mathbb{Z}'$ with $v_1 \neq 0$ (notations as in Lemma 4.1) approximating the desired direction and then take $m$ large. The result is a new basis $a'_{12}, a'_{12}, a_3, \ldots, a_r$ in which the direction of the second basis vector approximates the desired direction. Repeating this method one realigns the other basis vectors too, except for the first one. This completes the proof of Proposition 3.6.
5. QUASITRUNCATED MONOIDS

Let $C$ be a simplicial cone in $\mathbb{R}^r$ (for some $r \in \mathbb{N}$) in the sense of Section 3. Then if we write $C = C_0 + C_1$, we mean that $C$ is represented as a union of some $r$-dimensional simplex $C_0$ one of the vertices of which is 0 and some convex subset $C_1 \subset \mathbb{R}^r$ for which $C_0 \cap C_1$ is the $(r-1)$-dimensional face of $C_0$ not containing 0. The subsets of $\mathbb{R}^r$ of type $C_1$ will be called (the first word that comes to mind is just “truncated”) quasisimplices.

**Definition 5.1.** A $\Phi$-simplicial monoid $N$ will be called quasitruncated (quasinormal) if there exists a truncated (normal) monoid $M$ with integral extension $N \subset M$ (thus $M$ must be $\Phi$-simplicial because $C(M) = C(N)$) such that $C(M)$ admits the representation $C(M_0) = C_0 + C_1$ of the aforementioned type for which $C_1 \cap M = C_1 \cap N$.

The aim of this section is to prove the following corollary of Theorem 3.1:

**Corollary 5.2.** Let $N$ be a $\Phi$-simplicial monoid. Then $\text{int}(N)$ is a filtered union of quasitruncated monoids.

For any monoid $M$, $\bar{M}$ will denote the integral closure of $M$ (= the smallest integrally closed (normal) intermediate monoid $M \subset L \subset K(M)$). Of course, $\bar{M} = \{ m \in K(M) \mid \exists a_n \in \mathbb{N}, a_m m \in M \}$ (the monoid structure is written additively). Let $c_{\bar{M}/M} = \{ m \in M \mid m + \bar{M} \subset M \}$.

**Lemma 5.3.** Let $M$ be a finitely generated monoid with $U(M) = 0$. Then $c_{\bar{M}/M} \neq \emptyset$.

**Proof.** Since the extension $M \subset \bar{M}$ is integral we obtain $C(M) = C(\bar{M})$. By Gordan’s Lemma 3.3 $\bar{M}$ is finitely generated. Consider the integral extension of monoid rings $\mathbb{Z}[M] \subset \mathbb{Z}[\bar{M}]$. $\mathbb{Z}[\bar{M}]$ is a finitely generated $\mathbb{Z}[M]$-algebra. It is a standard fact of commutative algebra that in this situation $\mathbb{Z}[\bar{M}]$ is a finitely generated $\mathbb{Z}[M]$-module. Let $\{ f_1, \ldots, f_k \}$ generate $\mathbb{Z}[\bar{M}]$ as $\mathbb{Z}[M]$-module ($k \in \mathbb{N}$). We have the following inclusions $\mathbb{Z}[M] \subset \mathbb{Z}[\bar{M}] \subset \mathbb{Z}[K(M)]$. Hence, there exist elements $m_1, \ldots, m_k \in M$ for which $m_1 f_1, \ldots, m_k f_k \in [\mathbb{Z}[M]]$, (here the monoid structure is written multiplicatively). Finally, returning to our additive notations, we have $m_1 + \cdots + m_k \in c_{\bar{M}/M}$.

**Proof of Corollary 5.2.** By Theorem 3.1, $\text{int}(\bar{N})$ can be represented as a filtered union of truncated monoids; say $\text{int}(\bar{N}) = \bigcup_{i}^{\infty} M_i$, where $M_i$’s are truncated and $M_i \subset M_j$ for $i < j$. We have $\text{int}(N) = \bigcup_{i} \left( M_i \cap \text{int}(N) \right)$. Let us prove that for any $i \in \mathbb{N}$ the monoid $M_i \cap \text{int}(N)$ is quasitruncated. In what follows $C$ will denote the cone $C(N)$ and $C_i$ will denote $C(M_i)$ ($i \in \mathbb{N}$).
For any \( x \in \mathbb{R}' \) and any \( Y \subset \mathbb{R}' \), put \( x + Y = \{ x + y \mid y \in Y \} \). Since \( C'_i \)'s are contained in the interior of \( C \), we easily come to the conclusion that for any \( x \in \mathbb{R}' \) the sets \( C_i \setminus (x + C_i) \) are bounded. Let \( m \in C_{K/N} \) (previous lemma). It follows from the boundedness of \( C_i \setminus (m + C_i) \) that we will be done if we show that

\[
(C_i \cap (m + C_i)) \cap M_i \subset \mathbb{M}_i \cap \text{int}(\mathbb{N}).
\]

We have \( m + \mathbb{N} \subset \mathbb{N} \) and \( M_i \subset \text{int}(\mathbb{N}) \). Thus \( (m + \mathbb{N}) \cap M_i \subset \mathbb{M}_i \setminus \text{int}(\mathbb{N}) \).

On the other hand \( \mathbb{M}_i \cap C_i = M_i \) and \( K(M_i) = K(\mathbb{N}) \) (by Lemma 3.4). Hence, \( (C_i \cap (m + C_i)) \cap M_i = (m + \mathbb{N}) \cap M_i \) (here we use the following obvious fact: for any finitely generated normal monoid \( L \) with \( U(L) = 0 \) we have \( L = K(L) \cap C(L) \)). The corollary is proved.

6. **Monic Elements in Quasitruncated Monoid Rings**

All considered rings are assumed to be commutative. As usual for a ring \( \mathbb{A} \) the set of unimodular \( n \)-rows will be denoted by \( Um_n(\mathbb{A}) \). For \( \lambda, \mu \in Um_n(\mathbb{A}) \) we will write \( \lambda \sim \mu \) iff \( \exists x \in E_n(\mathbb{A}) \) (the group of elementary \( n \times n \)-matrices). \( U(\mathbb{A}) \) denotes the group of units, \( \text{Spec}(\mathbb{A}) \) is the prime spectrum of \( \mathbb{A} \) and \( \text{max}(\mathbb{A}) \) is the maximal one. The monoid structure will be written multiplicatively. A commutative monoid ring \( R[M] \) will be called \( \Phi \)-simplicial (truncated, quasitruncated) if \( M \) is so.

Let \( M \) be a \( \Phi \)-simplicial monoid (we always assume that in this situation \( M \) is finitely generated). In the following, the integral extension \( M \subset \mathbb{Z}'_+ \) (\( r = \text{rank}(M) \)), mentioned in Proposition 1.1 will be fixed. Let \( \{ t_1, \ldots, t_r \} \) be the free basis of \( \mathbb{Z}'_+ \). Thus \( M \) can be thought as a monoid consisting of monomials. As usual we will say that \( x \) is lower than \( y \) for some \( x, y \in \mathbb{Z}'_+ \) if \( x = t_1^{a_1} \cdots t_r^{a_r} \) and \( y = t_1^{b_1} \cdots t_r^{b_r} \) with \( a_i < b_i \) for some index \( i \in [1, r] \) and \( a_j = b_j \) for all \( j > i \) (thus \( t_i \) is lower than \( t_j \) iff \( i < j \)). Let \( f \in R[\mathbb{Z}'_+] \), then the highest member \( H(f) \) of \( f \) is defined as follows: \( H(f) = am \), where \( f = am + a_1m_1 + \cdots + a_km_k \) with \( m, m_1, \ldots, m_k \in M \), \( a \in R \setminus \{0\} \), \( a_1, \ldots, a_k \in R \) and \( m_i \)'s are strictly lower (\( i \in [1, k] \) ) than \( m \).

**Definition 6.1.** An element \( f \in R[\mathbb{Z}'_+] \) will be called monic if \( H(f) = at_i \) for some \( a \in U(R) \) and \( s \in \mathbb{Z}'_+ \); an element \( f \in E[M] \) for some \( \Phi \)-simplicial monoid \( M \) will be called monic if \( f \) is monic in \( R[\mathbb{Z}'_+] \) (via the embedding \( R[M] \subset R[\mathbb{Z}'_+] \)).

**Lemma 6.2.** Let \( N \) be a \( \Phi \)-simplicial quasinormal monoid, \( M \) be the enveloping normal monoid mentioned in Definition 5.1 (of course, \( M = \mathbb{N} \)), \( f \in R[N] \), \( J \) be the ideal of \( R[N] \) generated by \( N \setminus \{1\} \), where \( R \) is an
arbitrary commutative ring, and \( \tilde{J} \) be the image of \( J \) in \( R[N]/(f) \). Then for the natural homomorphism \( \psi: R[N]/(f) \to R[M]/(f) \) the ideal \( J \cap \text{Ker } \psi \subset R[N]/(f) \) is nilpotent.

**Proof.** By the definition of quasinormal monoid we obtain that for any \( n \in N \setminus \{1\} \) and \( g \in R[M] \) there exists \( c \in N \) for which \( n^c g \in R[N] \). Now let \( a_1 m_1 + \cdots + a_k m_k \in J \) and \( a_1 m_1 + \cdots + a_k m_k = fh \) for some \( h \in R[M] \). By our remark \( (fh)^c h \in R[N] \) for some \( c \in N \). Consequently, \( (fh)^{c+1} \) maps into 0 in \( R[N]/(f) \).

In the following, for arbitrary \( \Phi \)-simplicial monoid \( L, L_0 \) will denote the submonoid

\[
\{ t_1^{r_1} \cdots t_{r-1}^{r_{r-1}} | s_i \geq 0 \} \cap L \subseteq L \ (r \geq 1);
\]

of course, \( L_0 \) is \( \Phi \)-simplicial as well.

**Lemma 6.3.** Let \( R \) be a commutative ring and \( N \) be a \( \Phi \)-simplicial quasinormal monoid of ran \( \geq 1 \). Then for any monic element \( f \in R[N] \setminus U(R) \) the natural homomorphism \( R[N_0] \to R[N]/(f) \) is an integral extension (the map being considered if obviously mono).

**Proof.** Consider the special case when \( N \) is normal. In this situation \( N = \mathcal{K}(N) \cap \mathbb{Z}_+^r \) (we identify \( N \) with its image in \( \mathbb{Z}_+^r \)). Hence, \( f R[\mathbb{Z}_+^r] \cap R[N] = f R[N] \). Now, the lemma in the special case of normality of \( N \) follows from the commutative square

\[
\begin{array}{ccc}
R[N_0] & \xrightarrow{\alpha} & R[\mathbb{Z}_+^{r-1}] \\
\downarrow & & \downarrow \beta \\
R[N]/(f) & \xrightarrow{\gamma} & R[\mathbb{Z}_+^r]/(f)
\end{array}
\]

with integral extensions \( \alpha \) and \( \beta \) and injective \( \gamma \) (here \( \mathbb{Z}_+^{r-1} = \{ t_1^{r_1} \cdots t_{r-1}^{r_{r-1}} | s_i \geq 0 \} \)). Consider the general case. Let \( M = \bar{N} \). Without loss of generality we have to show that for any \( f_0 \in J \) (we use the notations from Lemma 6.2) its image \( \tilde{f_0} \) in \( RT[N]/(f) \) is integral over \( R[N_0] \). By the above consideration \( \psi(\tilde{f_0}) \) is integral over \( R[M_0] \). Since the composition \( R[N_0] \subset R[M_0] \subset R[M]/(f) \) is integral we obtain that there exists a monic polynomial (in the usual sense) \( F \in R[N_0][X] \), where \( X \) is a variable, for which \( F(\psi(\tilde{f_0})) = 0 \), or equivalently \( F(\tilde{f_0}) \in \text{Ker } \psi \). We can assume that \( F \) does not have a nonzero constant term (it suffices to consider \( XF \) instead of \( F \)). In this situation \( F(\tilde{f_0}) \in J \). By Lemma 6.2, \( F(\tilde{f_0}) \) is nilpotent. Thus \( (F(\tilde{f_0}))^c = 0 \) for some \( c \in N \). In particular, \( \tilde{f_0} \) is integral over \( R[N_0] \).
We recall the notion of height of any ideal $I$ in a commutative ring $A$ [L, Chap. 3]:

$$ht(I) = \min \{ht(p) \mid I \subseteq p \text{ and } p \in \text{spec}(A)\}.$$

**Lemma 6.4** [L, Chap. 3]. Let $\lambda \in Um_n(A)$ for some noetherian $A$ and some $n \in \mathbb{N}$. Then there exists $\mu \in Um_n(A)$ with $\lambda \sim \mu$ and $ht(\mu_1 A + \cdots + \mu_k A) \geq k$ for any $k \in \mathbb{N}$ where $(\mu_1, \ldots, \mu_n) = \mu$.

Let $I$ be an ideal in a $\Phi$-simplicial monoid ring $R[M]$. Then (similar to [L]) $\gamma(I)$ will denote “the ideal of leading coefficients,” more precisely,

$$\gamma(I) = \{a \in R \mid \text{there exists } f \in I \text{ with } H(f) = am \text{ for some } m \in M\}.$$

It is not hard to prove that $\gamma(I)$ is an ideal in $R$.

**Lemma 6.5.** Let $R$ be a noetherian ring and $M$ be a $\Phi$-simplicial monoid (i.e., $R[M]$ is noetherian and $\Phi$-simplicial). Then for any ideal $I \subseteq R[M]$ we have $ht(\gamma(I)) \geq ht(I)$.

**Proof.** Is based on some standard facts of commutative algebra.

**Step 1.** For an integral extension of rings $A \subseteq B$ and a prime ideal $p \subseteq B$ we have $ht(A \cap p) \geq ht(p)$.

**Proof.** In the considered situation there is no inclusion relation between the prime ideals of $B$ lying over a fixed prime ideal of $A$ [Ma, Chap. 1]. Therefore, any chain $p_0 \subsetneq \cdots \subsetneq p_k = p$ in spec($B$) induces the chain $A \cap p_0 \subsetneq \cdots \subsetneq A \cap p_k$ (in spec($A$)) of the same length. Hence, $ht(A \cap p) \leq ht(p)$.

**Step 2.** Let $A \subseteq B$ be an integral extension of rings with $B$ noetherian. Then for any ideal $I \subseteq B$ we have $ht(A \cap I) \geq ht(I)$.

**Proof.** By the previous step is just suﬃces to show that for any $q \in$ spec($A$) containing $A \cap I$ there exists $p \in$ spec($B$) containing $I$ for which $A \cap p \subset q$. Since $B$ is noetherian $\sqrt[q]{I} = \bigcap_{j=1}^{s} p_j$ for some $p_1, \ldots, p_s \in$ spec($B$); therefore $\bigcap_{j=1}^{s} (A \cap p_j) = A \cap \sqrt[q]{I} = A \cap I \subset q$. Since $q$ is prime we obtain $A \cap p_{j_0} \subset q$ for some $j_0 \in [1, s]$.

**Step 3.** Let $R$ be a noetherian ring and $I$ be an ideal in $R[Z'_+]$. Then $ht(\gamma(I)) \geq ht(I)$.

**Proof.** The case $r = 1$ is considered in [L, Chap. 3]. Now the desired inequality follows from the iteration process.

**Step 4.** Let $R, M,$ and $I$ be as in our lemma. Then for each $i \in [1, r]$ there exists natural $c_i$ for which $t_i^c \in M$ (since the extension $M \subseteq Z'_+$ is
integral). The $R$-algebra $A$ in $R[M]$ generated by these elements $t_i^c$ is isomorphic to $R[Z_t^+]$. $\text{ht}(\gamma(I)) \leq \text{ht}(\gamma(A \cap I))$ (since $\gamma(I) \supseteq \gamma(A \cap I)$), $\text{ht}(\gamma(A \cap I)) \geq \text{ht}(A \cap I)$ (Step 3), $\text{ht}(A \cap I) \geq \text{ht}(I)$ (Step 2, the extension $A \subset R[M]$ is integral and $R[M]$ is noetherian since $M$ is finitely generated). The lemma is proved.

In the following $N$ will denote a quasitruncated monoid and $M$ will denote the covering monoid mentioned in Definition 5.1 (thus $M = \tilde{N}$). In addition we will assume that the enumeration of $t_i$'s satisfies the following condition: there exists a truncated triple $(t, M, M')$ with free $M'$ for which $\Phi(t) = \Phi(t_i)$.

**Lemma 6.6.** Let $R[N]$ be a quasitruncated monoid ring and let $\eta$ be an $R$-automorphism of $R[Z_t^+]$ of type

$$
\eta(t_i) = \begin{cases} 
t_r & \text{if } i = r, \\
t_i + t_r^c & \text{if } i < r, c_i \in \mathbb{N}. 
\end{cases}
$$

Then there exist $c, c_i \in \mathbb{N}$ ($i \in [1, r]$) for which the restriction of $\eta$ on $R[N]$ (for appropriate embedding $N \subset \mathbb{Z}_t^+$) is an $R$-automorphism of $R[N]$ whenever $c_i > c$ and $c_i = d_ia_r - a_i$ for some $d_i \in \mathbb{Z}_t^+$.

**Proof.** Let $(t, M, M')$ be a truncated triple of the aforementioned type with $M'$ free. By Lemma 2.6 there exists a truncated triple $(t, M'', M)$ with $M''$ free. In this situation we know that it can be assumed that, writing additively, $M = \langle a_1, \ldots, a_r \rangle \cap \mathbb{Z}_t^+$, where

$$
\begin{align*}
\alpha_1 &= (1, 0, \ldots, a_1), \\
\alpha_2 &= (0, 1, \ldots, a_2), \\
&\quad \vdots \\
\alpha_r &= (0, 0, \ldots, a_r),
\end{align*}
$$

with $a_r > 0$; $\langle x_1, \ldots, x_r \rangle$ denotes the subgroup in $\mathbb{Z}_t$ generated by $x_i$'s (see also the proof of Lemma 2.5). Returning to multiplicative notations we see that $R[M]$ is an $R$-subalgebra in $R[t_1, \ldots, t_r]$ generated by monomials of type $(t_1t^c_r)^{b_1}(t_2t^c_r)^{b_2}\cdots(t_{r-1}t_1^{a_{r-1}})^{b_{r-1}}t_{r}^{b_{r}} \in R[t_1, \ldots, t_r]$ (we can assume $a_1, \ldots, a_{r-1} > 0$) for all possible $b_1, \ldots, b_{r-1}, b_r \in \mathbb{Z}$. Consider the $R$-automorphism of $R[t_1, \ldots, t_r]$ defined as follows: $\eta(t_r) = t_r$ and $\eta(t_i) = t_i + t_r^c$, $i \in [1, r-1]$, where the natural numbers $c_i$ are of the form $c_i = d_ia_r - a_i$ for some $d_i \in \mathbb{N}$. We claim that $\eta(R[M]) \subset R[M]$. Let
Let \( m = (\prod_{i=1}^{r-1} (t_i t_r^{a_i})^{b_i}) t_r^{a_r b_r} \) be an arbitrary element from \( M \), where \( b_1, \ldots, b_r \) are some integral numbers (see above). Then

\[
\eta(m) = \left( \prod_{i=1}^{r-1} (t_i t_r^{a_i} + t_r^{d_{i,r}})^{b_i} \right) t_r^{a_r b_r} \in S^{-1} R[M],
\]

where \( S \subset R[M] \) is the multiplicative subset generated by \( t_r^{a_r} \); indeed, we only have to show that \( b_1, \ldots, b_{r-1} \) cannot be negative numbers, but this condition obviously holds since \( m \in M \subset \mathbb{Z}_+^r \). On the other hand since \( \eta \in \text{Aut}_R(R[t_1, \ldots, t_r]) \) we obtain \( \eta(m) \in R[t_1, \ldots, t_r] \cap S^{-1} R[M] \). Hence, \( \eta(R[M]) \subset R[M] \). Since the same reasonings can be applied to the automorphism

\[
\eta(t_i) = \begin{cases} 
  t_r & \text{if } i = r, \\
  t_i & \text{if } i < r,
\end{cases}
\]

we obtain that the restriction \( \eta|_{R[M]} \) belongs to the group \( \text{Aut}_R(R[M]) \).

Now for our quasitruncated monoid by \( L \) denote its submonoid \( N(\Phi(t_i)) \subset N \). Let \( c \) be a sufficiently large natural number and \( \eta \) be the aforementioned element from \( \text{Aut}_R(R[M]) \) for which \( c_i > c \). Then for arbitrary \( n \in N \setminus L \) we have \( \eta(n) = n + f \), where \( f \) is divisible in \( (R[M]) \) by \( t_r^{d'} \) with sufficiently large \( d' \); indeed, it suffices to check this statement for \( G \cap (N \setminus L) \), where \( G \) denotes the finite generating set for \( N \), but this is a direct consequence of the definition of \( \eta \). Note that \( \eta(n) = n \) for any \( n \in L \).

It follows from the definition of a quasitruncated monoid that if \( c \) is sufficiently large then \( \eta(N) \subset R[N] \). In other words \( \eta|_{R[N]} \in \text{Aut}_R(R[N]) \).

In the following by \( \tilde{H}(f) \) will be denoted the highest term of \( f \) in correspondence with the order on \( \mathbb{Z}_+^r \) when \( t_i \) is lower than \( t_j \) iff \( i > j \) (the word "monic" will be used in the usual sense).

**Lemma 6.7.** Let \( R[N] \) be a quasitruncated monoid ring and \( f \in R[N] \) with \( \tilde{H}(f) = ut_i^{c_1} \cdots t_i^{c_r} \) for some \( u \in U(R) \). Then there exists an \( R \)-automorphism \( \eta \) of \( R[N] \) for which \( \eta(f) \) is monic.

**Proof.** It is a standard fact that there exists a sequence of natural numbers \( 0 \leq c_r \leq c_{r-1} \leq \cdots \leq c_2 \leq c_1 \) for which \( \eta(f) \) is monic (see [1, Chap. 3]), where \( \leq \) means sufficiently greater and

\[
\eta(t_i) = \begin{cases} 
  t_r & \text{if } i = r, \\
  t_i + t_r^{c_i} & \text{if } i > r.
\end{cases}
\]

By Lemma 6.6, \( c_i's \) can be chosen so that \( \eta|_{R[N]} \in \text{Aut}_R(R[N]) \).
LEMMA 6.8. Let $R$ be a noetherian ring with Krull dimension $d < \infty$, $N$ be a quasitruncated monoid and $F = (f_1, \ldots, f_n) \in \text{Um}_n(R[N])$, where $n \geq d + 2$. Then there exists $\eta \in \text{Aut}_R(R[N])$ and $G = (g_1, \ldots, g_n) \in \text{Um}_n(R[N])$ with $g_n$ monic for which $\eta(F) = (\eta(f_1), \ldots, \eta(f_n)) \sim G$.

Proof. By Lemma 6.4 there exists $\alpha \in E_n(R[N])$ for which $\text{ht}(h_1 R[N] + \cdots + h_k R[N]) \geq k$, $k \in [1, n]$, where $H = (h_1, \ldots, h_n) = F\alpha \in \text{Um}_n(R[N])$. Denote by $I$ the ideal $h_1 R[N] + \cdots + h_{n-1} R[N] < R[N]$. Thus by Lemma 6.5 we obtain $\text{ht}(\hat{\gamma}(I)) \geq \text{ht}(I) \geq n-1 \geq d + 1$ (here $\hat{\gamma}$ corresponds to $\hat{H}$), $\hat{\gamma}(I)$ must coincide with $R$. It just means that there exists $v \in I$ with $\hat{H}(v) = ut^n \cdots t^n$, where $u \in U(R)$. By Lemma 6.7 there exists $\eta \in \text{Aut}_R(R[N])$ for which $\eta(v)$ is monic. It is obvious that $\eta(v)^m + \eta(h_n)$ will be monic as well for sufficiently large natural $m$. In conclusion we have

$$\eta(F) \sim \eta(H) = (\eta(h_1), \ldots, \eta(h_{n-1}), \eta(h_n))$$

$$\sim (\eta(h_1), \ldots, \eta(h_{n-1}), \eta(v)^m + \eta(h_n)).$$

So $\eta$ and $G = (\eta(h_1), \ldots, \eta(h_{n-1}), \eta(v)^m + \eta(h_n))$ are the desired objects.

LEMMA 6.9. Let $R$ be a noetherian ring with Krull dim $R = d < \infty$ and $F \in \text{Um}_n(R[N])$ for some quasitruncated $N$ and $n \geq \max(d + 2, 3)$. Then there exists $\eta \in \text{Aut}_R(R[N])$ such that $\eta(F) \sim (1, 0, \ldots, 0)$ over $R[N]_{\text{gr}}$ for each $\mathfrak{M} \in \max(R[N_0])$.

Proof. By previous lemmas $\eta(F) \sim (f_1, \ldots, f_n)$ with $f_n$ monic for appropriate $\eta$. Denote by $\tilde{f}_i$ ($i \in [1, n-1]$) the image of $f_i$ in $R[N]_{\text{gr}}/(f_n)$. By Lemma 6.3 the extension $R[N_0]_{\text{gr}} \subset R[N]_{\text{gr}}/(f_n)$ is integral and since the considered rings are noetherian and $R[N_0]_{\text{gr}}$ is local we obtain the semilocality of $R[N]_{\text{gr}}/(f_n)$. But the elementary action on the set of unimodular rows of arbitrary length $>1$ over a semilocal ring is transitive. Consequently,

$$(\tilde{f}_1, \ldots, \tilde{f}_{n-1}) \sim (1, 0, \ldots, 0).$$

Therefore,

$$(f_1, \ldots, f_{n-1}, f_n) \sim (1 + g_1 f_n, \ldots, g_{n-1} f_n, f_n)$$

over $R[N]_{\text{gr}}$ for some $g_1, \ldots, g_{n-1} \in R[N]_{\text{gr}}$. It remains to note that

$$(1 + g_1 f_n, \ldots, g_{n-1} f_n, f_n) \sim (1, 0, \ldots, 0).$$
7. Quillen's Patching for Surjective $K_1$-Stabilizations

One knows that Quillen's patching theorem concerning the projective modules over a polynomial ring $[Q]$ admits the natural generalization for a graded situation [Ch2, G1, G2, L]:

**Proposition 7.1.** Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring $P$ be a finitely generated projective $A$-module. Then $P$ is extended from $A_0$ iff $P_\mu = (A_0)_\mu^{-1}P$ are extended from $(A_0)_\mu$ for all $\mu \in \text{max}(A_0)$.

In [G2] we proved the analogous generalization of Quillen's patching's $K_1$-analog established in [Su] (it should be noted that these generalized versions do not make use of any essential changes in the proofs). Now we need the same patching for surjective $K_1$-stabilization topics over a graded ring.

First, some words about notations. For arbitrary ring $A$ and its ideal $I$, $E_n(A, I)$ denotes the normal subgroup in the elementary matrix group $E_n(A)$ spanned by $e_{ij}(x)$, where $e_{ij}$ means the elementary matrix with unique nondiagonal component on $i$th row and $j$th column and $x$ ranges over $I$. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring, $f \in A$ and $a \in A_0$, then \( f^+(a) \) denotes $\xi(f)(a)$ where $\xi$ is the ring homomorphism $A \to A[X]$ ($X$ is a variable) for which $\xi(a_0 \oplus a_1 \oplus a_2 \oplus \cdots) = a_0 + a_1 X + a_2 X^2 + \cdots$ and $\xi(f)(a)$ is the image of $\xi(f)$ under $X \mapsto a$. For $a \in GL_n(A)$ and $a \in A_0$ analogously is defined $a^+(a)$. $A^+ = 0 \oplus A_1 \oplus A_2 \oplus \cdots$. The image of some matrix $\delta$ under the localization map relative to some element $x$ will be denoted by $\delta_x$.

**Lemma 7.2 [Su].** Let $A$ be a ring, $a \in A$, $n \geq 3$, $y \in GL_n(A_a)$, $z$ be a variable and $f \in A_a[z]$. Put $\sigma(z) = y e_i(z f) y^{-1}$ for some $i \neq j$. Then for all sufficiently large $s \in \mathbb{N}$ there exists $\tau \in E_n(A[z], zA[z])$ with $\tau_a = \sigma(a^+ z)$.

For a ring $A$, $R_k(A)$ will denote the subset in $GL_n(A)$ ($k, n \in \mathbb{N}, k \leq n$) consisting of those matrices which are "reducible" to some $k \times k$ matrices. More precisely: $a \in R_k(A)$ iff there exists $\epsilon \in E_n(A)$ with

$$\alpha = \epsilon \begin{pmatrix} 1_{n-k} & 0 \\ 0 & \alpha' \end{pmatrix},$$

where $1_{n-k}$ means the identity $(n-k) \times (n-k)$ matrix and $\alpha' \in GL_k(A)$. Actually in the case $n \geq 3$ the set $R_k(A)$ turns out to be a subgroup in $GL_n(A)$. This is a consequence of the fact that $E_n(A)$ (and $E_n(A, I)$ as well) is a normal subgroup in $GL_n(A)$ whenever $n \geq 3$ [Su].
Proposition 7.3. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring, $n \geq 3$, $k \leq n$ and $z \in GL_n(A, A^+)$. Then $z \in R_k(A)$ if and only if $z \in R_k(A_\mu)$ for all $\mu \in \max(A_0)$.

Here $A_\mu = (A_0 \setminus \mu)^{-1}A$.

Proof. Step 1. Let $\beta \in GL_n(A)$ and $\beta_{a} \in R_k(A_a)$ for some $a \in A_0$. Then there exists a natural number $s$ for which $\beta^+(c)(\beta^+(d))^{-1} \in R_k(A)$ whenever $c = d \mod a^s (c, d \in A_0)$.

Proof. Put $\beta(y, z) = \beta_{a}^+(y)(\beta_{a}^+(y + z))^{-1}$, where $y$ and $z$ are variables (we use the fact that $R_k$ is a group). We have $\beta(y, z) \in R_k(A_a[y, z]) \cap GL_n(A_a[y, z], (z))$. Thus $\beta(y, z) = (\prod_{k=1}^{n} e_{i_{\mu}}(b_k + z f_k)) \beta_{1}(y, z)$ for some $b_k + A_a[y]$, $f_k \in A_a[y, z]$ and

$$\beta_{1}(y, z) = \begin{pmatrix} 1_{n-k} & 0 \\ 0 & \beta_{0}(y, z) \end{pmatrix},$$

where $\beta_{0}(y, z) \in GL_k(A_a[y, z])$. For each $p \in [1, m]$ put $\gamma_p = \prod_{k=1}^{p} e_{i_{\mu}}(b_k) \in E_n(A_a[y])$. We have $\beta(y, z) = (\prod_{k=1}^{m} \gamma_k e_{i_{\mu}}(z f_k)) \gamma_k^{-1} \cdot \gamma_{m} \cdot \beta_{1}(y, z)$. By Lemma 7.2 there exists $s \in \mathbb{N}$ with

$$\prod_{k=1}^{m} \gamma_k e_{i_{\mu}}(a^s z f_k) \gamma_k^{-1} = \tau_{0}(y, z)$$

for some $\tau(y, z) \in E_n(A[y, z], (z))$. Therefore,

$$\beta(y, a^m z) = \tau_\sigma(y, z) \gamma_m \beta_1(y, a^m z).$$

Since $\beta(y, 0) = \tau_\sigma(y, 0) = 1_n$ we obtain that

$$\gamma_m = \begin{pmatrix} 1_{n-k} & 0 \\ 0 & \beta_{0}(y, 0)^{-1} \end{pmatrix}$$

and $\gamma_m \beta_1(y, a^m z) \in GL_n(A_a[y, z], (z))$. From this observation we see that for sufficiently large $s \in \mathbb{N}$

$$\gamma_m \beta_1(y, a^{n+m} z) = \begin{pmatrix} 1_{n-k} & 0 \\ 0 & \beta_{a}^+(y, z) \end{pmatrix},$$

where $\beta'(y, z) \in GL_k(A[y, z])$. Indeed, it just suffices to note that for arbitrary ring $A$, $x \in A$, $\sigma_1(x) \in GL_n(A[x], (z))$, $\sigma_2(x) \in GL_n(A[z], (z))$ ($z$ is a variable) with $\sigma_1(x) = \sigma_2(x)$, we have $\sigma_1(x^i z) = \sigma_2(x^i z)$ for all sufficiently large $s \in \mathbb{N}$. Therefore,

$$\beta(y, a^{n+m} z) = \tau_\sigma(y, a^m z) \begin{pmatrix} 1_{n-k} & 0 \\ 0 & \beta_{a}^+(y, z) \end{pmatrix}.$$
and (again using the same arguments)

\[ x^+(y)(x^+(y + a^{s_1 + s_2 + s_3}))^{-1} = \tau(y, a^{s_2 + s_3}z) \begin{pmatrix} 1 - 1 & 0 \\ 0 & \beta'(y, z) \end{pmatrix} \]

for some sufficiently large \( s_3 \in \mathbb{N} \). So the number \( s_1 + s_2 + s_3 \) possesses the desired property.

**Step 2.** Let \( n \geq 3 \), \( k \leq n \), \( x \in GL_n(A,A^+) \), \( a, b \in A_0 \) with \( uA_0 + bA_0 = A_0 \). Assume \( a \in R_k(A_a) \) and \( b \in R_k(A_b) \). Then \( a \in R_k(A) \).

**Proof:** By the previous step there exists a natural number \( s \) for which \( x^+(c)(x^+(d))^{-1} \in R_k(A) \) whenever one of the conditions \( c = d \mod a^s \) and \( c = d \mod b^s \) is satisfied (\( c, d \in A_0 \)). Since \( a \) and \( b \) are comaximal \( a', b' \) will be also. Thus \( a'g + b'h - 1 \) for some \( g, h \in A_0 \). We have

\[ x = x(a(0))^{-1} = (x(1)(x(a'g))^{-1}) = (x(a'g)(x(0))^{-1}) \in R_k(A) \]

(just here we use the aforementioned remark that \( R_k(A) \) is multiplicatively closed in \( GL_n(A) \) when \( n \geq 3 \)).

**Step 3.** Let \( n \geq 3 \), \( k \leq n \) and \( x \in GL_n(A,A^+) \). Then the set \( I = \{ a \in A_0 | x_a \in R_k(A_a) \} \) is an ideal in \( A_0 \).

**Proof:** It suffices to show that \( a + b \in I \) whenever \( a, b \in I \). But \( aA_{a+b} + bA_{a+b} = A_{a+b} \). By Step 1 we have \( (x_{a+b})_a \in R_k((A_{a+b})_a) \), \( (x_{a+b})_b \in R_k((A_{a+b})_b) \Rightarrow x_{a+b} \in R_k(A_{a+b}) \Rightarrow a + b \in I \).

**Step 4.** Let \( A, n, k, \alpha \) be as in the proposition. Then for each \( \mu \in \max(A_0) \) there exists \( a(\mu) \in A_0 \setminus \mu \) for which \( x_{a(\mu)} \in R_k(A_{a(\mu)}) \). Since \( \{ a(\mu) | \mu \in \max(A_0) \} \) generates the unit ideal in \( A_0 \), Step 3 implies \( x \in R_k(A) \). Q.E.D.

It remains to note that our proof is a variation of the corresponding proof from [Su]. The following corollary for \( A = A_0[t] \) is contained in [R].

**Corollary 7.4.** Let \( A = A_0 \oplus A_1 \oplus \cdots \) be a graded ring, \( n \geq 3 \), \( F \in \text{Um}_n(A) \), the natural image of \( F \) in \( \text{Um}_n(A_0) \) be \( (1, 0, \ldots, 0) \) and \( F \sim (1, 0, \ldots, 0) \) over each \( A_\mu \), where \( \mu \) varies over \( \max(A_0) \). Then \( F \sim (1, 0, \ldots, 0) \) (over \( A \)).

**Proof:** Since \( F \) defines a stably free \( A \)-module which is locally extended from a free \( A_0 \)-module by Proposition 7.1 this module must be free; consequently there exists \( x \in GL_n(A) \) with \( Fx = (1, 0, \ldots, 0) \), or equivalently \( F \) can
be completed to some invertible matrix $\beta$ (for details see, for example, [L]). Using the elementary actions on the first column of $\beta$ we see that without loss of generality it can be assumed that

$$\beta = \begin{pmatrix} 1 + a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad F = (1 + a_{11}, a_{12}, \ldots, a_{1n}),$$

where $a_{ij}, a_{ii} \in A^+$ ($i, j \in [1, n]$). Denote by $\pi$ the natural homomorphism $A \to A_0$ with $\text{Ker } \pi = A^+$. Then

$$\pi(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & \beta' \end{pmatrix}$$

for some $\beta' \in GL_{n-1}(A_0)$. Consider the matrix $\gamma = \beta \pi(\beta)^{-1} \in GL_n(A, A^+)$. Let us show that $\gamma \in R_{n-1}(A)$. By Proposition 7.3 it suffices to show that $\gamma \in R_{n-1}(A_\mu)$ for each $\mu \in \text{max}(A_0)$. Using the condition $F \sim (1, 0, \ldots, 0)$ over each $A_\mu$ the normality of $E_n(A_\mu)$ in $GL_n(A_\mu)$ implies $\beta \in R_{n-1}(A_\mu)$. Therefore, $\gamma \in R_{n-1}(A_\mu)$. Thus the first row of $e\beta \pi(\beta)^{-1}$ for some $e \in E_n(A)$ is $(1, 0, \ldots, 0)$. But then the first row of $e\beta$ is also $(1, 0, \ldots, 0)$. Again by the normality of $E_n$ in $GL_n$ there exists $e' \in E_n(A)$ for which $e\beta = e' \beta' \Rightarrow F e' = (1, 0, \ldots, 0) \Leftrightarrow F \sim (1, 0, \ldots, 0)$.

8. MAIN THEOREM

**Theorem 8.1.** Let $M$ be a submonoid in $Q^r_+$ for some $r \geq 0$ with the extension $M \subset Q^r_+$ integral. Then for arbitrary noetherian ring $R$ with Krull dimension $d < \infty$ the group $E_n(R[M])$ acts transitively on $Um_n(R[M])$ whenever $n \geq \text{max}(d + 2, 3)$.

**Question.** Let $R$ be as in the theorem and $M$ be any (commutative, cancellative) monoid of some finite rank $r$. Is the action of $E_n(R[M])$ on $Um_n(R[M])$ transitive where $n \geq \text{max}(d + 2, 3)$?

In the first special case, not covered by our theorem when $M$ is a group the action being considered actually is transitive by [Su].

**Corollary 8.2.** Let $R, M, d,$ and $n$ be the same as they were in the theorem. Then

(a) the natural homomorphism $GL_{n-1}(R[M]) \to K_1(R[M])$ is surjective,
(b) all finitely generated projective $R[M]$-modules of rank more than $d$ which are stably extended from $R$ are actually extended from $R$.

(c) let $R$ be a noetherian 1-dimensional ring; then all finitely generated stably free $R[M]$-modules are free.

Proof. (a) is obvious. (b) and (c) easily follow from Theorem 8.1 if one will use Quillen’s patching (for projective modules) and the elementary observation that any unimodular 2-row over an arbitrary ring is completable to an element from $GL_2$.

**Lemma 8.3.** Let Theorem 8.1 hold for some natural $r \geq 1$. Then for any $\Phi$-simplicial monoid $M$ the group $E_n(R[\int(M)])$ acts transitively on $Um_n(R[\int(M)])$, where $R$ and $M$ are the same as they were in Theorem 8.1.

Proof. The lemma follows from the following obvious observation: the interior of $\Phi(M)$ can be represented as a filtered union of “rational” closed simplices which (by Gordan’s lemma) excise finitely generated submonoids in $\int(M)$.

**Lemma 8.4.** Let Theorem 8.1 hold for monoids of rank $<r$ for some $r \geq 1$, $M$ be a $\Phi$-simplicial monoid of rank $=r$, $d$ and $n$ be as in Theorem 8.1, and $F \in Um_n(R[M])$. Then there exist $\varepsilon \in E_n(R[M])$ and $G \in Um_n(R[\int(M)])$ for which $F\varepsilon = G$.

Proof. Since for $r = 1$ there is nothing to prove we will assume that $r > 1$. Let $P$ be any vertex of $\Phi(M)$ and put $M' = M(\Phi(M) \setminus \{P\})$. We have the $R$-retraction $R[M(P)] \rightarrow R[M]$ induced by

$$
\pi(m) = \begin{cases} m, & m \in M(P), \\ 0, & m \notin M(P). \end{cases}
$$

Under our conditions there exists $\varepsilon' \in E_n(R[M(P)]) \subset E_n(R[M])$ for which $F\varepsilon' (R[M'])^*$. Let us show that actually $F\varepsilon' \in Um_n(R[M'])$. Since $F\varepsilon' \in Um_n(R[M])$ there exist $g_1, \ldots, g_n$ for which $f_1 g_1 + \cdots + f_n g_n = 1$, where $F\varepsilon' = (f_1, \ldots, f_n)$. Thus for any $m \in M' \setminus \{1\}$ we have

$$m = f_1(mg_1) + \cdots + f_n(mg_n), \quad mg_1, \ldots, mg_n \in R[M'].$$

In other words $M' \setminus \{1\} = f_1 R[M'] + \cdots + f_n R[M']$. Since the constant terms of $f_i's$ generate the unit ideal we must have $F\varepsilon' \in Um_n(R[M'])$. By the same arguments we easily see that there exists $\varepsilon'' \in E_n(R[M])$ for which $F\varepsilon'' \subset Um_n(R[M''])$, where $M'' = M(\Phi(M) \setminus$ the set of all vertices of $\Phi(M))$. By virtue of Lemma 8.3 the analogous arguments show that there exists $\varepsilon''' \in E_n(R[M])$ for which $F\varepsilon''' \subset Um_n(R[M(\Phi(M) \setminus$ the union of all 1-dimensional faces of $\Phi(M))])$ (here we have to consider $R$-retractions of
the type $R[N] \rightarrow R[M']$, where $\Phi(N)$ are the interiors of 1-dimensional faces of $\Phi(M)$. By "killings" of the interiors of high dimensional faces of the simplex $\Phi(M)$ we complete the proof.

Proof of Theorem 8.1. Since the monoids considered in the theorem can be represented as filtered unions of $\Phi$-simplicial monoids the task to be solved at once reduces to the special case of finite generation. We carry out the proof by induction on $r$. The case $r = 0$ follows from the corresponding classical results from $[B, V]$. Assume $r > 0$. Let $R$, $M$, $d$, $n$ be as in the theorem and $F \in Um_n(R[M])$. We claim that $F \sim (1, 0, \ldots, 0)$. By Lemma 8.4 it can be assumed that $F \in Um_n(R[\text{int}(M)])$. By Corollary 5.2 we can assume that $F \in Um_n(R[N])$ for some quasitruncated $N$. Let $N_0$ be as in Lemma 6.3. By Lemma 6.9 there exists $\varphi : Aut_R(R[N])$ such that $\varphi(F) \sim (1, 0, \ldots, 0)$ over $R[N]_{\text{max}}$ for each $\varphi \in \text{max}(R[N_0])$. By induction hypothesis there exists $\varepsilon \in E_n(R[N_0])$ for which the natural image of $\varepsilon(F)$ in $R[N_0]_{\text{max}}$ is $(1, 0, \ldots, 0)$. Since $R[N]$ is a graded $R[N_0]$-algebra ($R[N] = R[N_0] \oplus A_1 \oplus A_2 + \cdots$) we obtain that Corollary 7.4 implies the desired result. Q.E.D.

9. The Failure of the Direct $K_1$-Analog of Anderson's Conjecture

By the direct analog (for the functor $K_1$) of Anderson's conjecture we mean the hypothesis concerning natural isomorphisms of the type $K_1(R) \rightarrow K_1(R[M])$, where $R$ is a regular ring and $M$ is a normal monoid without nontrivial units. Somehow unexpectedly (from the main theorem of $[G1]$) it turns out that this isomorphism $K_1(R) \rightarrow K_1(R[M])$ does not hold even for the simplest representatives of $R$ and $M$ of the aforementioned type. For example, if we consider the normal $M \subset \mathbb{Z}_+^2$ generated by (writing additively) $\{(2, 0), (1, 1), (0, 2)\}$ then, according to $[Sr]$, $SK_1(\mathbb{C}[M])$ turns out to be an infinite abelian group ($\mathbb{C}$ means the complex numbers). It should be noted that $M$ actually is a simplest nonfree finitely generated normal monoid: $\text{Cl}(M) = \mathbb{Z}_2$. In this section we will show that for arbitrary regular ring $R$ there exist infinitely many finitely generated normal submonoids $M \subset \mathbb{Z}_+^2$ for which the natural homomorphisms $K_1(R) \rightarrow K_1(R[M])$ are not isomorphisms. It should be noted that it seems very probable that $K_1(R) \rightarrow K_1(R[M])$ if and only if $M$ is free.

Let $X$ and $Y$ be variables, for a ring $A$, let $A[x, y]$ be the quotient ring $A[X, Y]/(XY)$. According to $[D-Kr, Sw]$ we have

**Proposition 9.1.** Let $A$ be any ring. Then the natural homomorphism $K_2(A) \rightarrow K_2(A[x, y])$ is not an isomorphism.
In what follows the submonoids in $\mathbb{Z}_+^2$ will be thought of as multiplicative monoids consisting of the corresponding monomials in $X$ and $Y$.

**Proposition 9.2.** Let $R$ be a regular ring. Then the natural homomorphism $K_1(R) \to K_1(R[\int(\mathbb{Z}_+^2)])$ is not an isomorphism.

**Proof.** Consider the cartesian square

$$
\begin{array}{ccc}
A & \xrightarrow{f} & R[X, Y] \\
\downarrow & & \downarrow \\
R[X, Y] & \longrightarrow & R[x, y].
\end{array}
$$

We have $K_1(R[X, Y], [XY]) = \ker(K_1(f))$ [Mi]. The sequence

$$
0 \to (XY) \to R[X, Y] \to R[x, y] \to 0
$$

induces the following exact sequence [Mi]:

$$
K_2(R[X, Y]) \to K_2(R[x, y]) \to K_1(R[X, Y], (XY)).
$$

By the regularity of $R$ we have $K_2(R[X, Y]) = K_2(R)$; So by Proposition 9.1, $K_1(R[X, Y], (XY)) \neq 0$.

Assume $K_1(R) \cong K_1(R[\int(\mathbb{Z}_+^2)])$. Let us prove that in this situation $K_1(R[X, Y], (XY)) = 0$. We will prove that $K_1(f): K_1(A) \to K_1(R[X, Y])$ is an isomorphism. By virtue of the regularity of $R$ we have $K_1(R) = K_1(R[X]) = K_1(R[Y]) = K_1(R[X, Y])$. Consider the $R$-homomorphisms $\pi_1: R[X, Y] \to R[X]$ with $\pi_1(X) = X$, $\pi_1(Y) = 0$ and $\pi_2: R[X, Y] \to R[Y]$ with $\pi_2(X) = 0$, $\pi_2(Y) = Y$. Let $[\mu]$ be an arbitrary element from $K_1(A)$, $\mu \in GL(A)$. Then $\mu = (\alpha, \beta)$ for some $\alpha, \beta \in GL(R[X, Y])$ with $\pi_i(\alpha) = \pi_i(\beta)$, $i = 1, 2$. Since $K_1(R) = K_1(R[X])$ there exists $e_1 \in E(R[X])$ with $e_1 \pi_1(\alpha) = e_1 \pi_1(\beta) \in GL(R)$. Analogously $e_2, e_3, e_4, e_5, e_6 \in GL(R)$ for some $e_i \in E(R[Y])$. Therefore, $\pi_i(e_\mu) \in GL(R)$ (i = 1, 2) where $e = (e_2, e_1, e_2, e_1) \in E(A)$ (here we identify $GL(R)$ with its natural image in $GL(A)$). Let $e\mu = (\alpha', \beta')$ for some $\alpha', \beta' \in GL(R[X, Y])$. By the aforementioned remarks we have $\alpha', \beta' \in GL(R[\int(\mathbb{Z}_+^2)])$. By our assumption $e_1 \alpha' \in GL(R)$ for some $e_1 \in E(R[\int(\mathbb{Z}_+^2)])$. Let $\pi$ be the natural augmentation $R[\int(\mathbb{Z}_+^2)] \to R$ (i.e., $\pi(x^a y^b) = 0$, $a > 0$, $b > 0$). Then $(e_1, \pi(e_1)) \in E(A)$. Put $e_1 = (e_1, \pi(e_1))$. Analogously $e_2, e_3, e_4, e_5, e_6 \in GL(R)$ for some $e_2 \in E(R[\int(\mathbb{Z}_+^2)])$. Put $e_2 = (\pi(e_1), e_5)$. We have $e_2 \in E(A)$ and $e_2 e_1, e_4, e_6 \in GL(R)$. Hence, $[\mu] \in \text{Im}(K_1(R) \to K_1(A))$ and the natural homomorphism $K_1(R) \to K_1(A)$ is surjective. On the other hand, $R$ is a retract of $A$; finally $K_1(R) \to K_1(A)$. So we conclude that $K_1(R[X, Y], (XY)) = 0$, a contradiction.
COROLLARY 9.3. Let \( \text{int}(\mathbb{Z}^+_{\times}) \) be represented as a filtered union of finitely generated normal monoids \( M_i \) and \( R \) be a regular ring. Then the homomorphisms \( K_1(R) \to K_1(R[\mathbb{Z}^+_M]) \) are not isomorphisms for all but finite \( M_i \)'s.

The mentioned representation \( \text{int}(\mathbb{Z}^+_{\times}) = \bigcup_{i=1}^{\infty} M_i \) can be obtained, for example, by an arbitrary representation of the open interval which corresponds to \( \text{int}(\mathbb{Z}^+_{\times}) \) (via \( \Phi \)-correspondence) as the filtered union of closed "rational" segments.

10. NONDENSIY OF \( \text{Im}(\theta_1) \)

PROPOSITION 10.1. The image of \( \theta_1: GL_2(\mathbb{Z}) \to S^1 \times S^1 \) is not dense in \( S^1 \times S^1 \).

Proof. Of course, we could use direct (elementary) geometric arguments but I will show how this proposition follows from Proposition 9.2. If \( \text{Im}(\theta_1) \) is dense then free monoids are distributed densely in \( \mathbb{Z}^+_{\times} \). Hence, \( \text{int}(\mathbb{Z}^+_{\times}) \) will be a filtered union of free monoids of rank 2. Since \( K_1(R) = K_1(R[\mathbb{Z}^+_{\times}]) \) we shall have \( K_1(R) \to K_1(R[\text{int}(\mathbb{Z}^+_{\times})]) \), a contradiction.

Remarks. It can be analogously shown that

\[
\theta_1: GL_r(\mathbb{Z}) \to S^{r-1} \times \cdots \times S^{r-1}
\]

has not a dense image as well where \( r > 2 \). On the other hand, if we consider the analogous map

\[
\theta: GL_r \left( \mathbb{Z} \left[ \begin{array}{c} 1 \\ c \end{array} \right] \right) \to S^{r-1} \times \cdots \times S^{r-1}
\]

for arbitrary natural \( c > 1 \) then it turns out that \( \text{Im(\theta)} \) is dense. This statement merely coincides with the approximation theorem A from [G2], where we proved that \( K_1(R) \to K_1(R[M]) \) for regular \( R \) and any monoid \( M \) with \( U(M) = 0 \), for which there exists a natural \( c > 1 \) satisfying the condition (writing additively): \( \forall m \in U, \exists n \in U, \ c n = m \). The analogous isomorphisms for \( K_2 \) under the additional condition of \( M \in \mathbb{Q}^+ \) being integral are also established in [G2].

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REFERENCES


[V] L. N. VASERShteIN, On the stabilization of the general linear group over a ring, Math. USSR-Sb. 8 (1968), 383–400.