

# NORMALITY AND COVERING PROPERTIES OF AFFINE SEMIGROUPS

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## 1. INTRODUCTION

An *affine semigroup* is a finitely generated subsemigroup of a finitely generated free abelian group  $\mathbb{Z}^n$ . Let  $\text{gp}(S)$  denote the subgroup of  $\mathbb{Z}^n$  generated by  $S$ . Then the *normalization* of  $S$  is the subsemigroup

$$\bar{S} = \{x \in \text{gp}(S) \mid mx \in S \text{ for some } m > 0\}.$$

One calls  $S$  *normal* if  $S = \bar{S}$ . For simplicity we will often assume that  $\text{gp}(S) = \mathbb{Z}^n$ ; this is harmless because we can replace  $\mathbb{Z}^n$  by  $\text{gp}(S)$  if necessary. The *rank* of  $S$  is the rank of  $\text{gp}(S)$ . We will only be interested in the case in which  $S \cap (-S) = 0$ ; such affine semigroups will be called *positive*. The positivity of  $S$  is equivalent to the pointedness of the cone  $C(S) = \mathbb{R}_+ S$  generated by  $S$  in  $\mathbb{R}^n$ ; one has  $\text{rank } S = \dim C(S)$ . (All the cones appearing in this paper have their apex at the origin.)

The normality of  $S$  can now be characterized geometrically: one obviously has  $\bar{S} = C(S) \cap \mathbb{Z}^n$ , and so  $S$  is normal if and only if  $S = C(S) \cap \mathbb{Z}^n$ . Conversely, every convex, pointed, finitely generated rational cone  $C \subset \mathbb{R}^n$  yields a normal affine semigroup  $S(C) = C \cap \mathbb{Z}^n$  (this semigroup is finitely generated by Gordan's lemma). If we simply speak of a cone  $C$  in the following, then it is always assumed that  $C$  is convex, pointed, finitely generated, and rational. Since  $\dim C(S) = \text{rank } S$ , we often call the dimension of a cone its *rank*.

Each positive affine semigroup  $S$  can be embedded into  $\mathbb{Z}_+^m$ ,  $m = \text{rank } S$  (one uses  $m$  linearly independent integral linear forms representing support hyperplanes of  $C(S)$ ). Therefore every element can be written as a sum of irreducible elements, and the set of irreducible elements of  $S$  is finite since  $S$  is finitely generated. For a cone  $C$  the set of irreducible elements of  $S(C)$  is often called the *Hilbert basis*  $\text{Hilb}(C)$  of  $C$  in the combinatorial literature, a convention we will follow. More generally, we also call the set of irreducible elements of a positive affine semigroup  $S$  its Hilbert basis and denote it by  $\text{Hilb}(S)$ .

This paper is devoted to a discussion of sufficient and potentially necessary conditions for the normality of  $S$  in terms of combinatorial properties of  $\text{Hilb}(S)$ . As far as the necessity is concerned, these conditions can of course be formulated in terms of Hilbert bases of rational cones. The first property under consideration is the existence of a *unimodular (Hilbert) cover*:

**(UHC)** *Let  $C \subset \mathbb{R}^n$  be a finite, rational and pointed convex cone. Then  $\text{Hilb}(C)$  has a family of subsets  $\{X_i\}$  satisfying the following conditions:*

- (i) for each  $i$  the elements of  $X_i$  form part of a basis of the free abelian group  $\mathbb{Z}^n$ ,
- (ii)  $C$  is covered by the simplicial cones spanned by the  $X_i$ .

It would be enough to require that  $C \cap \mathbb{Z}^n$  is contained in the union of the cones  $\mathbb{R}_+ X_i$ . Furthermore (UHC) can be formulated for arbitrary positive affine semigroups  $S$ : one replaces  $\text{Hilb}(C)$  by  $\text{Hilb}(S)$ , and the condition (ii) by the requirement that  $S$  is the union of the semigroups generated by the  $X_i$ . However, it is evident that this condition is sufficient for the normality of  $S$ .

As a conjecture, (UHC) appears first in Sebö [Se, Conjecture B]. It is the main result of this paper that (UHC) does not hold for all cones. We will present a 6-dimensional counterexample to Sebö's conjecture in Section 3, in which we also describe an algorithm deciding (UHC). (See Firla and Ziegler [FZ] for previous attempts to disprove (UHC).)

The major positive result supporting (UHC) had been shown by Sebö [Se] and, independently, by Aguzzoli and Mundici [AM] and Bouvier and Gonzalez-Sprinberg [BoGo]: every 3-dimensional rational cone admits a *triangulation* (or *partition*) into unimodular simplicial subcones generated by elements of  $\text{Hilb}(C)$ . This is, of course, a much stronger property than (UHC). (However, [BoGo] also describes a 4-dimensional cone without such a triangulation.)

Another positive result for an important special case has been proved by Bruns, Gubeladze, and Trung [BGT]. Let  $P \in \mathbb{Z}^n$  be a lattice  $n$ -polytope, i. e. an  $n$ -dimensional polytope with vertices in  $\mathbb{Z}^n$ . It is natural to associate with such a polytope the subsemigroup  $S_P$  of  $\mathbb{Z}^{n+1}$  generated by the elements  $(x, 1)$ ,  $x \in P \cap \mathbb{Z}^n$ . One calls  $P$  *normal* if  $S_P$  is normal. As above we may assume that  $\text{gp}(S_P) = \mathbb{Z}^{n+1}$ . For  $S_P$  the property (UHC) then amounts to the following question [BGT, 1.2.3]: is every normal lattice polytope covered by its unimodular lattice subsimplices (has a unimodular cover, for short)? A lattice simplex in  $\mathbb{Z}^n$  with vertices  $v_0, \dots, v_n$  is unimodular if the vectors  $v_i - v_0$ ,  $i = 1, \dots, n$ , form a basis of  $\mathbb{Z}^n$ , or equivalently,  $(v_0, 1), \dots, (v_n, 1)$  form a basis of  $\mathbb{Z}^{n+1}$ . By [BGT, 1.3.1] the homothetic multiple  $cP$  has a unimodular cover for  $c \gg 0$ , regardless of  $\dim P$ . (It is even known that  $dP$  has a triangulation into unimodular simplices for *some*  $d$ , but the question whether such a triangulation exists for all sufficiently large  $d$  seems to be open; see Kempf, Knudsen, Mumford, and Saint-Donat [KKMS].) For elementary reasons one can take  $c = 1$  in dimension 1 and 2, and it was communicated by Ziegler that  $c = 2$  suffices in dimension 3. However, in higher dimension no effective lower bound for  $c$  seems to be known. (In contrast,  $cP$  is normal for  $c \geq \dim P - 1$ ; see [BGT, 1.3.3].) Our counterexample to (UHC) is in fact a normal semigroup of type  $S_P$  where  $P$  is a 5-dimensional lattice polytope. Thus the question about unimodular covers of normal polytopes has a negative answer.

A natural variant of (UHC), and weaker than (UHC), is the existence of a *free Hilbert cover*:

**(FHC)** Let  $C \subset \mathbb{R}^n$  be as in (UHC). Then  $\text{Hilb}(S)$  has a family of subsets  $\{X_i\}$  satisfying the following conditions:

- (i) for each  $i$  the elements of  $X_i$  are linearly independent,

(ii)  $S(C)$  is covered by the free subsemigroups generated by the  $X_i$ .

Again one can formulate (FHC) for semigroups, but for (FHC) – in contrast to (UHC) – it is not evident that it implies the normality of the semigroup. Nevertheless it does so, as we will see in Section 6. A formally weaker property is the *integral Carathéodory property*:

**(ICP)** Let  $C \subset \mathbb{R}^n$  be as in (UHC). Then every element of  $x \in S(C)$  has a representation  $x = a_1 s_1 + \cdots + a_m s_m$  with  $a_i \in \mathbb{Z}_+$ ,  $s_i \in \text{Hilb}(C)$ , and  $m \leq n$ .

Here we have borrowed the well-motivated terminology of [FZ]: (ICP) is obviously a discrete variant of Carathéodory’s theorem for convex cones. It was first asked in Cook, Fonlupt, and Schrijver [CFS] whether all cones have (ICP) and then conjectured in [Se, Conjecture A] that the answer is ‘yes’.

In joint work with M. Henk, A. Martin and R. Weismantel it has been shown that our counterexample to (UHC) also disproves (ICP) (see [BGHMW]). Thus none of the covering properties above is necessary for the normality of affine semigroups.

In order to describe the further results of this paper we introduce the *representation length*

$$\rho(x) = \min\{m \mid x = a_1 s_1 + \cdots + a_m s_m, a_i \in \mathbb{Z}_+, s_i \in \text{Hilb}(S)\}$$

for an element  $x$  of an affine semigroup  $S$ . If  $\rho(x) \leq m$ , we also say that  $x$  is *m-represented*.

In Section 6 we will show that  $S$  is in fact normal and  $C(S)$  satisfies (FHC) if the inequality  $\rho(x) \leq \text{rank } S$  holds with ‘probability 1’ for the elements  $x$  of  $S$ . (It will be made precise below what ‘probability 1’ means.) Especially, (FHC) and (ICP) are equivalent. Moreover this result shows that one should very well be able to recognize a counterexample to (ICP) (or (FHC)) by random methods. Its proof indicates an algorithm deciding (FHC).

In order to measure the deviation of  $S$  from (ICP), we will introduce the notion of *Carathéodory rank* of an affine semigroup  $S$ ,  $\text{CR}(S) = \max\{\rho(x) \mid x \in S\}$ , and two variants, *asymptotic* and *virtual* Carathéodory rank for which we allow an exceptional set of ‘measure zero’ or, respectively, only finitely many exceptions among the elements of  $S$ . For normal  $S$  we will establish inequalities for the asymptotic Carathéodory rank that improve the known inequalities for  $\text{CR}(S)$  itself (Section 4), and under special assumptions on  $S$  we can derive these inequalities also for the virtual Carathéodory rank (Section 5).

In Section 2 we will introduce the class of *tight* cones  $C$  which is distinguished by the property that, roughly speaking, for any element  $s \in \text{Hilb}(S)$  the set  $\text{Hilb}(S) \setminus \{s\}$  is *not* the Hilbert basis of the cone it generates. Each of (UHC), (FHC), and (ICP) can be reduced to the class of tight cones, since a counterexample of minimal dimension to any of these properties can be ‘shrunk’ to a tight one. Despite of extensive calculations (which are much faster than checking (UHC)) we have not been able to find a non-trivial tight cone in dimension  $\leq 3$ . However, there exist tight cones in dimension 4, and in dimension  $\geq 5$  one can easily give examples. Our 6-dimensional counterexample is tight.

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## 2. TIGHT CONES

In this section we introduce the class of tight cones and semigroups and show that they play a crucial rôle for (UHC) and the other covering properties.

**Definition 2.1.** Let  $S$  be a normal affine semigroup,  $x \in \text{Hilb}(S)$ , and  $S'$  the semigroup generated by  $\text{Hilb}(S) \setminus \{x\}$ . We say that  $x$  is *non-destructive* if  $S'$  is normal and  $\text{gp}(S')$  is a direct summand of  $\text{gp}(S)$  (and therefore equal to  $\text{gp}(S)$  if  $\text{rank gp}(S) = \text{rank gp}(S')$ ). We say that  $S$  is *tight* if every element of  $\text{Hilb}(S)$  is destructive. A cone  $C$  is *tight* if  $S(C)$  is tight.

It is clear that only those elements of  $\text{Hilb}(S)$  can be non-destructive that generate the extremal rays of  $C(S)$ ; we will call them *extremal* elements or generators. Suppose that  $x$  is an extremal element of  $\text{Hilb}(S)$ . Then  $S[-x]$  (the subsemigroup of  $\text{gp}(S)$  generated by  $S$  and  $-x$ ) splits into a product  $\mathbb{Z}x \times S_x$  where  $S_x$  is again a normal affine semigroup without nontrivial units (for example, see [Gu1, Theorem 1.8]). As a consequence one has  $C(S)[-x] \approx \mathbb{R} \times C(S_x)$ .

**Lemma 2.2.** *Let  $S$  be a normal positive affine semigroup and  $x \in \text{Hilb}(S)$  a non-destructive element. Let  $S'$  be the semigroup generated by  $\text{Hilb}(S) \setminus \{x\}$  and  $S_x$  the quotient  $S[-x]/(\mathbb{Z}x)$  introduced above.*

- (i) *If  $S'$  and  $S_x$  both satisfy (UHC), then so does  $S$ .*
- (ii) *If  $S'$  and  $S_x$  both satisfy (FHC), then so does  $S$ .*
- (iii) *One has  $\text{CR}(S) = \max(\text{CR}(S'), \text{CR}(S_x) + 1)$ .*

*Proof.* Suppose  $S'$  and  $S_x$  both satisfy (UHC). Since  $\text{gp}(S')$  is a direct summand of  $\text{gp}(S)$  and  $\text{Hilb}(S') = \text{Hilb}(S) \setminus \{x\}$  by the hypothesis on  $x$ , it is clear that all elements of  $S'$  are contained in subsemigroups of  $S$  generated by subsets  $X_i$  of  $\text{Hilb}(S)$  such that  $X_i$  generates a direct summand of  $\text{gp}(S)$ . In proving that  $S$  satisfies (UHC), it is therefore enough to consider  $S \setminus S'$ .

Let  $z \in S \setminus S'$ . By hypothesis on  $S_x$ , the residue class of  $z$  in  $S_x$  has a representation  $\bar{z} = a_1\bar{y}_1 + \cdots + a_m\bar{y}_m$  with  $a_i \in \mathbb{Z}_+$  and  $\bar{y}_i \in \text{Hilb}(S_x)$  for  $i = 1, \dots, m$  such that  $\bar{y}_1, \dots, \bar{y}_m$  span a direct summand of  $\text{gp}(S_x)$ . Next observe that  $\text{Hilb}(S)$  is mapped onto a system of generators of  $S_x$  by the residue class map. Therefore we may assume that the preimages  $y_1, \dots, y_m$  belong to  $\text{Hilb}(S)$ . Furthermore,  $z = a_1y_1 + \cdots + a_my_m + bx$  with  $b \in \mathbb{Z}$ .

It only remains to show that  $b \in \mathbb{Z}_+$ . There is a representation of  $z$  as a  $\mathbb{Z}_+$ -linear combination of the elements of  $\text{Hilb}(S)$  in which the coefficient of  $x$  is positive. Thus, if  $b < 0$ ,  $z$  has a  $\mathbb{Q}_+$ -linear representation by the elements of  $\text{Hilb}(S) \setminus \{x\}$ . This implies  $y \in C(S')$ , and hence  $y \in S'$ , a contradiction.

This proves (i), and (ii) and (iii) follow similarly.  $\square$

We say that a semigroup  $S$  as in the lemma *shrinks* to the semigroup  $T$  if there is a chain  $S = S_0 \supset S_1 \supset \cdots \supset S_t = T$  of semigroups such that at each step  $S_{i+1}$  is

generated by  $\text{Hilb}(S_i) \setminus \{x\}$  where  $x$  is non-destructive. An analogous terminology applies to cones.

**Corollary 2.3.** *A counterexample to (UHC) that is minimal with respect to first dimension and then  $\#\text{Hilb}(C)$  is tight. A similar statement holds for (FHC).*

In fact, suppose that the cone  $C$  is a minimal counterexample to (UHC) with respect to dimension, and that  $C$  shrinks to  $D$ . Then  $D$  is also a counterexample to (UHC) according to Lemma 2.2. (For (FHC) the argument is the same.) It is therefore clear that one should search for counterexamples only among the tight cones. We will discuss the algorithmic aspects of this strategy in Section 3.

**Remark 2.4.** It is not hard to see that there are no tight cones of dimension  $\leq 2$ . Furthermore, the cone  $C_P$  over a lattice polytope  $P$  of dimension 2 is never tight, since each vertex of  $P$  represents a non-destructive element of  $\text{Hilb}(C_P)$ . However, we cannot prove that all 3-dimensional cones  $C$  are non-tight; in general, an extremal element of  $\text{Hilb}(C)$  can very well be destructive, even if  $\dim C = 3$ . In dimension 4 there exist tight cones but none of the examples we have found is of the form  $C_P$  with a 3-dimensional lattice polytope  $P$ . In dimension  $\geq 5$  one can easily describe a class of tight cones: let  $W$  be a cube whose lattice points are its vertices and its barycenter; then the cone  $C_W$  is tight if  $\dim W \geq 4$ .

### 3. THE COUNTEREXAMPLE TO (UHC)

Before we give the counterexample, we outline the strategy of the search. It consists of 4 steps:

- (G) the choice of the generators of the cone  $C$  to be tested;
- (T) the shrinking of  $C$  to a tight cone;
- (C) the computation of several covers of  $C$  by simplicial subcones;
- (U) the verification that  $C$  has a unimodular cover or otherwise.

There is not much to say about step (G). Either the generators of  $C$  have been chosen by a random procedure depending on some parameters, especially the dimension, or they have been chosen systematically in order to exhaust a certain class. Many of the examples tested have been cones  $C_P$  over lattice polytopes  $P$ . Especially we have worked with random parallelepipeds  $P$  of a given dimension. Parallelepipeds are automatically normal:

**Proposition 3.1.** *Let  $P \subset \mathbb{R}^n$  be a lattice polytope. Suppose that for all  $n \in \mathbb{N}$  the multiple  $(n+1)P$  is the union of the sets  $x + nP$  where  $x$  runs through the vertices of  $P$ . Then  $P$  is normal.*

The proposition follows immediately from the definitions.

Step (T) is carried out as follows. First the Hilbert basis of  $C$  is computed and among its elements the set  $E$  of extremal ones. Then successively each element  $x$  of  $E$  is tested for being non-destructive by checking whether (i)  $\text{Hilb}(C) \setminus \{x\}$  is a Hilbert basis of the cone  $C'$  it generates and (ii) whether the group generated by  $\text{Hilb}(C) \setminus \{x\}$  is a direct summand of  $\mathbb{Z}^n$ . If so,  $C$  is replaced by  $C'$ . Otherwise the

next element of  $E$  is tested in the same way. The procedure stops with a tight cone (which often is  $\{0\}$ ).

It is clear that (T) depends on an efficient algorithm for the computation of Hilbert bases. Such an algorithm has been developed by R. Koch and the first author (see [BK]).

For each of the covers mentioned in step (C) we first compute a triangulation  $T$  depending on the order in which  $\text{Hilb}(C)$  is given, and for the other covers this order is permuted randomly. None of the simplicial subcones  $\sigma \in T$  contains an element of  $\text{Hilb}(C)$  different from the extremal generators of  $\sigma$ . Many of the simplicial subcones  $\sigma$  of  $T$  will be unimodular and others non-unimodular. We then try to improve the situation as follows: for each non-unimodular  $\sigma$  we look at the cones  $\sigma_v$  generated by  $\sigma$  and  $v + \sum(w - v)$  where  $v$  is an extremal generator of  $\sigma$  and  $w$  runs through the set  $R$  of the remaining extremal generators of  $\sigma$ . For each element  $y \in \text{Hilb}(C) \cap \sigma_v$  the cone  $\sigma$  is covered by the union of the  $n - 1$  cones  $\sigma_1, \dots, \sigma_{n-1}$  generated by  $v, y$  and  $n - 2$  elements from  $R$ . We try to choose  $y$  in an ‘optimal’ way, replace  $\sigma$  by  $\sigma_1, \dots, \sigma_{n-1}$ , and iterate the procedure. Unfortunately the effect of this step depends on the probability that a cone  $\sigma_i$  is unimodular. In dimension 6 (or higher) it does usually not improve the situation.

The quality  $Q(B)$  of each of the (say, 50) coverings  $B$  computed is measured as follows: we sum the absolute values of the determinants of the non-unimodular simplicial subcones of  $B$ . Among the coverings we choose the 3 best ones  $B_1, B_2, B_3$ , and they are the basis for the last step (U) (the number 3 can be varied). First a list of all intersections  $\gamma = \sigma_1 \cap \sigma_2 \cap \sigma_3$  is formed where  $\sigma_i$  runs through the non-unimodular simplicial subcones of  $B_i$ . Then each ‘critical subcone’ obtained in this way is compared to the list  $L$  of unimodular simplicial subcones generated by elements of  $\text{Hilb}(C)$ . First, if  $\gamma$  is contained in one of the elements of  $L$ , then it is discarded. Second, if the interior of  $\gamma$  is intersected by some  $\sigma \in L$ , one of the support hyperplanes of  $\sigma$  splits  $\gamma$  into two subcones that are then checked recursively. Third, if no unimodular simplicial subcone intersects the interior of  $\gamma$ , then we have found the desired counterexample. The algorithm stops since the number of unimodular simplicial subcones, and therefore the number of hyperplanes available for the splitting of the critical subcones, is finite.

The output of our implementation of step (U) is a list of subcones  $\delta$  such that the interior of their union (with respect to  $C$ ) is the complement of the union of the unimodular subcones.

The basis of all computations involved is the dual cone algorithm (see Burger [Bu]) that for a given cone  $C \subset \mathbb{R}^n$  computes a system of generators of the dual cone

$$C^* = \{\varphi \in (\mathbb{R}^n)^* \mid \varphi(x) \geq 0 \text{ for all } x \in C\}.$$

Note that the intersection  $C \cap D$  of cones  $C$  and  $D$  is the dual of the cone generated by the union of  $C^*$  and  $D^*$ .

The counterexample finally emerged when we directed our search to cones over 5-dimensional parallelepipeds and the tight cones produced from them. Even for generators with ‘small’ coefficients, the Hilbert bases of these cones can be quite

large. We have tried to select examples that are not too ‘big’. Nevertheless the task is usually formidable, both in computing time and memory requirements. A typical example:  $\#\text{Hilb}(C) = 38$ , the minimal value of  $Q(B) = 324$ , computing time about 24 hours, memory requirement  $> 100$  MB.

Thus we were quite surprised by finding the following ‘small’ counterexample  $C_6$  to (UHC) whose Hilbert basis consists of the following 10 vectors:

$$\begin{aligned} z_1 &= (0, 1, 0, 0, 0, 0), & z_6 &= (1, 0, 2, 1, 1, 2), \\ z_2 &= (0, 0, 1, 0, 0, 0), & z_7 &= (1, 2, 0, 2, 1, 1), \\ z_3 &= (0, 0, 0, 1, 0, 0), & z_8 &= (1, 1, 2, 0, 2, 1), \\ z_4 &= (0, 0, 0, 0, 1, 0), & z_9 &= (1, 1, 1, 2, 0, 2), \\ z_5 &= (0, 0, 0, 0, 0, 1), & z_{10} &= (1, 2, 1, 1, 2, 0). \end{aligned}$$

The first (isomorphic) version of this counterexample produced by our programs is of the form  $C_Q$  where  $Q$  is a 5-dimensional normal lattice polytope. Since  $C_6$  does not have (UHC),  $Q$  is not covered by its unimodular lattice subsimplices, and therefore the question [BGT, 1.2.3] has a negative answer. (We are grateful to M. Jibladze and R. Firla for checking our first computation indicating that  $C_6$  violates (UHC), and to M. Henk, A. Martin and R. Weismantel who independently verified that  $z_1, \dots, z_{10}$  form the Hilbert basis of  $C_6$  by the the program SIP [MW].)

The cone  $C_6$  and the semigroup  $S_6 = S(C_6)$  have several remarkable properties:

- (1)  $C_6$  has 27 facets, of which 5 are not simplicial.
- (2) The automorphism group  $\text{Aut}(S_6)$  of  $S_6$  has order 20, and it operates transitively on  $\text{Hilb}(S_6)$ . In particular this implies that  $z_1, \dots, z_{10}$  are all extremal generators of  $S_6$ .
- (3) The embedding above has been chosen in order to make visible the subgroup  $U$  of those automorphisms that map each of the sets  $\{z_1, \dots, z_5\}$  and  $\{z_6, \dots, z_{10}\}$  to itself;  $U$  is isomorphic to the dihedral group of order 10. However,  $C_6$  can even be realized as the cone over a 0-1-polytope in  $\mathbb{R}^5$ .
- (4) The vector of lowest degree disproving (UHC) is  $t_1 = z_1 + \dots + z_{10}$ . Evidently  $t_1$  is invariant for  $\text{Aut}(S_6)$ , and it can be shown that its multiples are the only such elements.
- (5) The Hilbert basis is contained in the hyperplane  $H$  given by the equation  $-5\zeta_1 + \zeta_2 + \dots + \zeta_6 = 1$ . Thus  $z_1, \dots, z_{10}$  are the vertices of a normal 5-dimensional lattice polytope  $P_6$  (isomorphic to the polytope  $Q$  mentioned above) that is not covered by its unimodular lattice subsimplices (and contains no other lattice points).
- (6) If one removes all the unimodular subcones generated by elements of  $\text{Hilb}(C_6)$  from  $C_6$ , then there remains the interior of a convex cone  $N$ . While  $P_6$  has normalized volume 25, the intersection of  $N$  and  $P_6$  has only normalized volume  $1/1080$ .
- (7) The binomial ideal defining the semigroup ring  $K[S_6]$  over an arbitrary field  $K$  is generated by 10 binomials of degree 3 and 5 binomials of degree 4 (the latter correspond to the non-simplicial facets).

- (8) The  $h$ -vector of  $P_6$  is  $(1, 4, 10, 10)$  and the  $f$ -vector (supplied by the referee) is  $(1, 10, 40, 80, 75, 27)$ .  
 (9) One has  $\text{CR}(C_6) = 7$ .

In particular  $C_6$  is even a counterexample to (ICP). This has been shown in cooperation with Henk, Martin and Weismantel; see [BGHMW], which also gives more detailed information on properties 2, 3, and 9.

We have found only one more counterexample to (UHC) essentially different from  $C_6$ . It is also of dimension 6, but its Hilbert basis contains 12 elements. As Henk, Martin, and Weismantel have verified, it violates (ICP), too. Thus the question whether there exist examples satisfying (ICP), but violating (UHC), remains open.

#### 4. ESTIMATES FOR ASYMPTOTIC CARATHÉODORY RANKS

Recall from the introduction that the representation length  $\rho(x)$  of an element  $x$  in a positive affine semigroup is the minimal number of irreducible elements needed to represent  $x$ .

**Definition 4.1.** Let  $S$  be a positive affine semigroup. Then the *Carathéodory rank* of  $S$  is the number

$$\text{CR}(S) = \max\{\rho(x) \mid x \in S\}.$$

For a cone  $C$  we set  $\text{CR}(C) = \text{CR}(S(C))$ .

The following result of Sebö [Se] seems to be the best available estimate for Carathéodory ranks of cones.

**Theorem 4.2.** *For every cone  $C$  one has  $\text{CR}(C) = \text{rank}(C)$  if  $\text{rank}(C) \leq 3$ , and  $\text{CR}(C) \leq 2 \cdot \text{rank}(C) - 2$  if  $\text{rank}(C) > 3$ .*

Next we introduce two variants of the notion of Carathéodory rank. In the following  $B(\delta)$  is the standard Euclidean closed ball of radius  $\delta$ .

**Definition 4.3.** Let  $S \subset \mathbb{Z}^n$  be a positive affine semigroup.

- (a) The *asymptotic Carathéodory rank* of  $S$ , denoted by  $\text{CR}^a(S)$ , is defined as the smallest natural number  $m$  such that the following limit exists and satisfies the equality

$$\lim_{\delta \rightarrow \infty} \frac{\#(m\text{-represented points in } S \cap B(\delta))}{\#(S \cap B(\delta))} = 1.$$

(Since  $m = \#\text{Hilb}(S)$  satisfies the condition,  $\text{CR}^a(S)$  is well-defined.)

- (b) The *virtual Carathéodory rank* of  $S$ , denoted by  $\text{CR}^v(S)$ , is the smallest natural number  $m$  such that there exists  $d > 0$  for which all elements of  $(S \cap \mathbb{Z}^n) \setminus B(d)$  are  $m$ -represented (in other words, up to a finite number of exceptions, all elements of  $S$  are  $m$ -represented).

For a cone  $C$  we set  $\text{CR}^a(C) = \text{CR}^a(S(C))$  and  $\text{CR}^v(C) = \text{CR}^v(S(C))$ .

The following sequence of inequalities is immediate from the definitions

$$(1) \quad \text{rank}(S) \leq \text{CR}^a(S) \leq \text{CR}^v(S) \leq \text{CR}(S) \leq \#\text{Hilb}(S).$$

*Observation.*  $\text{CR}^a(S)$  and  $\text{CR}^v(S)$  are intrinsic invariants of  $S$ , i. e. they do not depend on the embedding  $S \rightarrow \mathbb{Z}^n$ .

One easily shows

**Lemma 4.4.** *Let  $S_1$  and  $S_2$  be positive affine semigroups. Then*

$$\text{CR}^a(S_1 \times S_2) = \text{CR}^a(S_1) + \text{CR}^a(S_2).$$

While the analogous equality for  $\text{CR}(S)$  is obvious, one evidently has  $\text{CR}^v(S_1 \times S_2) > \text{CR}^v(S_1) + \text{CR}^v(S_2)$  if  $\text{CR}^v(S_1) > \text{CR}(S_1)$  and  $S_2 \neq 0$ .

Let us next give examples of semigroups demonstrating that the inequalities in (1) may be strict.

**Examples 4.5.** (a) Let  $p_1, \dots, p_r$  be the first  $r$  prime numbers. Put  $a_i = \prod_{j \neq i} p_j$ ,  $i \in [1, r]$ , and consider the numerical semigroup  $N_r \subset \mathbb{Z}_+$  generated by the  $a_i$ . Since  $\text{gcd}(a_1, \dots, a_r) = 1$ , we conclude that  $N_r$  contains all sufficiently large natural numbers. On the other hand any proper subset of  $\{a_1, \dots, a_r\}$  is not coprime. Hence  $\text{CR}^a(N_r) = \text{CR}^v(N_r) = r$ . (This example has been taken from [CFS].)

As we will show in Theorem 6.1, the equality  $\text{CR}^a(S) = \text{rank}(S)$  implies that  $\text{CR}(S) = \text{rank}(S)$ . Since the semigroup  $S_6$  of Section 3 does not satisfy (ICP), there even exist normal affine semigroups  $S$  with  $\text{CR}^a(S) > \text{rank}(S)$ .

(b) Now consider the affine semigroup

$$S_r = \mathbb{Z}_+(1, 0) + \mathbb{Z}_+(1, 1) + \mathbb{Z}_+(0, a_1) + \dots + \mathbb{Z}_+(0, a_r) \subset \mathbb{Z}^2.$$

It follows from the discussion in (a) that  $\text{CR}^v(S_r) \geq r$  (actually,  $\text{CR}^v(S_r) = r + 2$ ). Set  $v = (0, a_i)$  for some  $i \in [1, r]$  and let  $\sigma \subset \mathbb{R}_+^2$  be the cone spanned by  $(1, 0)$  and  $(1, 1)$ ; then elementary geometric arguments show

$$\lim_{\delta \rightarrow \infty} \frac{\#(\bigcup_{k=1}^{\infty} (kv + \sigma) \cap \mathbb{Z}_+^2 \cap B(\delta))}{\#(\mathbb{Z}_+^2 \cap B(\delta))} = 1 \quad \text{and} \quad \bigcup_{k=1}^{\infty} (kv + \sigma) \cap \mathbb{Z}_+^2 \subset S_r.$$

Therefore  $\text{CR}^a(S_r) \leq 3$  (actually, one has equality). So we have produced an affine semigroup  $S_r$  for which  $\text{CR}^a(S_r) < \text{CR}^v(S_r)$  (if  $r > 3$ ).

(c) As for the inequality  $\text{CR}^v(S) < \text{CR}(S)$ , one considers the numerical semigroup  $S = \mathbb{Z}_+7 + \mathbb{Z}_+8 + \mathbb{Z}_+10 \subset \mathbb{Z}_+$ .

That the other inequalities in (1) may in general be strict is easily seen. (There are probably examples for which all the inequalities are simultaneously strict.) Nevertheless these inequalities are not independent of each other. We will see in Theorem 6.1 that the equality  $\text{rank } S = \text{CR}^a(S)$  forces the equality of all the numbers in (1) except  $\# \text{Hilb}(S)$ .

Our main result on estimating asymptotic Carathéodory ranks is the next theorem. In particular, it shows that Sebö's bound can be improved by 1 for asymptotic Carathéodory ranks.

**Theorem 4.6.** (i) *For a cone  $C \subset \mathbb{R}^n$  one has*

$$\text{CR}^a(C) \leq \max\{\text{CR}^a(F) + 2 \mid F \text{ a facet of } C\}.$$

(ii) *For a cone  $C \subset \mathbb{R}^n$  of rank  $\geq 3$  one has  $\text{CR}^a(C) \leq 2 \cdot \text{rank}(C) - 3$ .*

Part (ii) of the theorem follows immediately from part (i) by induction since 3-dimensional cones satisfy (UHC). The next two lemmas are auxiliary steps in the proof of part (i).

Let  $F$  be a facet of the  $n$ -dimensional cone  $C \subset \mathbb{R}^n$ . The subgroup  $H_F$  of  $\mathbb{Z}^n$  generated by the semigroup  $F \cap \mathbb{Z}^n$  has rank  $n - 1$ , and there is a unique group isomorphism  $v_F : \mathbb{Z}^n / H_F \rightarrow \mathbb{Z}$  such that  $S(C)$  is mapped to  $\mathbb{Z}_+$ . It is clear that there is an element  $y \in C \cap \mathbb{Z}^n$  with  $v_F(y) = 1$ . Writing  $y = x_1 + \cdots + x_k$  with  $x_1, \dots, x_k \in \text{Hilb}(C)$  we see that  $v_F(x_j) = 1$  for exactly one  $j$ .

**Lemma 4.7.** *For each facet  $F \subset C$  there is an element  $x \in \text{Hilb}(C)$  such that  $v_F(x) = 1$ .*

**Remark 4.8.** This lemma guarantees the existence of at least one unimodular subcone of  $C$  generated by elements of  $\text{Hilb}(C)$ , as follows immediately by induction on  $\dim(C)$ . In particular, any normal lattice polytope contains a unimodular lattice subsimplex.

For each facet  $F \subset C$  fix an element  $x_F \in \text{Hilb}(C)$  with the property  $v_F(x_F) = 1$ . Consider the subcone  $C(x_F, F) \subset C$  spanned by  $F$  and  $x_F$ . Clearly,

$$C(x_F, F) \cap \mathbb{Z}^n \approx \mathbb{Z}_+ x_F \times (F \cap \mathbb{Z}^n).$$

So, by Lemma 4.4,

$$(*) \quad \text{CR}^a(C(x_F, F)) = \text{CR}^a(F) + 1.$$

It is also clear, that

$$(**) \quad \text{Hilb}(C(x_F, F)) \subset \text{Hilb}(C).$$

There is an affine hyperplane that avoids 0 and meets all the rays  $\mathbb{R}_+ x$ ,  $x \in C \setminus \{0\}$ . Fix such a hyperplane  $H \subset \mathbb{R}^n$ . Then the cross-section  $\Phi(C) = C \cap H$  is a finite convex polytope. For each point  $x \in C \setminus \{0\}$  we let (according to [Gu1])  $\Phi(x)$  denote the point of intersection of the radial ray of  $x$  with  $H$ . Likewise, for any subcone  $C' \subset C$  we let  $\Phi(C')$  denote the sub-polytope  $C' \cap H \subset \Phi(C)$ .

Let  $z \in C \cap \mathbb{Z}^n$  be any extremal element, i. e.  $z \neq 0$  and  $z$  lies on an edge of  $C$ . For any facet  $F \subset C$  not containing  $z$  we denote by  $C(z, F)$  the subcone of  $C$  spanned by  $z$  and  $F$ . The polytope  $\Phi(C)$  decomposes into pyramids

$$\Phi(C) = \bigcup_F (\Delta(z, F)),$$

where  $F$  runs through the facets of  $C$  avoiding  $z$  and  $\Delta(z, F) = \Phi(C(z, F))$ .

Let  $\Phi(F)^0$  be the union of all the facets of  $\Delta(z, F)$  except  $\Phi(F)$ . For a real number  $\varepsilon > 0$  we put

$$\Delta_\varepsilon(z, F) = \Delta(z, F) \setminus \Phi(F)_\varepsilon^0,$$

where the subscript  $\varepsilon$  is used to denote the  $\varepsilon$ -neighborhood in  $\Delta(z, F)$ .

The crucial argument for the proof of Theorem 4.6 is

**Lemma 4.9.** *For every real number  $\varepsilon > 0$  there is a  $\delta > 0$  such that*

$$\{y \in \mathbb{Z}^n \setminus B(\delta) \mid \Phi(y) \in \Delta_\varepsilon(z, F)\} \subset \mathbb{Z}_+ z + (C(x_F, F) \cap \mathbb{Z}^n).$$

*Proof.* If  $y$  is as above and does not belong to  $C(x_F, F)$  itself, then a ‘big’ initial subset of the sequence  $\{y - kz\}_{k=1}^\infty$  lies in the cone spanned by  $z$  and  $F$ , the  $y - kz$  ‘slowly’ approximate the set  $\Phi(F) \cap \Delta_\varepsilon(z, F)$ , and eventually  $\Phi(y - k_0z) \in \Phi(C(x_F, F))$  for some  $k_0$ . The situation is illustrated by Figure 1.  $\square$

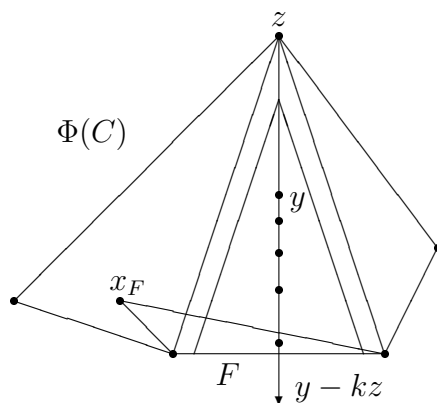


FIGURE 1

Set  $m = \text{CR}^a(F) + 2$ . Lemma 4.9 together with (\*) and (\*\*) ensure that

$$\lim_{\delta \rightarrow \infty} \frac{\#\{x \in \mathbb{Z}^n \cap B(\delta) \mid \rho(x) \leq m, \Phi(x) \in \Delta_\varepsilon(z, F)\}}{\#\{x \in \mathbb{Z}^n \cap B(\delta) \mid \Phi(x) \in \Delta_\varepsilon(z, F)\}} = 1,$$

where  $\rho(x)$  is measured with respect to  $\text{Hilb}(C)$ . Finally, since  $\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon(z, F) = \Delta(z, F)$ , standard arguments complete the proof of Theorem 4.6.

## 5. ESTIMATES FOR VIRTUAL CARATHÉODORY RANKS

For an interesting class of cones  $C$  we can replace asymptotic by virtual Carathéodory rank. Let  $S = S(C)$ . We have observed in Section 2 that for every extremal generator  $x \in \text{Hilb}(C)$  the semigroup  $S[-x]$  splits into a product  $\mathbb{Z}x \times S_x$ .

**Definition 5.1.** A cone  $C \subset \mathbb{R}^n$  is called *smooth* if for every extremal element  $x \in \text{Hilb}(C)$  the semigroup  $S_x$  is free.

This terminology is motivated as follows. Let  $K$  be a field,  $R = K[S(C)]$ , and consider the affine toric variety  $V = \text{Spec } R$ . The non-zero elements of  $S(C)$  generate a maximal ideal  $\mathfrak{m}$ . Moreover, those elements of  $\text{Hilb}(C)$  that lie in the edges of  $C$  generate an ideal  $I$  with  $\text{Rad } I = \mathfrak{m}$ . Therefore  $V \setminus \{\mathfrak{m}\}$  is smooth if and only if the localizations of  $R$  with respect to the extremal elements of  $\text{Hilb}(C)$  are smooth semigroup rings, and the latter condition is exactly the definition of ‘smooth’ above. Further, in the homogeneous case, i. e. when  $C$  is the cone over some normal lattice polytope  $P \subset \mathbb{Z}^{n-1}$ , the smoothness of  $V \setminus \{\mathfrak{m}\}$  is equivalent to the smoothness of the projective variety  $\text{Proj}(R)$ . (For details on toric varieties we refer the reader to Fulton [Fu].)

**Theorem 5.2.** (i) *If the cone  $C \subset \mathbb{R}^n$  is smooth, then*

$$\mathrm{CR}^v(C) \leq \max\{\mathrm{CR}^v(F) + 2 \mid F \text{ a facet of } C\}.$$

(ii) *For a smooth cone  $C \subset \mathbb{R}^n$  of rank  $\geq 3$  one has  $\mathrm{CR}^v(C) \leq 2 \cdot \mathrm{rank}(C) - 3$ .*

Again (i) implies (ii). The proof of (i) is decomposed into several lemmas.

**Lemma 5.3.** *Let  $C \subset \mathbb{R}^n$  be a smooth cone of rank  $n$ . Then for any vertex  $p$  of the polytope  $\Phi(C)$  there is a basis  $X_p$  of  $\mathbb{Z}^n$  satisfying the following conditions:*

- (1)  $X_p \subset \mathrm{Hilb}(C)$ ,
- (2)  $\Phi(C(X_p))$  is a neighborhood of  $p$  in  $\Phi(C)$ .

*Proof.* Let  $S = S(C)$  and  $x \in \mathrm{Hilb}(C)$  be the extremal element with  $\Phi(x) = p$ . The residue class map  $S \rightarrow S_x$  maps  $\mathrm{Hilb}(C)$  to a system of generators of  $S_x$ . Especially the irreducible generators of  $S_x \approx \mathbb{Z}_+^{n-1}$  have preimages  $y_1, \dots, y_{n-1} \in \mathrm{Hilb}(C)$ . Clearly  $x, y_1, \dots, y_{n-1}$  is a basis of  $\mathbb{Z}^n$ , and it also satisfies condition (2).  $\square$

For a cone  $C \subset \mathbb{R}^n$  and a subset  $W \subset \Phi(C)$  we will write

$$\mathrm{CR}^v(C, W) \leq m$$

if all elements  $x \in C \cap \mathbb{Z}^n$ , sufficiently far from the origin, are  $m$ -represented whenever  $\Phi(x) \in W$ .

The next lemma is a weak analogue of Lemma 4.4 for virtual Carathéodory ranks.

**Lemma 5.4.** *Let the cone  $D$  be the product of the cones  $C_1$  and  $C_2$ . Then for any neighborhood  $U$  of  $\Phi(C_2)$  in  $\Phi(D)$  one has*

$$\mathrm{CR}^v(D, \Phi(D) \setminus U) \leq \mathrm{CR}(C_2) + \mathrm{CR}^v(C_1).$$

*Proof.* Set  $S_i = S(C_i)$ . The only elements of  $S(C)$  that are not  $\mathrm{CR}(C_2) + \mathrm{CR}^v(C_1)$ -represented must be in the sets  $s_1 + S_2, \dots, s_r + S_2$  where  $s_1, \dots, s_r$  are the finitely many elements of  $S_1$  that are not  $\mathrm{CR}^v(C_1)$ -represented. Only finitely many elements  $x \in s_i + S_2$  have  $\Phi(x) \notin U$ .  $\square$

For a positive real number  $\varepsilon$  the  $\varepsilon$ -neighborhood in  $\Phi(C)$  of the boundary  $\partial\Phi(C)$  will be denoted by  $\partial_\varepsilon\Phi(C)$ .

**Lemma 5.5.** *Let  $C \subset \mathbb{R}^n$  be a smooth cone. Then there is a real number  $\varepsilon > 0$  such that*

$$\mathrm{CR}^v(C, \partial_\varepsilon\Phi(C)) = \max\{\mathrm{CR}^v(F)\} + 1 \mid F \text{ facet of } C\}.$$

*Proof.* For a pair of faces  $E \subset F$  we set  $\mathrm{codim}(E, F) = \dim(F) - \dim(E)$ . One then has

$$(\dagger) \quad \mathrm{CR}^v(E) + \mathrm{codim}(E, F) \leq \mathrm{CR}^v(F).$$

Consider a point  $t \in \partial\Phi(C)$ . There is a unique face, say  $E \subset C$ , such that  $t$  is contained in the relative interior of  $\Phi(E)$ .

Assume  $\mathrm{codim}(E, C) = 1$ . By Lemma 4.7 there is an element  $x \in \mathrm{Hilb}(C)$  such that  $v_E(x) = 1$ . It is clear that there is a neighborhood  $U_t$  of  $t$  in  $\Phi(C)$  which lies entirely in the pyramid spanned by  $\Phi(x)$  and  $\Phi(E)$ . By Lemma 5.4 this neighborhood can be chosen small enough to have  $\mathrm{CR}^v(C, U_t) = \mathrm{CR}^v(E) + 1$ .

Now assume  $d = \dim(E) < n - 1$ . Let  $p \in E$  be any vertex and  $X_p$  be a free basis of  $\mathbb{Z}^n$  satisfying the conditions of Lemma 5.3. We can write  $X_p = \{x_1, \dots, x_d, x_{d+1}, \dots, x_n\}$ , where  $\{x_1, \dots, x_d\}$  freely generates the group  $\text{gp}(E \cap \mathbb{Z}^n)$ . Let  $\Gamma$  denote the convex hull of  $E$  and  $\{x_{d+1}, \dots, x_n\}$ . Then the sub-semigroup of  $C \cap \mathbb{Z}^n$  determined by the subset  $\Gamma \subset \Phi(C)$  (the semigroup of all elements ‘passing through’  $\Gamma$ ) is naturally isomorphic to the product  $\mathbb{Z}_+^{n-d} \times (E \cap \mathbb{Z}^n)$ . On the other hand, there is a neighborhood  $U_t$  of  $t$  in  $\Phi(C)$  which lies entirely in  $\Gamma$ . By Lemma 5.4  $U_t$  can thus be chosen small enough to have

$$\text{CR}^v(C, U_t) = \text{CR}^v(E) + n - d.$$

Now pick an intermediate facet  $E \subset F \subset C$ . By (†) we get

$$\text{CR}^v(C, U_t) \leq \text{CR}^v(F) - \text{codim}(E, F) + n - d = \text{CR}^v(F) + 1.$$

Thus we have shown that any point  $t \in \partial\Phi(C)$  has a neighborhood  $U_t \subset \Phi(C)$  such that

$$\text{CR}^v(C, U_t) \leq \max_F \{\text{CR}^v(F)\} + 1.$$

Since  $\partial\Phi(C)$  is compact we arrive at the conclusion that

$$\text{CR}^v(C, \partial_\varepsilon\Phi(C)) \leq \max_F \{\text{CR}^v(F)\} + 1$$

for a sufficiently small real number  $\varepsilon > 0$ . It is also clear that the inequality cannot be strict.  $\square$

Now we complete the proof of Theorem 5.2. Assume  $C \subset \mathbb{R}^n$  is a smooth cone. Fix a real number  $\varepsilon > 0$  such that

$$\text{CR}^v(C, \partial_\varepsilon\Phi(C)) = \max_F \{\text{CR}^v(F)\} + 1,$$

where  $F$  runs through the facets of  $C$ . Let  $E \subset C$  be an edge. Elementary geometric arguments ensure that for any element  $y \in C \cap \mathbb{Z}^n$ , sufficiently far from the origin and satisfying the condition  $\Phi(y) \notin \partial_\varepsilon\Phi(C)$ , there is an element  $z' \in E \cap \mathbb{Z}^n$  such that  $\Phi(y - z') \in \partial_\varepsilon\Phi(C)$  and, simultaneously,  $y - z'$  is far enough from the origin. Since  $z' = kz$  for the minimal generator  $z$  of  $E \cap \mathbb{Z}^n \approx \mathbb{Z}_+$  we conclude that  $\text{CR}^v(C) \leq \max_F \{\text{CR}^v(F)\} + 2$ . We illustrate the situation by Figure 2.

**Remark 5.6.** The hypothesis that  $C$  is smooth can be replaced by the assumption that all cones  $C'$  such that  $\dim C' < \dim C$  have (UHC). In this case it is not hard to show that the boundary  $\partial\Phi(C) \subset \Phi(C)$  has a neighborhood contained in the union of the unimodular subcones of  $C$  generated by elements of  $\text{Hilb}(C)$ . In particular,  $\text{CR}^v(C) \leq 5$  if  $\dim C = 4$ .

## 6. A PROBABILISTIC VERSION OF (ICP)

In this section we show that a probabilistic version of (ICP) implies normality and even (FHC).

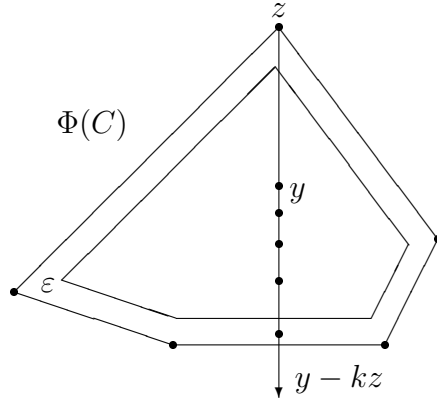


FIGURE 2

**Theorem 6.1.** *Let  $S$  be a positive affine semigroup. If  $\text{rank}(S) = \text{CR}^a(S)$  then  $S$  is normal, one has*

$$\text{rank}(S) = \text{CR}^a(S) = \text{CR}^v(S) = \text{CR}(S),$$

*and  $S$  satisfies (FHC). Especially (ICP) and (FHC) are equivalent and they imply the normality of  $S$ .*

*Proof.* We may assume that  $\text{gp}(S) = \mathbb{Z}^n$  (replacing the ambient lattice by  $\text{gp}(S)$ ). Consider the system of all hyperplanes in  $\mathbb{R}^n$  containing 0 that are spanned by  $(n-1)$ -tuples of linearly independent elements of  $\text{Hilb}(S)$ . These hyperplanes subdivide the cone of  $S$  into a family of subcones:

$$C(S) = \bigcup_{\alpha} C_{\alpha}.$$

For each index  $\alpha$  we let  $\Sigma_{\alpha}$  denote the system of those subsets  $\sigma \subset \text{Hilb}(S)$  which satisfy the conditions:

- (i)  $\sigma$  is a basis of  $\mathbb{R}^n$ ,
- (ii) the simplicial cone spanned by  $\sigma$  contains (equivalently, intersects in its interior) the cone  $C_{\alpha}$ .

For each  $\sigma \in \Sigma_{\alpha}$  the free sub-semigroup

$$\mathbb{Z}_+x_1 + \cdots + \mathbb{Z}_+x_n \subset S, \quad \{x_1, \dots, x_n\} = \sigma,$$

will be denoted by  $F_{\sigma}$ .

Choose  $x \in C(S) \cap \mathbb{Z}^n$ . Then there is an index  $\alpha$  such that  $x \in C_{\alpha}$ . By Lemma 6.2 below we can find a  $\sigma \in \Sigma_{\alpha}$  such that  $x \in \text{gp}(F_{\sigma})$ . But, since  $C_{\alpha} \cap \text{gp}(F_{\sigma}) = F_{\sigma}$ , we conclude that  $x \in F_{\sigma}$ . Since  $F_{\sigma} \subset S$  this shows first that  $S$  is normal and second that every element of  $S$  can be represented by linearly independent elements of  $\text{Hilb}(S)$ .  $\square$

As just seen, the crucial argument in the proof of 6.1 is the following lemma.

**Lemma 6.2.** *Let  $S$  be a positive affine semigroup with  $\text{CR}^a(S) = \text{rank } S$ . With the notation introduced above, one then has*

$$\bigcup_{\sigma \in \Sigma_\alpha} \text{gp}(F_\sigma) = \mathbb{Z}^n$$

for each index  $\alpha$ .

*Proof.* For subsets  $M \subset N$  of  $\mathbb{Z}^n$  we set

$$\pi(M | N) = \lim_{\delta \rightarrow \infty} \frac{\#(M \cap B(\delta))}{\#(N \cap B(\delta))},$$

provided that the limit exists.

Now assume that there exists  $z \in \mathbb{Z}^n$  not contained in the union of the subgroups  $\text{gp}(F_\sigma)$ . Then

$$(*) \quad z + H_\alpha \subset \mathbb{Z}^n \setminus \left( \bigcup_{\sigma \in \Sigma_\alpha} \text{gp}(F_\sigma) \right) \quad \text{where} \quad H_\alpha = \bigcap_{\sigma \in \Sigma_\alpha} \text{gp}(F_\sigma).$$

Clearly,  $H_\alpha$  is a proper subgroup of  $\mathbb{Z}^n$  of finite index  $[\mathbb{Z}^n : H_\alpha]$  (because  $H_\alpha$  has rank  $n$ ). In particular  $\pi(z + H_\alpha | \mathbb{Z}^n) = 1/[\mathbb{Z}^n : H_\alpha]$ .

Straightforward arguments show that this equation remains valid after restriction to  $C_\alpha$  (or an arbitrary cone  $C \subset \mathbb{R}^n$ ), i. e.

$$(**) \quad \pi((z + H_\alpha) \cap C_\alpha | \mathbb{Z}^n \cap C_\alpha) = 1/[\mathbb{Z}^n : H_\alpha] > 0.$$

Let us show that

$$(***) \quad \pi(S \cap C_\alpha | \mathbb{Z}^n \cap C_\alpha) = 1.$$

We need the well-known fact (for instance, see [Gu2, Lemma 5.3]) that for any affine semigroup  $T$  the conductor-ideal  $\mathfrak{c}_{\bar{T}/T} = \{t \in T \mid t + \bar{T} \subset T\}$  is not empty, where  $\bar{T}$  denotes the normalization of  $T$ .

Pick  $s \in \mathfrak{c}_{\bar{S}/S}$ . In our situation  $\bar{S} = C(S) \cap \mathbb{Z}^n$ . Thus  $s + (C(S) \cap \mathbb{Z}^n) \subset S$ . Now (\*\*\*) follows from the easily verified equality

$$\pi(s + (C(S) \cap \mathbb{Z}^n) \cap C_\alpha | \mathbb{Z}^n \cap C_\alpha) = 1.$$

The proof of the lemma is completed as follows. By similar arguments with conductor ideals one shows that

$$\pi(S \cap (z + H_\alpha) \cap C_\alpha | (z + H_\alpha) \cap C_\alpha) = 1.$$

Observe that the set of those elements of  $S$  that can be represented by linearly dependent elements of  $\text{Hilb}(S)$  is contained in the union of a finite set of hyperplanes. Then it follows easily from the last formula, condition  $\text{CR}^a(S) = n$  and (\*\*\*) that

$$\pi((z + H_\alpha) \cap C_\alpha | \mathbb{Z}^n \cap C_\alpha) = 0,$$

a contradiction to (\*\*). □

**Remark 6.3.** The proof of Theorem 6.1 suggests an algorithm deciding (FHC) (or (ICP)) for a cone  $C$ . In addition to the steps (G)–(U) outlined in Section 3 one applies the following recursive procedure to each of the non-unimodularly covered subcones  $\delta$  resulting from step (U): (i) if the union of the subgroups  $F_\sigma$ ,  $\sigma \in \Sigma_\delta$ , is  $\mathbb{Z}^n$ , then  $\delta$  is ‘freely’ covered and can be discarded (we use notation analogous to that in the proof of Lemma 6.2); (ii) otherwise, if there is a simplicial subcone  $\sigma$  generated by elements of  $\text{Hilb}(C)$  intersecting  $\delta$  in its interior, then we split  $\delta$  into two subcones along a suitable support hyperplane of  $\sigma$ ; (iii) if there is no such  $\sigma$ , then one has found a counterexample to (FHC).

It is crucial for this algorithm that the question whether  $\mathbb{Z}^n$  is the union of subgroups  $U_1, \dots, U_m$  can be decided algorithmically. For that one forms their intersection  $V$  and checks that for each residue class modulo  $V$  a representative is contained in one of the  $U_i$ .

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