CLASSICAL ALGEBRAIC K-THEORY OF MONOID ALGEBRAS

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§ 0. Introduction

Grothendieck's well known theorem asserts, that the natural homomorphism $K_0(R) \to K_0(R[x_1, \ldots, x_n, y_1, \ldots, y_m])$ is an isomorphism (where $x_j$ and $y_j$ are variables) whenever $R$ is regular. The nonstable analog of this in the case when $R$ is a field and $m = 0$ is known as Serre's problem (settled affirmatively in 1976, [16, 18]). It is natural to set the problem of the description of such monoids $L$ for which $K_0(R) \to K_0(R[L])$ is an isomorphism for regular rings $R$ (and the analogous problem for the nonstable case as well). The conjectures in this spirit were proposed in [2, 3, 9]. Solving these conjectures in [12] we succeed in describing all such commutative cancellative monoids $L$ for which projective $R[L]$-modules are free whenever $R$ is a principal ideal domain. It turned out that the class of such monoids coincides with the class of all commutative, cancellative, torsionfree, seminormal monoids (the last means $2m \in L$, $3m \in L \implies m \in L$, $m$ being the element from the group of quotients). The easy adaptation of the proof, given in [12], to the stable situation shows that the same class of monoids can be described as the maximal class of such commutative cancellative monoids $L$, for which the natural homomorphism $K_0(R) \to K_0(R[L])$ is an isomorphism whenever $R$ is a (commutative) regular ring. Thus we obtain the maximal generalization of the Grothendieck's aforementioned theorem.

gereralisation of the Bass-Heller-Swan's theorem concerning the isomorphism $K_4(R) \cong K_4(R[x_1, \ldots, x_n])$ for regular $R$ (it is clear, that we have to require the absence of the nontrivial invertible elements in the monoid). But in this direction the difficulties arise from the very beginning. It turns out that there exists a normal monoid $L$ of rank 2 ($m \in L \implies m \in L$, where $n \geq 1$ and $m$ is in the group of the quotients) for which $SK_4(R[L]) \neq \phi$ ([13]). Such counterexamples exist for other regular rings of coefficients (including some fields) as well. The simplest explicit example of the normal monoid $L$ for which $SK_4(R[L]) \neq \phi$ is given in [17]; it is the submonoid in $\mathbb{Z} \oplus \mathbb{Z}$ generated by the elements $(2,0)$, $(1,1)$, $(0,2)$ ($\mathbb{C}$ means the complex numbers). On the other hand, if we consider the sub-monoids in $Q$-spaces ($Q$ is the field of the rational numbers), which are "densely distributed", then we can establish the exact $K_4$-analogue of the aforementioned results concerning the functor $K_0$. More precisely, in this paper we prove

a) For any euclidean ring $R$ (such is any field and the ring of integers $\mathbb{Z}$) and any $c$-divisible (commutative, cancellative, torsionfree) monoid $L$ for some $c > 1$ ($\forall m \in L \exists n \in L \; c \cdot n = m$), we have $SL_4(R[L]) = E_{2,2}(R[L])$ whenever $c > 3$ ; note that all $c$-divisible monoids are seminormal.

b) The maximal class of (commutative, cancellative, torsionfree) $c$-divisible monoids $L$ (for some $c > 1$), for which the natural homomorphism $K_4(R) \to K_4(R[L])$ is an isomorphism, coincides with the class of all (commutative, cancellative, torsionfree) $c$-divisible monoids with the trivial subgroups of the invertible elements; here $R$ is any regular ring;
e) The analogous statement concerning the functor $K_L$ for all those $a$-divisible monoids $L$ in which there exists a system of linearly independent elements $m_1, \ldots, m_d$ such that for any $m \in L$ there exist nonnegative rational numbers $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ satisfying the equality $m = \lambda_1 m_1 + \cdots + \lambda_d m_d$ (in $\mathbb{Q} \otimes K(L)$).

Our proofs are based on the geometric interpretation of monoids, which was introduced in [I2]. The results concerning $K_L$ are obtained by combining of these (geometric) methods with those ones from [I3].

Some words about notations:
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote integers, rationals and reals (resp.);
$\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ denote the additive monoids of corresponding nonnegative numbers;

All the considered monoids are assumed to be commutative, cancellative and torsionfree (i.e. there does not exist a torsion element in the group of quotients);

For any monoid $L$ its group of quotients is denoted by $K(L)$;

All the considered rings are assumed to be commutative, with 1, and the homomorphisms preserving 1 |

For any ring $R$ and its ideal $I$ by $E_\gamma(R, I)$ is denoted the normal subgroup in the group of elementary matrices $E_\gamma(R)$ obtained by $e_i^{(a)}(x)$, $i \neq j, a \in I$, where $e_i^{(a)}(x)$ means the standard elementary matrix with one nonzero non-diagonal component $a$ in the $i$-th row and $j$-th column; $E_\gamma(R, I)$ is normal in $GL_\gamma(R)$ for $\gamma \geq 3$ ([93]). By $x_i^{(a)}(x)$, $i \neq j$, will be denoted the standard generators of the Steinberg group $SL(R)$;

Our other $\gamma$-theoretic notations follow [4, 5, 115];

By max $(R)$ will be denoted the maximal spectrum of $R$ and by spec $(R)$ the prime one.

§ I. Combinatorial geometry and monoids

I. For any natural $d$ by $S^{d-1}$ we denote the standard unit sphere in the euclidean space $\mathbb{R}^d$. A subset $X \subseteq S^{d-1}$ is said to be convex if for any $x, y \in X$, which are not opposite on $S^{d-1}$, the shortest line (obviously uniquely determined) on $S^{d-1}$ connecting $x$ and $y$ is a subset of $X$. In this sense the pair of opposite points on $S^{d-1}$ turns out to be convex. For any subset $Y \subseteq S^{d-1}$, there exists the smallest convex subset in $S^{d-1}$ containing $Y$ (it will be called the convex envelope of $Y$). For any natural $0 < \gamma < d$ we define an $\gamma$-dimensional open convex subset in $S^{d-1}$ as a subset of geometric dimension $\gamma$, for which there exists a real subspace in $\mathbb{R}$ of dimension $\gamma + 1$, intersection of which with the sphere $S^{d-1}$ contains our subset as a convex open subset. In these terms $0$-dimensional open convex subsets in $S^{d-1}$ are just points or pairs of opposite points. An $\gamma$-dimensional convex closed polytope is defined as an $\gamma$-dimensional intersection of the finite family of closed hemispheres of the sphere $E \cap S^{d-1}$, where $E$ is some real subspace in $\mathbb{R}$ of dimension $\geq \gamma + 1$. Analogously is defined an open convex polytope. A point $x \in S^{d-1}$ is called rational if it is determined by the intersection of some rational radial direction with the sphere $S^{d-1}$. A convex subset $X \subseteq S^{d-1}$ is said to be rational if it is spanned by the rational points. The verification of the following statements is straightforward:

a) Any proper open convex subset (of arbitrary dimension) can be represented as a union of closed polytopes (as well as open ones) of the same dimension, which are embedded in each other;

b) Any proper convex subset in $S^{d-1}$ is contained in some
closed hemisphere of $S^{d-1}$ in the case when this convex subset is closed and without a pair of opposite points there exists an open enveloping hemisphere.

We have the following "rationalizations" of these statements:

a) Any proper open convex $(d-1)$-dimensional subset in $S^{d-1}$ is rational and such subset can be represented as a union of rational closed polytopes of dimension $(d-1)$, embedded in each other;

b) Any rational closed convex proper subset without a pair of opposite points is contained in the open hemisphere with the rational closure.

I.2. The monoid structure (throughout this paragraph) will be written additively.

Definition I.1.

a) An extension of monoids $\mathcal{N} \subseteq \mathcal{M}$ is said to be integral, if for any $m \in \mathcal{M}$ there exists $n \in \mathcal{N}$, such that $nm \in \mathcal{N}$

b) A submonoid $\mathcal{N} \subseteq \mathcal{M}$ is called integrally closed in $\mathcal{M}$ if $\mathcal{N}$ coincides with the largest submonoid in $\mathcal{M}$ which is integral over $\mathcal{N}$.

c) If a monoid $\mathcal{M}$ is integrally closed in its group of quotients $K(\mathcal{M})$ then $\mathcal{M}$ is called integrally closed; in the case of finite generation of $\mathcal{M}$ we'll use the word "normal" as well.

d) A monoid $\mathcal{M}$ is said to be seminormal if for any $x \in K(\mathcal{M})$ the following implication holds: $2x, 3x \in \mathcal{M} \Rightarrow x \in \mathcal{M}$

We recall that a domain $R$ is called seminormal of its multiplicative monoid of nonzero elements is seminormal.

The following statement is well known (see, for example, [4, 11, 12]):

Theorem I.2. For any domain $R$ and any monoid $\mathcal{M}$ the integrally closedness (seminormality) of the monoid domain $R[\mathcal{M}]$ is equivalent to the integrally closedness (seminormality) of $R$ and $\mathcal{M}$ simultaneously.

Consider the monoid $\mathcal{M}$ for which $\text{rank}(\mathcal{M}) \leq \dim_{Q}(\mathcal{Q} \otimes K(\mathcal{M})) < \infty$.

Denote this rank by $d$ and fix one of the embeddings obtained by the composition $\mathcal{M} \to \chi(\mathcal{M}) \to \mathcal{Q} \otimes K(\mathcal{M}) \to \mathcal{Q} \to \mathcal{R}$.

Assume $\mathcal{M} \neq \{0\}$. Then for any $m \in \mathcal{M}\setminus\{0\}$ by $\Phi(m)$ denote the intersection of the radial ray, obtained by $\mathcal{M}$ (via the embedding $\mathcal{M} \hookrightarrow \mathcal{Q} \otimes \mathcal{R}$), with the sphere $S^{d-1}$. By $\Phi(\mathcal{M})$ denote the convex envelope of the set $\{\Phi(m) | m \in \mathcal{M}\setminus\{0\}\}$. In case $\mathcal{M} = \{0\}$ put $\Phi(\mathcal{M}) = \emptyset$. It should be noted that $\Phi(\mathcal{M})$ is the dense subset in $\Phi(\mathcal{M})$, more precisely — it coincides with the set of all rational points of the rational convex subset $\Phi(\mathcal{M}) = S^{d-1}$.

Proposition I.3.

a) An extension of monoids $\mathcal{N} \subseteq \mathcal{M}$ is integral if $\Phi(\mathcal{N}) = \Phi(\mathcal{M})$.

b) A submonoid $\mathcal{N}$ is integrally closed in $\mathcal{M}$ if the following implication holds: $m \in \mathcal{M}\setminus\{0\}$, $\Phi(m) \in \Phi(\mathcal{N}) \Rightarrow m \in \mathcal{N}$.

It should be noted that if we consider a system of monoids $\{\mathcal{M}_1, \ldots, \mathcal{M}_{n}\}$ which are submonoids of some monoid $\mathcal{M}$, then we consider the $\Phi$-correspondences for $\mathcal{M}_1, \ldots, \mathcal{M}_{n}$, which are induced from the $\Phi$-correspondence for $\mathcal{M}$.

Proof of the proposition I.3 is straightforward.

The following is known as Gordan's lemma.

Lemma I.4 (see [10]). For any natural $\tau$ and any integ-
rally closed submonoid $\mathcal{M} \subseteq \mathbb{Z}^d$ the finite generation of $\mathcal{M}$ is equivalent to the fact that either $\Phi(\mathcal{M})$ is a (finite) closed polytope or $\Phi(\mathcal{M})$ is congruent to some $S^{d-1}$, $d < \infty$.

**Remark I.6.** In fact Lemma 4 holds for arbitrary submonoids $\mathcal{M} \subseteq \mathbb{Z}^d$ without any requirement of the integral closedness.

If the contrary is not said all the considered monoids are assumed to have a finite rank.

**Notation I.6.**

a) Let $\mathcal{W}$ be any convex subset in $\Phi(\mathcal{M})$ (for some monoid $\mathcal{M}$), then by $\mathcal{M}(\mathcal{W})$ we denote the "$\Phi$-submonoid):

$$\{m \in \mathcal{M} \mid (\forall w \in \mathcal{W}) \exists q \in \mathcal{W}, q \cdot w = m\};$$

b) For any monoid $\mathcal{M} \neq \{0\}$ by $\mathcal{M}$ we denote the "$\Phi$-submonoid" $\mathcal{M}(\text{int} \Phi(\mathcal{M})) = \mathcal{M}$ where $\text{int} \Phi(\mathcal{M})$ is the interior of the convex set $\Phi(\mathcal{M})$ (of course, here the relative interior is meant — that is we consider the usual interior of $\Phi(\mathcal{M})$ in the sphere $E \cap S^{d-1}$, where $E$ is a real subspace in $\mathcal{M}$ for which $\Phi(\mathcal{M}) = E$ and $\dim E = \dim \Phi(\mathcal{M}) + 1$);

c) For any multiplicative subset $S \subseteq \mathbb{Z}$ (the ring of integers), consisting of the positive numbers, and for any monoid $\mathcal{M}$ by $S^{-1}\mathcal{M}$ we denote the universal monoid which contains $\mathcal{M}$ and is $S$-divisible for any $s \in S$ (i.e. for any element $x$ there exists $y$ such that $sy = x$). Constructively $S^{-1}\mathcal{M}$ can be obtained as follows

$$S^{-1}\mathcal{M} = \bigcap_{s \in S} (\mathcal{M} \cdot s^{-1}), s \in S;$$

in the case $S = \{1, c_1, c_2, \ldots\}$ for some $c \in \mathbb{N}$ we'll write $c^{-1}\mathcal{M}$ instead of $S^{-1}\mathcal{M}$.

\[d\) Let $\mathcal{N}$ be a submonoid in $\mathcal{M}$, then by $\mathcal{N}^{-1}\mathcal{M}$ we denote the localization of $\mathcal{M}$ relatively $\mathcal{N}$, i.e. $\mathcal{N}^{-1}\mathcal{M}$ is the submonoid in $\mathcal{K}(\mathcal{M})$ generated by $\mathcal{K}(\mathcal{N}) \cup \mathcal{M}$.

**Proposition I.7.** ([22]) Let $\mathcal{M}$ be an integrally closed monoid and $\mathcal{W}$ be any rational convex subset in $S^{d-1}$ for which $\dim \mathcal{W} = \dim (\mathcal{W} \cap \Phi(\mathcal{M}))$. Then there exists a unique integrally closed submonoid $\mathcal{L} \subseteq \mathcal{K}(\mathcal{M})$ such that $\mathcal{L} \cap \mathcal{M}$ is integrally closed in $\mathcal{M}$ and in $\mathcal{L}$ simultaneously and $\Phi(\mathcal{L}) = \mathcal{W}$.

It is obvious, that $\mathcal{L} = \mathcal{K}(\mathcal{M})(\mathcal{W})$. In this situation the monoid $\mathcal{L}$ will be denoted by $\mathcal{M}(\mathcal{W})$. This notation generalizes the notation, introduced in Not. I.6, a.

In the following (according to [22]) we will say that $\mathcal{N}$ is an $\Phi$-submonoid in $\mathcal{M}$ if $\mathcal{N}$ is integrally closed in $\mathcal{M}$.

**Proposition I.8.** ([22]). Let $\mathcal{M}$ be any normal monoid of $\text{rank}(\mathcal{M}) = 1$ and without nontrivial invertible elements ($\mathcal{M}$ is finitely generated). Then for any vertex $\alpha$ of the polytope $\Phi(\mathcal{M})$ we have

$$\mathcal{M}(\{\alpha\})^{-1}\mathcal{M} \cong \mathbb{Z} \oplus \mathcal{M}';$$

where $\mathcal{M}'$ is a normal monoid without nontrivial invertible elements and with $\text{rank}(\mathcal{M}') = \text{rank}(\mathcal{M}) - 1$.

**Proposition I.9.** ([22]). Let $\mathcal{M}$ be any monoid. Then $\mathcal{M}$ is seminormal iff for any open convex rational subset $\mathcal{W} \subset \Phi(\mathcal{M})$ of arbitrary dimension the monoid $\mathcal{M}(\mathcal{W})$ is integrally closed.

In the following we'll also use the following

**Proposition I.10.** For any natural $d$ and any integrally closed submonoid $\mathcal{M} \subseteq \mathbb{Z}^d$, for which the extension $\mathcal{M} \subseteq \mathbb{Z}^d$
is integral, there exists a system of the type
\[ a_1 = (a_{i1}, a_{i2}, \ldots, a_{i\hat{d}}) \in \mathbb{Z}^d_+ , \]
\[ a_2 = (0, a_{i2}, \ldots, a_{i\hat{d}}) \in \mathbb{Z}^d_+ , \]
\[ \ldots \]
\[ a_d = (0, 0, \ldots, a_{i\hat{d}}) \in \mathbb{Z}^d_+ , \]

where \( a_{i\hat{d}} > 0 \) (where \( i \in \{1, \ldots, \hat{d}\} \)), such that \( \mathcal{M} = \mathbb{Z}^d_+ \cap G \), where \( G \) is the subgroup in \( \mathbb{Z}^d_+ \) generated by the elements \( a_1, \ldots, a_d \).

Proof (Sketch). From the integrally closedness of \( \mathfrak{M} \) and the integrality of the extension \( \mathcal{M} = \mathbb{Z}^d_+ \), we obtain that \( \mathcal{M} = k(\mathfrak{M}) \cap \mathbb{Z}^d_+ \) (the intersection is considered in the group \( \mathbb{Z}^d_+ \)). Now, for any \( i \in \{1, \ldots, \hat{d}\} \) consider the set of all elements from \( \mathcal{M} \), which have the type \( (0, \ldots, 0, b_{i1}, \ldots, b_{i\hat{d}}) \) with \( b_{i\hat{d}} = 0 \), and choose among them the element with minimal \( b_{i\hat{d}} \) (it is obvious, that the set mentioned here is not empty).

In the following if the monoid \( \mathcal{M} \) satisfies the condition of the existence of a rational closed hemisphere in \( S^{d-1} \), for which \( \Phi(\mathcal{M}) \) is a subset of the interior of this hemisphere, we will consider the intersections of the radial rays, obtained by the elements of \( \mathcal{M} \), with the tangent to this hemisphere hyperplane (of dimension \( d-1 \)), which is parallel to the boundary of our hemisphere. Thus, in the mentioned situation, our \( \Phi \)-correspondence will be assumed to be "flat". Let us note that the mentioned hyperplane is "rational" (it is obtained by the rational linear form) and the points of the type \( \Phi(m) \) become rational modulo certain factor. The geometric facts, listed in the section I.I, in this "flat" situation become more obvious.

Proposition I.II ([12]).

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a) For any monoid \( \mathcal{M} \) the group of its invertible elements \( \mathcal{U}(\mathcal{M}) \) is trivial iff \( \Phi(\mathcal{M}) \) does not possess a pair of opposite points.

b) A \( \Phi \)-submonoid of an integrally closed (semiregular) monoid is so.

c) For any monoid \( \mathcal{M} \) and any convex subset \( \mathcal{W} \) of \( \Phi(\mathcal{M}) \) for which \( \dim \mathcal{W} = \dim \Phi(\mathcal{M}) \) we have \( k(\mathcal{M}) = k(\mathcal{M}(\mathcal{W})) \).

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I.3. Approximation theorems for \( \mathbb{C} \)-divisible monoids. In this paper we always assume that \( \mathbb{C} \) denotes a natural number not equal to 1. Throughout this section the \( \Phi \)-correspondence will be assumed to be "flat". We define a (closed) simplex as a polytope for which the number of vertices coincides with the geometric dimension of this polytope + 1. A convex set will be called an open simplex if it is the interior of some closed simplex.

We recall that a monoid \( \mathcal{M} \) is \( \mathbb{C} \)-divisible if \( \mathbb{C}^+ \mathcal{M} = \mathcal{M} \).

Theorem 1. Let \( \mathcal{M} \) be any \( \mathbb{C} \)-divisible (for some \( \mathbb{C} > 1 \)) integrally closed monoid for which \( \Phi(\mathcal{M}) \) is an open simplex. Then \( \mathcal{M} \) can be represented as a limit of a directed diagram of free monoids.

Proof. It suffices to show that for any finite subset \( \mathcal{M}_0 = \mathcal{M} \) there exists an intermediate free monoid \( \mathcal{M}_0 = \mathcal{M} \subseteq \mathcal{M} \). By \( \Phi_0 \), denote the convex envelope of the finite set \( \{ \Phi(m) \mid m \in \mathcal{M}_0 \} \) (we assume that \( 0 \notin \mathcal{M}_0 \)). By elementary geometric reasonings we have the existence of a rational closed simplex \( \Delta = \Phi_0(\mathcal{M}_0) \), for which the set \( \{ \Phi(m) \mid m \in \mathcal{M}_0 \} \) consists of the internal points of \( \Delta \). By \( G_0 \), denote the group of quotients of the submonoid of \( \mathcal{M} \), generated by \( \mathcal{M}_0 \). Of course \( G_0 = k(\mathcal{M}(\mathcal{M}_0)) \), and thus there exists a finitely generated submonoid \( \mathcal{W} = \mathcal{M}(\mathcal{M}) \) for which \( G_0 = k(\mathcal{M}) \). Without
the loss of generality we can assume $\Phi(N) = \Delta$. By Gordon's lemma (Lemma I.4) the integral closure of $N$ remains finitely generated; so, in addition, we can assume that $N$ is integrally closed. By the suitable choice of the coordinates in $\Phi(G)$ we can also assume that $N \subseteq \mathbb{Z}_d^d$ and $\Phi(N) = \Phi(\mathbb{Z}_d^d)$ (here $d = \text{rank}(N)$, $d \geq 1$). Let $a_1, \ldots, a_d \in \mathbb{Z}_d^d$ be the elements from $N$ mentioned in the proposition I.10 and $G$ be a subgroup of $\mathbb{Z}_d^d$, generated by $\{a_1, \ldots, a_d\}$, then $K(N) = G$ and, thus, $G_o = G$. The free submonoid $N' \subseteq N$, generated by the elements $a_1, \ldots, a_d$ is a $\Phi$-submonoid in $N$ (because $K(N') = K(N)$). Now, choose a sufficiently large natural $\tau$ and consider the element $a'_d = c^{-\tau} \cdot a_d$. We obtain a new system of the (linearly independent) elements $a_1, a_2, \ldots, a_d, a'_d$. For arbitrary natural numbers $x_1, \ldots, x_d$, $x_{d-1}$, we have a free monoid $N''$, generated by the elements $a_1, a_2, \ldots, a_d, a'_d$. Of course $\Phi(N') \subseteq \Phi(N'')$ (here the $\Phi$-correspondence may be not "flat"). By $\Delta_d$ denote the base (of dimension $d - 2$) of $\Delta$, which is opposite to the vertex $\Phi(a_d)$.

If $\tau$ is sufficiently large then the points $\Phi(a_1 + x_1 a'_1, \ldots, a_d + x_d a'_d)$ move sufficiently closely to the base $\Delta_d$, when $x_1 \to \infty$. We can choose the natural numbers $\tau, x_1, \ldots, x_{d-1}$ so that the points $\Phi(a_1 + x_1 a'_1, \ldots, a_d + x_d a'_d)$ will be sufficiently close to $\Delta_d$; at the same time remaining the internal points of $\Delta$. In this situation our $\Phi$-correspondence again can be assumed to be "flat" (because we don't exceed the boundary of $\Delta$). Furthermore, by suitable choice of the natural numbers mentioned above we can assume that the affine space spanned by the points $\Phi(a_1 + x_1 a'_1, \ldots, a_d + x_d a'_d)$ is "almost" parallel to $\Delta_d$ (with the sufficient exactness). The free monoid, generated by the elements $a_1 x_1 a'_1, \ldots, a_d x_d a'_d$, is denoted by $N'' = \Phi(N'')$. We have $\Phi(N) = \Phi(N'') = \Delta$ and $G = K(N'') = K(N)$.

Now repeat our procedure of the "blowing up" of free monoids relatively to the free $\Phi$-submonoid of $N''$, generated by the elements $a_1, a_2, \ldots, a_d$. Thus we obtain a new free monoid $N_3$, for which $\Phi(N_3) \subseteq \Phi(N'')$, $K(N_3) \subseteq K(N'')$ and one of the two-dimensional faces of the simplex $\Phi(N'')$ "almost" coincides with certain two-dimensional face of $\Delta$ (for the monoid $N_3$ it was so for some $1$-dimensional face). Continuing our procedure we obtain an increasing sequence of free monoids $N_1 \subseteq N_2 \subseteq \ldots \subseteq N''$ for which $\Phi(N_1) = \Phi(N_2) = \ldots = \Phi(N'')$ and $\Phi(N_2)$ is a sub-simplex in $\Delta$, almost covering the interior of $\Delta$ ($\Phi(N'')$ consists of the internal points of $\Delta$) with the sufficient exactness. Thus, we can assume $M = N''$. Theorem A is proved.

Remark I.12. It must be noted that the openness of $\Phi(M)$ in $\Theta$ is essential. The analogous statement for the closed simplices $\Phi(M)$ is not true. The simplest explicit example of this is the monoid $3^{-1} L$, where $L$ is the normal submonoid in $\mathbb{Z}_3^2$ generated by $(2, 0), (4, 1), (0, 2)$. Indeed, if $3^{-1} L = \ell_{\mathbb{X}_3^2}$, where diagram is directed, then we would have a presentation of the type $(4, 1) = a(0, 2, 3) + b(0, 2, 3)$ for some nonnegative integers $a, b, m, n$, which is impossible.

As before, the $\Phi$-correspondence is assumed to be flat.

Definition I.13.

I) A polarized monoid is a triple $(P, \Gamma, M)$ where $P$ is a rational point (the pole), $\Gamma$ is a convex rational finite closed polytope not containing $P$ and $M$ is a finitely generated integ-
rally closed monoid, for which the following conditions hold:

a) \( P \) does not belong to any affine space, spanned by arbitrary face of the polytope \( \Gamma \),

b) \( \Phi(\mathcal{M}) \) coincides with the convex envelope of \( \Gamma \cup \{P\} \) and \( \dim \Phi(\mathcal{M}) = \dim \Gamma \),

c) for any \((\dim \Gamma - 1)\)-dimensional face \( \gamma \) of \( \Gamma \), the \( \Phi \)-submonoid \( \mathcal{M}_\gamma = \mathcal{M} \), obtained by the subpolytope of \( \Phi(\mathcal{M}) \) spanned by \( \gamma \cup \{P\} \), is generated by \( \mathcal{M}(\gamma) \cup \mathcal{M}(\{P\}) \);

2) A quasipolarized monoid is a triple \((\mathcal{P}, \Gamma, \mathcal{M})\), where \( \mathcal{P} \) is a point and \( \Gamma \) is a (convex) finite closed polytope (may be nonrational), not containing \( \mathcal{P} \), and \( \mathcal{M} \) is an integrally closed monoid, for which \( \Phi(\mathcal{M}) \) coincides with the interior of the convex envelope spanned by \( \Gamma \cup \{P\} \) and \( \dim \Phi(\mathcal{M}) = \dim \Gamma \) (we don't require the finite generation for \( K(\mathcal{M}) \)).

Let \((\mathcal{P}, \Gamma, \mathcal{M})\) be a polarized monoid. In the following:

a) \( \mathcal{L} \) will be denoted the point opposite to \( P \) (on \( S^{d-1} \)),
b) \( \mathcal{L} = \mathcal{M}(\mathcal{M}(\gamma)) \) denoted the monoid in \( K(\mathcal{M}) \) generated by \( \mathcal{M}(\gamma) \cup \{-m \mid m \in \mathbb{Z} \} \),
c) a \((\dim \Gamma - 1)\)-dimensional face of \( \Gamma \) for which the pole \( P \) and the polytope \( \Gamma \) lie on the same side relatively to this face (in the affine space, spanned by \( \Phi(\mathcal{M}) \) or equivalently by \( \Gamma \)) will be called positive; other faces of \( \dim \Gamma - 1 \) will be called negative.

**Proposition I.III.** If \((\mathcal{P}, \Gamma, \mathcal{M})\) is polarized monoid, then \((\mathcal{L}, \Gamma, \mathcal{M})\) is so. The positive faces of \( \Gamma \) in the first triple become negative in the second triple and conversely.

**Proof.** Denote by \( \mathcal{L} \) the \( \Phi \)-submonoid \( \mathcal{M}(\mathcal{P}) = \mathcal{M} \) and by \( \Gamma_+ (\Gamma_-) \) the convex set \( \Phi(\mathcal{M}) \) \( (\Phi(\mathcal{M})(\mathcal{M}_0) \) resp.).

Let \( \{\gamma_i\} \) be the set of all positive \((\dim \Gamma - 1)\)-dimensional faces of \((\mathcal{P}, \Gamma, \mathcal{M})\) and \( \{\gamma_j\} \) be the set of negative ones. Then we have \( \Gamma_0 = \mathcal{L} \cup \mathcal{L} \) and \( \Gamma = \mathcal{M}(\mathcal{L}) \), where \( \mathcal{L} \) denotes the convex env. of \( \mathcal{L} \) and \( \mathcal{L} \) is the convex envelope of \( \{\gamma_i\} \). To show the normality of \( \mathcal{M} \) it suffices to show the normality of each \( \mathcal{M}(\gamma_i) \) (because, according to Prop. I.III. c, we have \( K(\mathcal{M}) = K(\mathcal{M}(\gamma_i)) \)).

Since \( \mathcal{M}(\Gamma) \) is a \( \Phi \)-submonoid of \( \mathcal{M} \) (because \( \mathcal{M} \) is normal) we have \( K(\mathcal{M}(\gamma_i)) = K(\mathcal{M}(\gamma_i)) \) and in its turn \( K(\mathcal{M}(\gamma_i)) \) coincides with \( K(\mathcal{M}(\gamma_i)) \) (see the definition I.III). By def. I.III. c, \( \mathcal{M}(\gamma_i) \approx \mathcal{M}(\gamma_i) \mathcal{A} \). Finally, we obtain \( \mathcal{M}(\gamma_i) \approx \mathcal{M}(\gamma_i) \mathcal{A} \), where \( \mathcal{A} = \{-m \mid m \in \mathbb{Z} \} \), and this monoid is obviously normal (moreover, since \( \gamma_i \approx \mathcal{A} \), we have \( \mathcal{M}(\gamma_i) \approx \mathcal{M}(\gamma_i) \)).

Now the proof of our statement can be easily completed.

**Theorem B.** Let \((\mathcal{P}, \Gamma, \mathcal{M})\) be any quasipolarized \( c \)-divisible monoid (for some \( c > 1 \)), then there exists a (countable) system of polarized monoids \((\mathcal{P}_i, \Gamma_i, \mathcal{M}_i)\) and a directed diagram \( \{\mathcal{L}_{ij} : \mathcal{M}_i \to \mathcal{M}_j\} \), such that:

a) \( \mathcal{M} = \lim (\mathcal{L}_{ij} : \mathcal{M}_i \to \mathcal{M}_j) \),
b) \( \mathcal{L}_{ij} (\mathcal{M}_i(\Gamma_i)) \subset \mathcal{M}_j(\Gamma_j) \),
c) \( \mathcal{M}_i(\Gamma_i) \) naturally maps into \( \mathcal{M}_i(\Gamma_i) \) (see Not. 6) so that \( \lim (\mathcal{M}_i(\Gamma_i)) = \mathcal{M}(\Gamma) \).

**Proof.** It just suffices to show that for any finitely generated submonoid \( \mathcal{M}_0 \approx \mathcal{M} \) there exists a polarized monoid \((\mathcal{P}_i, \Gamma_i, \mathcal{M}_i)\) such that \( \Gamma_i^* \) almost covers \( \Gamma \) (the interior of the polytope \( \Gamma \)), \( \mathcal{M}_i \approx \mathcal{M} \) and \( \mathcal{M}_i^* \approx \mathcal{M}_i \). Choose a (convex, finite, closed) subpolytope \( \tilde{\Gamma} \subset \Gamma \) and a rational point \( \tilde{P} \in \Phi(\mathcal{M}) \) so that \( \tilde{\Gamma} \) is included in the interior of \( \tilde{\Gamma} \), \( \tilde{P} \notin \tilde{\Gamma} \), \( \tilde{P} \) does not belong to any affine space spanned by any face of \( \tilde{\Gamma} \) (we assume that \( \dim \tilde{\Gamma} = \dim \Gamma \) and \( \Phi(\mathcal{M}_0) \) is a subset in the interior of the convex envelope of
The rationality of $\tilde{r}$ is not required. The existence of such objects is obvious: we have to choose $\tilde{P}$ sufficiently close to $P$ and choose $\tilde{\nu}$ so that it "almost" covers the interior of $\nu$, and then move slightly the point $\tilde{P}$ so that it remains rational and the conditions on the relation between the faces of $\tilde{\nu}$ and $\tilde{P}$ are satisfied. Now, choose any rational simplex $\Delta$, for which $\tilde{P}$ is one of the vertices and $\tilde{\nu} \subset \Delta$. Then $\nu_0 \subset \nu(0)$, since $\nu_0$ is finitely generated there exists a normal submonoid $\nu \subset \nu(0)$, such that $\nu_0 \subset \nu$ and $\mathcal{D}(\nu) = \Delta$. Here the Jordan's lemma is used. By the suitable choice of the coordinates in $\mathbb{Q}^{\mathbb{Q}}$, we can assume that $\Delta = \Phi(\mathbb{Z}^d_+) (d = \operatorname{rank}(\nu(0)))$, $\nu \subset \mathbb{Z}^d_+$ and this extension is integral. Let $a_1, \ldots, a_{d-1}, a_d$ be the elements mentioned in the proposition I.10, for which $\Phi(a_{d-1}) = \tilde{P}$. By similar reasoning, which were used in the proof of Th. A, we come to the conclusion that there exist the natural numbers $g, x_1, \ldots, x_{d-1}$, such that $\tilde{P}$ and $\tilde{\nu}$ lie on the same side (in the affine space, spanned by $\Phi(\nu)$) relatively to the affine space of dimension $\dim(\nu - l)$ spanned by the points $\Phi(a_{d-1} - x_{d-1} a_d), \ldots, \Phi(a_{d-1} + x_{d-1} a_d)$ where $a_d = c^{\nu} a_d$. Denote this (dim $\nu - l$) - dimensional affine space by $X$ and consider the polar projection $\pi$ of the polytope $\tilde{\nu}$ into $X$ (with the pole $\tilde{P}$). By our conditions on $\tilde{\nu}$, none of the faces of $\tilde{\nu}$ degenerate, i.e. $\tilde{\nu}$ preserves their dimensions. For any (dim $\tilde{\nu} - l$) - dimensional face $\tilde{\gamma}$ of $\tilde{\nu}$, the group of quotients of the monoid $(\mathbb{Z} a_1 + \cdots + \mathbb{Z} a_{d-1} + \mathbb{Z} a_d)$ is the same (according to the proposition I.11.c). For any such $\gamma$ there exists a base $B_\gamma = \{e_1, \ldots, e_d\}$ of this group $=$ $\mathbb{Z}(a_1 - x_{d-1} a_d) + \cdots + \mathbb{Z}(a_{d-1} - x_{d-1} a_d)$, satisfying the condition: $\{e_i\}_{i=1}^{d-1}$ is a subset of the interior of $\nu(\gamma)$. To show the existence of such bases we have to choose the rational closed simplices (of dimension $\dim X = \dim(\nu(\gamma))$) in the interiors of $\tilde{\nu}(\gamma)$ and then again apply the proposition I.10. There exists natural numbers $x_1^2, \ldots, x_{d-1}^2, \gamma'$, where $\gamma'$ varies over the set of all $(\dim \tilde{\nu} - l)$ - dimensional faces of $\tilde{\nu}$, such that for each $\gamma'$ the affine space $\gamma'$ spanned by the points $\Phi(\tilde{P} + x_1^2 \mathcal{C} \mathcal{C} a_d), \ldots, \Phi(\tilde{P} + x_{d-1}^2 \mathcal{C} \mathcal{C} a_d)$ is sufficiently close and sufficiently parallel to the affine space spanned by $\gamma'$ (if $\gamma'$ is sufficiently large then the points $\Phi(\tilde{P} + x_1 \mathcal{C} \mathcal{C} a_d), \ldots, \Phi(\tilde{P} + x_{d-1} \mathcal{C} \mathcal{C} a_d)$ move sufficiently slowly towards the point $\tilde{P}$ when $x_1 \to \infty$, and at the same time these points remain on the line, connecting $\Phi(\tilde{P})$ and $\tilde{P}$). Thus, we can achieve that the convex, closed, rational polytope $\tilde{\nu}'$, bounded by these affine spaces $\gamma'$ will be included in the interior of $\tilde{\nu}$ and will cover this interior with the sufficient exactness, so that we have the inclusion $\Phi(\nu) \subset \operatorname{conv. env.} \nu'$ when $\tilde{\nu}$ will not belong to any $\gamma'$. By $H$ denote the subgroup in $K(\nu)$, generated by $\{c^{\nu} a_d\} \cup \mathcal{G}$ (it does not depend on $\nu$) and by $\mathcal{M}$ denote the $\Phi$ - submonoid in $H$, obtained by the convex envelope of $\{e_i\}_{i=1}^d$. Then $\tilde{P}, \tilde{\nu}, \mathcal{M}$ will be a desired polarised monoid, Q.E.D.

I.4. In the case when the multiplicative subset $S \subset \mathbb{Z}$ coincides with the set of all positive integers, the monoid $S^{-1} \mathcal{M}$ (for any monoid $\mathcal{M}$) will be denoted by $\mathcal{M}_0 \mathcal{M}$. For any monoid $\mathcal{M}$ put $\mathcal{M}_0 = \mathcal{M} \setminus \{0\}$.

Proposition I.15. For any monoid $\mathcal{M}$ of rank $d$ the convex...
set $\Phi(\mathcal{M})$ is a simplex of dimension $d - 1$ ($d \geq 1$) iff $Q_c \cdot \mathcal{M} = Q_c^d$; all $c$-divisible monoids are seminormal ($c > 1$).

Proof is trivial ($2m, 3m \in \mathcal{M} \Rightarrow c m \in \mathcal{M} \Rightarrow m \in \mathcal{M}$).

We recall that a functor $F : \text{Rings} \rightarrow \text{Ab}{,}\text{Groups}$ is continuous if it preserves the limits of the directed diagrams; and $F$ is semiexact if for any cartesian square of rings

$$
\begin{array}{ccc}
R & \rightarrow & R_1 \\
\downarrow & & \downarrow i \\
R_2 & \rightarrow & R',
\end{array}
$$

with $i$ surjective, the natural sequence

$$F(R) \rightarrow F(R_1) \oplus F(R_2) \rightarrow F(R')$$

is exact; a ring is called $F$-regular if for any natural $n$ the natural homomorphism $F(R) \rightarrow F(R[\mathbf{x}_1, \ldots, \mathbf{x}_n])$ is an isomorphism ($\mathbf{x}_i$'s are variables).

Theorem I.16. Let $F : \text{Rings} \rightarrow \text{Ab}{,}\text{Groups}$ be any continuous semiexact functor, $R$ be a $F$-regular ring and $\mathcal{M}$ be a $C$-divisible monoid for some $c > 1$, such that $Q_c \cdot \mathcal{M} = Q_c^d$ ($d \in \mathbb{N}$). Then the natural homomorphism $F(R) \rightarrow F(R[\mathcal{M}])$ is an isomorphism.

Proof. Consider the following cartesian square

$$
\begin{array}{ccc}
R[\mathcal{M}_+^\times] & \rightarrow & R[\mathcal{M}] \\
R & \rightarrow & R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}].
\end{array}
$$

Since $\mathcal{M}_+^\times$ is integrally closed (Fr. I.9 and I.15) by the Approximation Theorem A we have $\mathcal{M}_+^\times = \lim_{\rightarrow} \mathbb{Z}_+^d$. By continuity of $F$ we obtain $F(R[\mathcal{M}_+^\times]) = \lim_{\rightarrow} F(R[\mathbb{Z}_+^d]) = F(R)$.

Hence, our cartesian square implies the exact sequence

$$F(R) \rightarrow F(R) \oplus F(R[\mathcal{M}]) \rightarrow F(R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}]).$$

Let $A_i$ be any $(\dim \Phi(\mathcal{M}) - 1)$-dimensional face of the simplex $\Phi(\mathcal{M})$. Then we have the cartesian square

$$
\begin{array}{ccc}
R[\mathcal{M}(A_i)_+] & \rightarrow & R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}] \\
\downarrow & & \downarrow \\
R & \rightarrow & R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}].
\end{array}
$$

Thus, as above, we obtain the exact sequence

$$F(R) \rightarrow F(R) \oplus F(R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}]) \rightarrow F(R[\mathcal{M}] / \mathcal{M}(A_i)_+^\times R[\mathcal{M}]).$$

Now, repeat the same procedure relatively to another $(\dim \Phi(\mathcal{M}) - 1)$-dimensional face of $\Phi(\mathcal{M})$, etc. After the "killing" of the interiors of all $(\dim \Phi(\mathcal{M}) - 1)$-dimensional faces we turn to the $(\dim \Phi(\mathcal{M}) - 2)$-dimensional faces, etc. Finally we'll descend to the coefficient ring $R$. Thus we have the exact sequences

$$F(R) \rightarrow F(R) \oplus F(A_i) \rightarrow F(A_i^+)$$

where $i \in C, \mathcal{M}$ and $A_i = R[\mathcal{M}], A_i^+ = R[\mathcal{M}] / \mathcal{M}_+^\times R[\mathcal{M}]$. 


$A_k = R[\mu]/(\mu(0), \mu(1))R[\mu]$, $A_h = R[\mu]/(\mu(0), \mu(1))R[\mu]$.

Let $R$ be a $K_l$-regular ring $(l=0,1)$. Then $R$ is any monoid for which $Q_k \cong Q_k^+(\mu)$ for any natural $\tau \geq 1$. Then $K_l(R) \cong K_l(R[\mu])$. It should be noted (according to [14]) the analogous statement for the functor $BR : Rings \rightarrow Ab : Groups$ for the suitable class of rings) is also true.

2. The equality $SL_\tau(R[\mu]) = E_\tau(R[\mu])$ for $c$-divisible monoids.

We define the class $F_\tau$ of rings as follows: a ring $R$ belongs to $F_\tau$ if $SL_\tau(R) = E_\tau(R)$ and $SL_\tau(R') = E_\tau(R')$ where $R'$ is an arbitrary ring for which there exists a finite sequence $R = R_1, R_2, \ldots, R_n$ such that $R_i = R_{i+1} = R_{i+2} = \ldots = R_n$ for some $\tau \leq \max(\tau_i)$ or $R_i = R_{i+1} = R_{i+2} = \ldots = R_n$ (here $R_i(\vec{x})$ denotes the localization of the polynomial ring $R[\vec{x}]$ with respect to the multiplicative set of monic polynomials.) Of course any field belongs to $F_\tau$. The class $E_\tau$ is defined analogously to $F_\tau$, we only require that the localizations $R_i(\vec{x})$ of $R_i(\vec{x})$ occur in alternating order. It turns out that any local PID belongs to $F_\tau$ for all $\tau \geq 3$. Indeed (since $R(\vec{x})$ is also PID) we have only to show that $S_k(\vec{x}) = 0$ (for being a local PID). But for any local regular ring $\Lambda$ and its arbitrary regular parameter $\tau$, we have $S_k(\vec{a}) = 0$ (this follows from the localization sequence).

It remains to note that $R(\vec{x})$ is of the type $\Lambda_\tau$ ([16]).

Theorem 2.1. Let $\tau$ be any $c$-divisible monoid for some $c \geq 1$ (may be rank($\mathbb{L}$) = $\infty$) and $R \in F_\tau$ for some $\tau \geq 3$. Then $SL_\tau(R[\mathbb{L}]) = E_\tau(R[\mathbb{L}])$ for some $\tau \geq 3$. Then $SL_\tau(R[\mathbb{L}]) = E_\tau(R[\mathbb{L}])$.

The rest of this chapter is devoted to the proof of this theorem. The proof, given here, is divided in two parts: in "algebraic" part and in "geometric" part. Both of them essentially use the technique, developed in [19] and generalizes it's theorems for the monoid algebra. Here a number of difficulties arises and we overcome them with the essential use of the polarized monoids.

2.2. Notations. In the following: $R$ is any commutative ring; $(\mathcal{P}, \Gamma, \mu)$ is a polarized monoid, which sometimes will be briefly denoted by $\mu$; for the monoid algebra $R[\mu]$ by $\mu$, we denote it's any maximal ideal containing $\mu$; $\mathcal{N} = \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6$ are the same what they were in the previous chapter; we'll use the notions of positive and negative faces (necessarily of dimension $\dim^{\mathcal{N}-1}$) of the polytope $\mathcal{F}$ as well, $\mathcal{N}_6$ will be the same as it was in the definition 1.13.

It is obvious that $\mathcal{N} \cong \mathbb{Z}^+$ and by $\tau$ will be denoted the generator of $\mathcal{N}$. Throughout this chapter the monoid structure is written multiplicatively.

Let us introduce some new notations, which are analogous or (sometimes) coincide with the notations from [19]: for any (commutative) ring $R$, and any natural $\tau \geq 1$ by $\tau(R)$ denote the subgroup in $SL_\tau(R)$, which consists of the matrices of type

$$
\begin{pmatrix}
1 & a_{12} & \ldots & a_{1\tau} \\
0 & 1 & \ldots & a_{2\tau} \\
0 & 0 & \ldots & a_{3\tau} \\
0 & 0 & \ldots & a_{\tau-1,\tau}
\end{pmatrix}
$$
by $D(R)$ denote the subgroup in $SL_k(R)$, consisting of the diagonal matrices;  \( \mathcal{T}(R) = T(R)D(R) = D(R)T(R) \);
\( \Pi(R) \) will be the group of all monomial matrices with the components \( \pm 1 \); \( \Pi(R) \) is generated by \( \omega_{i,j} = e_{i,j}e_{j,i} \); where \( e_{i,j}(a) \) denotes the elementary matrix with unique nonzero nondiagonal component \( (i \text{-th row and } j \text{-th column}) \); \( a \); 
\( \mathcal{M}(R) = \Pi(R)D(R) = D(R)\Pi(R) \); 
\( \delta_i = \text{diag}(1, \ldots, 1, a, 1, \ldots, 1) \in E_{i-1} \);
\( \varepsilon \in GL_\tau(R[\tau^{-1}\mu]) \) (here \( \mathcal{M} \) is a polarized monoid).

2.3. The first (algebraic) part of the proof.

\( \mathcal{M}_{\tau \times 5}(R) \) denotes the set of \( \tau \times 5 \) matrices over \( R \) and, as usually, \( \mathcal{U}_{\nu}(R) \) denotes the set of all unimodular \( \tau \times \) columns with the components from \( R \). As it was mentioned above \( \mathcal{M} \) will denote a polarized monoid (hence, \( \mathcal{M} \) has a finite rank).

Lemma 2.2 (Vaserstein, [I9]). Let \( \tau \) and \( 5 \) be any natural numbers, \( \mathcal{I} \) be an ideal in \( R \), \( \mathcal{M}_{\tau \times 5}(\mathcal{I}) \) (the components are from \( \mathcal{I} \)) and \( \mathcal{M} \in \mathcal{M}_{\tau \times 5}(R) \), such that \( A_5 + M_4 M_2 \in GL_5(R) \). Then
\[ a) \ A_5 + M_4 M_2 \in GL_5(R), \]
\[ b) \begin{pmatrix} A_5 + M_4 M_2 \end{pmatrix} = 0 \]
As a result of this we obtain

Lemma 2.3 ([I9]). Let \( \mathcal{U}_\nu(R) \subseteq \mathcal{M}_{\tau \times 5}(\mathcal{I}), \)

\( w^0 \) and one of the components of \( w \) is equal to zero. Then
\[ \mathcal{I}_\tau + v w \in E_{\tau}(R, \mathcal{I}). \]

Lemma 2.4. ([I9]). Let \( v \in \mathcal{U}_\nu(R), w \in \mathcal{M}_{\tau \times 5}(R), w^0 = 0 \). Then \( w = \sum \alpha_{ij,j} v_{i,j,j} w_{i,j} e_{i,j} \) for some \( \alpha_{ij,j} \in R \), where \( v = (v_{1, \ldots, \nu}; v \in \mathcal{I}) \) and \( e_{i,j} \) denotes the \( i \)-th component in the standard base of \( R^\tau \).

Lemma 2.5. Let \( R \) be any (comm.) ring, \( \mathcal{M} \) be any polarized monoid, \( \tau \geq 3 \) and \( \alpha \in E_{\tau}(R[\mathcal{M}]_\nu) \). Then \( \delta_i^\tau \alpha \delta_i \in E_{\tau}(R[\mathcal{M}]_\nu) \) and \( \delta_i^{-1} \alpha \delta_i \in E_{\tau}(R[\mathcal{M}]_\nu) \).

Proof. Without the loss of generality assume that \( i = r \).

Since \( \mathcal{M} \) is polarized \( a \) can be represented as \( \prod_{k=1}^n e_{i,k,k}(a_k) \), where \( a_k \in R[\mathcal{M}(\mu)] \) and \( f_k \in R[\mathcal{M}]_\mu \). Put \( \beta_k = \prod_{k=1}^n e_{i,k,k}(a_k) \in E_{\sum_i R[\mathcal{M}(\mu)]_\nu} \) (possibly). Then
\[ \mathcal{I}_\tau + v w \in E_{\tau}(R, \mathcal{I}); \]
where \( \nu \) denotes the \( i \)-th column of \( \beta_\nu \) and \( \nu^0 \) denotes the \( i \)-th column of \( \beta^{-1}_\nu \). To show that \( \delta_i \beta \delta_i^{-1}, \delta_i \beta \delta_i \in E_{\tau}(R[\mathcal{M}]_\nu) \) is sufficient to show the inclusions
\[ \delta_i \beta \delta_i^{-1} \mathcal{I}_\tau + v w \mathcal{I}_\tau \delta_i \delta_i \]
\[ \delta_i \beta \delta_i \mathcal{I}_\tau + v w \mathcal{I}_\tau \delta_i \delta_i \]
According to lemma 2.4 \( w = \sum \alpha_{ij,j} v_{i,j,j} w_{i,j} e_{i,j} \) for some \( \alpha_{ij,j} \in R[\mathcal{M}]_\nu \). Thus, \( \delta_i \mathcal{I}_\tau + v w = \prod \mathcal{I}_\tau + v w \delta_i \delta_i \mathcal{I}_\tau \delta_i \delta_i \)

Therefore, in this case we can assume that one of the components of \( w \) is equal to zero (instead of \( \mathcal{I}_\tau + v w \)).
Here the inequality \( r \geq 3 \) is used).

Now put \( \psi = (\psi_1, \ldots, \psi_{r-1}, \psi_r)^T \) (\( M^T \) for arbitrary matrix
\( M \) denotes the transposed to \( M \) matrix), \( \psi' = (\psi_1, \ldots, \psi_{r-1}, t \psi_r) \), \( \psi'' = (t \psi_1, \ldots, t \psi_{r-1}, \psi_r) \).

We have \( \psi' \psi'' = (1, \ldots, 1, 0) = 0 \) and one of the components of \( \psi' \) (resp. \( \psi'' \)) is equal to zero. Hence, according to the lemma 2.3, we obtain

\[
\begin{align*}
\alpha_1 & = (1 + v + t^2 w) \beta_1 = 1 + v + t^2 w \in E_2(R[M]^\mu) \\
\beta_1 & = (1 + v + t^2 w) \beta_1 = 1 + v + t^2 w \in E_2(R[M]^\mu).
\end{align*}
\]

Up to this moment the proof of lemma 2.5 in essence coincides with the proof of the corresponding lemma from [13]. But we have also to show that \( \beta_1 \beta_m \beta_1^{-1} \in E_2(R[M]^\mu) \), \( \beta_1 \beta_m \beta_1^{-1} \in E_2(R[M]^\mu) \).

In [13] \( P \) is merely absent. Let \( \gamma \) be any positive face of \( \Gamma \), then we have the \( R \)-retraction of monoid algebras

\[
R[M(\gamma)] \rightarrow R[M],
\]
where for any \( m \in M \)

\[
\exists (m) = \begin{cases} m, & m \in M(\gamma), \\ 0, & m \notin M(\gamma). \end{cases}
\]

By \( M(\gamma) \) denote the maximal ideal \( M \cap R[M(\gamma)] \in \text{max}(R[M(\gamma)]) \).

Thus we obtain the \( R \)-retraction \( R[M(\gamma)] \rightarrow R[M] \).

Put \( \beta_1(m) = \beta_1(\beta_m) \). Since \( \alpha \in GL_q(R[M]^\mu), tR[M]^\mu \) we obtain that \( \beta_1(m) = 1 \) for any positive \( \gamma \). Because \( \alpha \) is polarized we have

\[
\beta_1 \in \text{Sl}_1(R[M(\gamma)]^\mu) \cap GL_q(R[M(\gamma)]^\mu), tR[M(\gamma)]^\mu, \text{max}(R[M(\gamma)]^\mu) = E_2(R[M(\gamma)]^\mu) \cap GL_q(R[M]^\mu), tR[M]^\mu.
\]

(R[M]^\mu \text{ is local}).

\[
\beta_1 \beta_m \beta_1^{-1} \in E_2(R[M]^\mu).
\]

Hence

\[
\begin{pmatrix}
t + t(a_{11} & t(a_{12} & \cdots & t(a_{1r}) \\
\cdots & t(a_{11} & t(a_{12} & \cdots & t(a_{1r}) \\
t(a_{11} & t + t(a_{22} & \cdots & t(a_{2r}) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
t(a_{11} & t(a_{21} & t(a_{32} & \cdots & t(a_{3r})
\end{pmatrix}
\]

for some \( \alpha_{ij} \in R[M(\gamma)]^\mu \), \( i, j \in 1, \ldots, r \).

In this situation \( t(a_{ij}) = t \alpha \gamma \).

Consequently

\[
\begin{pmatrix}
1 + t(a_{11} & t(a_{12} & \cdots & t(a_{1r}) \\
\cdots & t(a_{11} & t(a_{12} & \cdots & t(a_{1r}) \\
t(a_{11} & t + t(a_{22} & \cdots & t(a_{2r}) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
t(a_{11} & t(a_{21} & t(a_{32} & \cdots & t(a_{3r})
\end{pmatrix}
\]

Maybe some of the elements \( t(a_{11}, \ldots, t(a_{1r}) \in R[M]^\mu \) don't belong to \( R[M(\gamma)]^\mu \). By the locality of \( R[M(\gamma)]^\mu \) it's obvious, that using elementary row and column transformations, which correspond to the left and right multiplications by the elements from \( E_2(R[M]^\mu) \), we can reduce \( \beta_1 \beta_m \beta_1^{-1} \) to the diagonal matrix (we have "to kill" the rows and columns in \( \beta_1 \beta_m \beta_1^{-1} \) in increasing order of their indices). But a diagonal matrix with \( \det = 1 \) is elementary itself. Analogously \( \beta_1 \beta_m \beta_1^{-1} \in E_2(R[M]^\mu) \).

Our lemma is proved.

By \( \mathcal{N}_2(R[M(\gamma)]^\mu) \) (\( M \) is of the aforementioned type) denote the subgroup of \( E_2(R[M(\gamma)]^\mu) \), which consists of the matrices \( \alpha \), such that \( \alpha = (m_1, \ldots, m_{r-1}, a, m_{r+1}, \ldots, m_r) \), where \( a \in U(R[M(\gamma)]^\mu) \) and the elements \( m_i \) (\( i = 1, \ldots, r \)) satisfy the following conditions: for arbitrary positive face \( \gamma \) of the polytope \( \Gamma \) the images \( t(v_i) \) are equal to zero, where

\[
\begin{align*}
v_i & = (v_{i1}, \ldots, v_{ir}) \in \mathcal{N}_2(R[M(\gamma)]^\mu) \\
v_i & \in \mathcal{N}_2(R[M(\gamma)]^\mu).
\end{align*}
\]
\( m, m \in \mathcal{M}(\gamma), \)
\( o, m \notin \mathcal{M}(\gamma). \)

(Here \( \mu(\gamma) \) is the intersection \( \mu(\mathcal{M}(\gamma))^\perp \)) and (as it was above) \( e_i^\perp \) denotes \( i \)-th element of the standard base of \( R^2 \).

Analogously, put
\[ \mathcal{N}^-(R[\mathcal{M}(\gamma)], \mu) = \{ a \in E_R(R[\mathcal{M}(\gamma)], \mu) | a < (m_1, \ldots, m_{-1}, a, m_{-1}, \ldots)^T \} \]
for some \( m, a \in R[\mathcal{M}(\gamma)], \mu \) of the aforementioned type.

Lemma 2.6. For any \( a \in \mathcal{N}^-(R[\mathcal{M}(\gamma)], \mu) \) (resp. \( a \in \mathcal{N}^+(R[\mathcal{M}(\gamma)], \mu) \)) we have \( b_i^\perp \) \( a \in E_R(R[\mathcal{M}(\gamma)], \mu) \).

Proof. By \( a \) denote the matrix from \( E_R(R[\mathcal{M}(\gamma)], \mu) \) which has the same image under the composition
\[ R[\mathcal{M}(\gamma)], \mu \xrightarrow{\iota} R[\mathcal{M}], \mu \xrightarrow{\iota} R[\mathcal{M}], \mu/\langle \rangle \]
as the matrix \( a \) modulo (1). The existence of \( a \) follows from the coincidence of the image of \( R[\mathcal{M}(\gamma)], \mu \) (under the mentioned composition) with \( R[\mathcal{M}], \mu/\langle \rangle \) (this coincidence, in its turn, follows from the fact that \( \mathcal{M} \) is polarized). The condition \( a \in \mathcal{N}^-(R[\mathcal{M}], \mu) \) exactly coincides with the condition \( b_1, \ldots, b_i \in E_R(R[\mathcal{M}], \mu) \).

Note, that \( a - a \equiv 1 \mod{\langle R[\mathcal{M}], \mu \rangle} \). Thus, lemma 2.7 follows from the lemmas 2.5 and 2.6.

The proofs of the following lemmas (2.10-2.13), in essence, repeat the corresponding proofs from [19], but for the convenience of the reader (and for completeness) we give the detailed proofs. \( \mathcal{K} \) will denote the field \( R[\mathcal{M}(\gamma)] \); let \( \gamma \) be the canonical homomorphism \( R[\mathcal{M}(\gamma)] \rightarrow \mathcal{K} \); \( \tau \) will denote any natural number \( \geq 3 \). By \( V \) denote the subset
\[ E_R(R[\mathcal{M}], \mu) \rightarrow (R[\mathcal{M}], \mu) \rightarrow (R[\mathcal{M}], \mu) \]
of the element \( E_R(R[\mathcal{M}], \mu) \).

Put \( \iota = (1, \tau) \),
\[ \mathcal{N}(\mathcal{K}(k) = \{ b \in E_R(\mathcal{K}(k)) | b < e_i, \exists \delta < e_i \} \text{ for some } k \in \mathcal{K}, k \neq e_i \} \]
\[ \mathcal{N}(\mathcal{K}(k) = \{ b \in E_R(\mathcal{K}(k)) | b < e_i, \exists k < e_i \} \in \mathcal{K}, k \neq e_i \} \]
Lemma 2.8. ([19]). Let \( I \) be an ideal in some ring \( B \) and \( \psi: B \to A \) be the canonical map, where \( A = B/I \).

Then:

a) for any \( \alpha \in T(A) \) (resp. \( \alpha \in \Pi(A) \)) there exists \( \beta \in T(B) \) (resp. \( \beta \in \Pi(B) \)), such that \( \psi(\beta) = \alpha; \)

b) if the natural homomorphism \( U(B) \to U(A) \) is surjective then for any \( \alpha \in D(A) \) (resp. \( \alpha \in U(A) \)) there exists \( \beta \in D(B) \) (resp. \( \beta \in U(B) \)), such that \( \psi(\beta) = \alpha; \)

c) if \( B \) is local and \( I \) is maximal, then for any \( \alpha \in E_{\gamma}(A) \) for which \( e_{\gamma} \alpha = \alpha e_{\gamma} \) (for some \( \alpha \in U(A) \)) there exists \( \beta \in E_{\gamma}(B) \) such that \( \psi(\beta) = \alpha \) and \( e_{\gamma} \beta = e_{\gamma} \) (for some \( \beta \in U(B) \))

d) if \( \beta \in T(B) \) and \( \psi(\beta) = 1 \), then \( \beta \in E_{\gamma}(B, I) \) (here \( U \) denotes the group of invertible elements).

Lemma 2.9. ([15, 19]). For arbitrary field \( k \), \( SL_{\gamma}(k) = \gamma^i \cdot \mu = \gamma^i \cdot \mu \cdot T \) and \( SL_{\gamma}(k) = \gamma^i \cdot \mu \cdot T \) (when \( i \leq \gamma \)).

Lemma 2.10. For any \( i \leq \gamma \) we have \( \gamma^i \cdot \mu \cdot T \subseteq V \) and \( \gamma^i \cdot \mu \cdot T \subseteq V \).

Proof. We'll show \( \gamma^i \cdot \mu \cdot T \subseteq V \). Since \( E_{\gamma}(R[N^{-1} \mu]_{\mu}) \) is a normal subgroup in \( GL_{\gamma}(R[N^{-1} \mu]_{\mu}) \), \( \gamma \cdot \mu \cdot T \) normalizes \( T(R[N^{-1} \mu]_{\mu}) \), it just suffices to show that \( \gamma \cdot \mu \cdot T \subseteq V \), where \( \mu \in E_{\gamma}(R[N^{-1} \mu]_{\mu}) \) and \( \mu \) be the same as it was in the proof of Lemma 2.7. Then \( \gamma = \gamma \cdot \mu \cdot T \) (just here we use that \( \mu \) is polarized).

According to the Lemma 2.5 it's obvious, that it suffices to show that \( \gamma \cdot \mu \cdot T \subseteq V \). By the Lemma 2.9 \( \gamma(\alpha) = \beta_1 \beta_2 \beta_3 \) for some \( \beta_1, \beta_2, \beta_3 \in \mu \cdot T \). Using the Lemma 2.9 we obtain \( \gamma = \gamma \cdot \mu \cdot T \), where \( \gamma \in E_{\gamma}(R[N^{-1} \mu]_{\mu}) \) (see above), \( \gamma \cdot \mu \cdot T \subseteq V \) and \( \mu \cdot T \subseteq V \) (see above), \( \gamma \cdot \mu \cdot T \subseteq V \) (see above), and \( \gamma \cdot \mu \cdot T \subseteq V \) (see above).
Consequently \( \psi(\beta_1) = \psi(\beta_2)^{-1} \) for all \( m \in E_q(\mathbb{R}(\mathbb{M})_\mu) \).

By the Lemma 2.8 there exists \( \xi \in \mathbb{R}(\mathbb{M})_\mu \) for which \( \psi(\beta_1) = \phi(\beta_1) \).

Now, \( \delta = (\beta_1, \beta_2) : (\beta_1, \beta_2) \in E_q(\mathbb{R}(\mathbb{M})_\mu) \), \( \xi \in \mathbb{R}(\mathbb{M})_\mu \), \( \phi(\beta_1) = \phi(\beta_2) \).

Again by the Lemma 2.8 (a) we obtain that \( \gamma \in \mathbb{R}(\mathbb{M})_\mu \), \( \phi(\beta_3) = \phi(\beta_4) \). Complete the proof of our lemma it suffices to put

\[
\delta = (\beta_1, \beta_2, \beta_3, \beta_4), \quad \delta_1 = (\beta_1, \beta_2), \quad \delta_2 = (\beta_3, \beta_4).
\]

Now we are ready to prove the analog of the Main Theorem from [19].

**Proposition 2.14.** Let \( t > 0, m \in E_q(\mathbb{R}(\mathbb{M})_\mu) \), then there exists \( \gamma > 0 \) such that \( \delta \in \mathbb{R}(\mathbb{M})_\mu \).

Let \( \delta = (\beta_1, \beta_2, \beta_3, \beta_4) \in E_q(\mathbb{R}(\mathbb{M})_\mu) \). Then \( \delta \in E_q(\mathbb{R}(\mathbb{M})_\mu) \).

**Step I.** Any element \( a \in G \) can be written as \( a = a_1 a_2 \),

where \( a_1, a_2 \in G_+ \) and \( a_1 = 1 + \mu \).

**Proof.** Let \( a = a_1 a_2 \), where \( a_1, a_2 \in G_+ \). Assume \( a = \prod_{i} e_{i+\delta} \)

for some \( e_i \in E_q(\mathbb{M})_\mu \). Each element \( e_i \)

can be represented as the sum \( g_i / S_i \) where \( g_i \in \mathbb{R}(\mathbb{M})_\mu \) and \( S_i \in \mathbb{R}(\mathbb{M})_\mu \).

**Step II.** The monoid structure is written multiplicatively. The possibility for such presentation is the direct consequence of the definition of the polarised monoid. Put \( a_0 = \prod e_{i+\delta} \), \( a_1 = \prod e_{i+\delta} \).

It is not hard to prove that \( a_0 a_1 = \prod e_{i+\delta} \).

Indeed, \( a_0 = \prod e_{i+\delta} \), \( a_1 = \prod e_{i+\delta} \).

Now, instead of \( a_0 a_1 \) put \( a_0 a_1 \) and instead of \( a_0 \) put \( a_0 a_1 \).

**Step 3.** By \( H \) denote the subset of \( E_q(\mathbb{R}(\mathbb{M})_\mu) \), consisting of the matrices \( H \), for which \( H \in \mathbb{R}(\mathbb{M})_\mu \).

For any \( a \in E_q(\mathbb{R}(\mathbb{M})_\mu) \), the matrix \( e_{i+\delta}(a^{-1}) \) belongs to \( H \).

**Proof.** For convenience assume \( i = 1 \), \( \mu = 1 \). Let \( \delta = \delta_1, \delta_2 \in \mathbb{R}(\mathbb{M})_\mu \) and \( a = 1 + \mu \).

\( 0 = (1 + \delta_1, 1 + \delta_2) \) and \( \delta_1 = (1 + \delta_2) \) denote the first and \( \mu \)-th rows (resp.) of \( a_0 \). Then the first row of \( e_{i+\delta}(a^{-1}) \) is equal to \( (1 + \delta_1, 1 + \delta_2) \) and \( c \).

By the definition of \( C \), \( c \in \mathbb{R}(\mathbb{M})_\mu \). Using the fact that \( \mathbb{R} \) is polarized it's obvious that \( \mathbb{R}(\mathbb{M})_\mu \) is spectrally \( \mathbb{R}(\mathbb{M})_\mu \) and, since \( \mu \in \mathbb{R}(\mathbb{M})_\mu \), we have \( g \in \mathbb{R}(\mathbb{M})_\mu \). As it was in step I, \( g = g_1 g_2 / S / S \), where \( g_1, g_2 \in \mathbb{R}(\mathbb{M})_\mu \).

Finally.
\[ e_{it}(a^{-1}) e_{i\xi}(-a^{-1}) = a_i [e_{it}(a(t + a(s'/s))^{-1})]^{-1} \]
\[ a_i e_{it}(a^{-1}) \in G, \quad G = G_i. \]

Step 3. According to the proposition I.14, the triples \((P, \rho, M)\) and \((P, \rho, M_i)\) are exactly symmetric. By the previous step (changing \(t \) by \(t^{-1}\)) we obtain
\[ e_{i\xi}(at)G_i e_{i\xi}(-at) \subseteq G_i G, \]
where \( a \) is any element from \( R[M(M)]_\mu \). Taking the inverse elements we have
\[ e_{i\xi}(at)G e_{i\xi}(-at) \subseteq G. \]

Because \( H \) is multiplicatively closed and the elements \( e_{i\xi}(at) \), \( e_{i\xi}(at^{-1}) \) \((a \in R[M(M)]_\mu)\) generate the whole group \( E_i(M(N^{-1})M) \) (here we again use that \( H \) and \( \mu \) are polarized), the aforementioned remarks prove that for arbitrary \( \xi \in E_i(R[N^{-1}]M) \)
\[ \bar{s} \xi s^{-1} \subseteq G \]

Step 4. \( G = E_i(R[N^{-1}]M) \cap GL_i(R[N^{-1}]M, \mu R[N^{-1}]M). \)

Proof. Let \( \xi \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M), \xi \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M), \)
and \( \xi \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M). \) Then according to the previous step
\[ s^{-1} \xi \xi s = s^{-1} \xi s \subseteq \bar{s}^{-1} G \bar{s} \subseteq G, \]
\[ s^{-1} \xi \xi s \subseteq \bar{s}^{-1} G \bar{s} \] (since \( \bar{s} \in E_i(R[N^{-1}]M) \))
\[ \bar{s}^{-1} G \bar{s} \subseteq G. \]

It's obvious that the elements \( s^{-1} \xi \xi s^{-1} \) and \( \bar{s}^{-1} \xi \xi \) generate \( E_i(R[N^{-1}]M, \mu R[N^{-1}]M). \) Therefore for any \( \xi \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M) \) we have \( \bar{a} \xi \subseteq G. \)

Now, assume
\[ \xi \in E_i(R[N^{-1}]M) \cap GL_i(R[N^{-1}]M, \mu R[N^{-1}]M). \]

Then, according to Lemma 2.13, \( \bar{a}_i \bar{a}_2 \bar{a}_1 \) where \( \bar{a}_i \in G_i \) and \( \bar{a}_i \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M). \)

Hence \( \bar{a} \bar{a}_i \bar{a}_2 \bar{a}_1 \subseteq G_i \subseteq G. \)

Step 5. Let \( \bar{a} \in SL_i(R[M(M)\mu), \bar{a}_2 \in SL_i(R[M(M)]_\mu) \)
and \( \bar{a}_2 \bar{a}_i \bar{a}_2 \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M). \)

Since \( M \) and \( M_i \) are polarized \( R[M_i]^M / (\mu) \cong \hat{k} \xi (\bar{x}^{-1}) \) and \( R[M]^M / (\mu) \cong \hat{k} \xi (\bar{x}^{-1}) \) where \( \hat{k} \) is the field \( R / \mu \) and \( \bar{x} \) is a variable. Consequently, multiplying \( a \) and \( \bar{a}_i \) by the corresponding elementary matrices, we obtain that \( \bar{a}_i \bar{a}_2 \bar{a}_1 \) and \( \bar{a}_2 \bar{a}_i \bar{a}_2 \) are \( \xi \in E_i(R[N^{-1}]M, \mu R[N^{-1}]M). \)

Hence \( \hat{k} \xi (\bar{x}^{-1}) \subseteq E_i(R[N^{-1}]M) \cap GL_i(R[N^{-1}]M, \mu R[N^{-1}]M). \)

Let's prove that this intersection coincides with \( SL_i(R[M(M)\mu]). \)

Thus we have to show that \( R[M(M)]_\mu R[M(M)]_\mu = R[M(M)]. \)

(the intersection is considered in \( R[N^{-1}]M_\mu \). Considering the ring \( R \) instead of \( R \) we reduce the general case to the case of the local coefficient ring. In terms of elements we have to show that if \( s = s' = 0 \) for some \( s, s', s' \in R[M(M)] \)
and \( \bar{a}_i \in R[M(M)], \bar{a}_i \in R[M(M)], \)
then \( s, s' \in R[M(M)]. \)

Without the loss of generality we can assume that \( s, s' \in R \). Let \( \varphi(s) \) \((\varphi(s)) \) denote the convex hull of the set \( \{ (m_n) | \xi = \sum a \xi = m \}, a \in R, \xi \neq 0, m = \xi \}
\((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \)
and \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \)
the inclusion \( \xi \in R[M(M)] \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \((\varphi(s)) \) \( \varphi(s) \) \( 

If \( \varphi \neq \mu \) then there exists the vertex of the finite closed polytope \( \varphi_\mu \), which does not
belong to the polytope \( \mathcal{P} \). Of course, the mentioned vertex will be the vertex of the polytope \( \mathcal{P}(\delta) \) as well. Assume this vertex is \( \mathcal{P}(m_{\alpha}) \) (recall, that \( f = \sum a_i m_i \)). We have
\[
\mu = (M(x))^u v R(M(x)), \quad \text{where} \quad v = \mu \cap R. \quad \text{Hence}
\]
\[
\mathcal{P}(\delta) = \bigcup_{\alpha} \mathcal{P}(m_{\alpha}), \quad \text{where} \quad \alpha \in A, \quad x \in [R].
\]

The fact that for arbitrary elements \( \alpha \) and \( \beta \) from the arbitrary monoid \( L \) (for finite rank), which are not equal to \( 1 \in L \) (monoid structure is written multiplicatively) and for which \( \mathcal{P}(\alpha) \) and \( \mathcal{P}(\beta) \) are not opposite points on \( S_{\text{rank}(L)} - 1 \), the point \( \mathcal{P}(x) \) is the internal point of the closed segment, connecting \( \mathcal{P}(\alpha) \) and \( \mathcal{P}(\beta) \), we come to the conclusion that \( \mathcal{P}(\delta) = \mathcal{P}(\delta^* \delta^*) \) (defined analogously to \( \mathcal{P}(\delta) \) and \( \mathcal{P}(\delta^* \delta^*) \)) is a nonempty finite closed subpolytope in \( \mathcal{P} \), for which one of the vertices coincides with the above mentioned vertex of \( \mathcal{P} \), (here we use, that \( S_{\text{rank}(L)} \in \mathcal{U}(R) \)). Analogously, \( \mathcal{P}(S_{\text{rank}(L)} - 1) \) is a subpolytope in \( \mathcal{P} \). It's obvious that \( \mathcal{P} \subseteq \mathcal{P} \) and \( \mathcal{P} \subseteq \mathcal{P} \). Consequently (since \( \mathcal{P}(\delta) \neq \mathcal{P}(\delta) \)) \( \mathcal{P}(\delta) \neq \mathcal{P}(\delta^* \delta^*) \) — a contradiction. Thus we obtain, that \( \delta^* \delta = \mathcal{E}_N(R(M(x))) \), and finally \( a \in A, \quad \delta^* \delta = \mathcal{E}_N(R(M(x))) \), Q.E.D.

2.4. The (second) geometric part of the theorem. For arbitrary ring \( R \) and arbitrary \( a \in A \), the image of \( a \) in \( M(a)(R_a) \), where \( a \) is any element from \( R \), will be denoted by \( a_a \). We follow the notations, introduced in the previous sections (2.2 and 2.3). By \( B_d \) briefly will be denoted the statement of the theorem 2.1 for \( c \)-divisible (\( c > 1 \)) monoids of rank \( d \). It should be noted that, solving Anderson's

conjecture, in \( [I_2] \), we introduced the analogous notation for the main theorem: \( A_d, d \) beside this \( B \) stresses that we investigate Bass \( K \)-groups for monoid algebras! We carry out the proof of the theorem 2.1 for arbitrary \( c \)-divisible seminormal monoids, having finite rank, with the induction process on \( d \). In the following all the considered polytopes are assumed to be "flat".

Definition 2.15 (\( [I_2] \)). A finite closed polytope \( \delta \) will be called a pyramid if it is the convex envelope of some finite set of points (from the euclidean space) \( \{ x_1, \ldots, x_n \} \) where \( x_1 \) does not belong to the affine space, spanned by the points \( x_2, \ldots, x_n \) (\( x_1 \) will be called the vertex of the pyramid \( \delta \)).

Let \( \mathcal{P} \) be any finite closed polytope. Then it admits the presentation \( \mathcal{P} = \delta U_\mathcal{P} \), where \( \delta \) is a pyramid and \( \mathcal{P} \) is a finite closed polytope, both of them having dimension \( \dim \mathcal{P} \), such that \( \delta \) and \( \mathcal{P} \) have a unique common (\( \dim \mathcal{P} - 1 \))-dimensional facet, namely \( \delta \). Moreover, for any preliminarily chosen vertex \( x \) of \( \mathcal{P} \) there exists the aforementioned presentation \( \mathcal{P} = \delta U_\mathcal{P} \) for which \( x \) is the vertex for \( \delta \) as well. In the following by \( \delta' \) will be denoted any subpyramid in \( \delta \) such that the (chosen) vertex of \( \delta' \) is an internal point of \( \delta \) and the base of \( \delta' \) coincides with the base of \( \delta \) (conv.env. of \( \{ x_2, \ldots, x_n \} \)).

Of course, \( \dim \delta' = \dim \delta \) and \( \delta' \subseteq \delta \).

Definition 2.16 (\( [I_2] \)). A sequence \( \{ \mathcal{P}_i \}_{i=1}^\infty \) of finite closed rational polytopes is called admissible if the following conditions hold:

a) \( \mathcal{P}_i = \mathcal{P}_i \), \( i \in N \);
b) either \( \mathcal{P}_i \neq \mathcal{P}_{i+1} \) or there exist rational polytopes
Proposition 2.17 ([12]). Let $\Phi$ be any rational finite closed polytope, $a$ be any rational point from the interior of $\Phi$, and $U$ be any neighbourhood (in the Euclidean topology) of $a$.

Then there exists an admissible sequence $\{ \Phi_i \}_{i=1}^{\infty}$, for which $\Phi_1 = \Phi$ and $\Phi_i \subset U$ for all sufficiently large indices $i$.

Since the importance of this proposition (for our goals) we sketch its proof. We carry out this proof with the induction on $\dim \Phi$. In the case $\dim \Phi = 0$ there is nothing to prove. Now assume that the case $\dim \Phi = 1$ is already proved. Consider any $(\dim \Phi - 1)$-dimensional face $\gamma$ of the polytope $\Phi$. By the induction hypothesis, there exists an admissible sequence $\{ \Psi_i \}_{i=1}^{\infty}$ such that $\Psi_1 = \Phi$ and $\Psi_i = \Delta$ for all sufficiently large indices $i$, where $\Delta$ is any preliminary chosen subsimplex in $\gamma$. It is not hard to prove that there exists an admissible sequence $\{ \Phi_i \}_{i=1}^{\infty}$ starting with $\Phi_1 = \Phi$, such that the sequence $\{ \gamma_i \}_{i=1}^{\infty}$ can be obtained by the intersections $\Phi_i \cap \gamma$ for some subsequence $\{ \Phi_i \}_{i=1}^{\infty}$.

Without the loss of generality we can assume, that $\Phi_i \cap \gamma = \Delta$. Now, we can construct a polytope $\Phi'$, which relates to $\Phi_i$ as $\gamma \cup \gamma'$ relates to $\gamma \cup \gamma'$, hence $\Phi' \cap \gamma = \Delta'$ where $\Delta'$ is some $(\dim - 1)$-dimensional face of $\Delta$.

Constructing a new polytope $\Phi''$ in the same way (relatively to the objects $\Delta'$ and $\Phi''$), etc., we come to the conclusion that there exists an admissible sequence $\{ \Phi_i \}_{i=1}^{\infty}$, for which $\Phi_i$'s with sufficiently large indices $i$ do not intersect with $\gamma$.

Repeating our procedure relatively to other faces of $\Phi$ and then joining the fragments of the so obtained admissible sequences we see, that there exists and admissible sequence, starting with $\Phi$, all members of which with the sufficiently large indices do not intersect with $\partial \Phi$ — the boundary of $\Phi$. Hence there exists a rational number $0 < \tau < 1$ and an admissible sequence starting with $\Phi$, for which $\tau \Phi, \tau^2 \Phi, \tau^3 \Phi, \ldots$, is a subsequence in this admissible sequence, where for arbitrary polytope $\Phi$ by $\tau \Phi$ is denoted the homothetic image of $\Phi$, obtained by the centre $a$ and by the coefficient $\tau$. This finishes the proof.

Let $L$ be any $c$-divisible integrally closed monoid ($c > 1$) for which $\Phi(L)$ is a finite closed polytope; $c \geq 3$, $R$ be a commutative ring from the class $\mathcal{F}_\tau$; assume $\Phi(L)$ is represented as the union $\cup U_\gamma$ (of the above type) and $\delta'$ be the subsimplex in $b$ of the aforementioned type. Thus $b, b'$ and $\delta'$ have a common face $b_{\delta}$.

Lemma 2.18. If $\mathcal{E}_{a-1}$ holds then, in the case when $R$ is local, for any $a \in \text{L}_\gamma(R[Liy])$ there exists $e \in E_\gamma(R[Liy])$ for which $e \in \text{L}_\gamma(R[Liy])$.

We'll use the following two lemmas from [19]:

Lemma 2.8. For arbitrary ring $A$ and matrix $a \in \text{L}_\gamma(A[\mathbf{2}])$ if $a \in E_\gamma(A[\mathbf{2}])$ and $a \in E_\gamma(A[\mathbf{2}^{-1}])$ then $a \in E_\gamma(A[\mathbf{2}^{-2}])$, where $\tau \geq 3, 2$ is a variable and $A(\mathbf{2})$ (resp. $A(\mathbf{2}^{-1})$) denotes the localization with respect to the multiplicative set of monic relatively to $2$ (resp. $2^{-1}$) polynomials.

Lemma 2.20. Let $A$ and $a, \delta'$ be as above, $a \in A$, $\gamma \in \text{G}_\delta(A_{a_n})$.

Put $b(z) = \gamma Z_{b_{\delta}}(z) \delta^{-1} (z \\neq \ell)$, where $z \in A_{\delta}(z)$. Then there exist a natural number $n$ and a matrix $\gamma \in E_\gamma(A[\mathbf{2}])$ for which $\tau \delta_{n} = \tilde{\gamma}(a')$.

Proof of the Lemma 2.18. By $\Phi$ denote the polytope $b' \cup U_\gamma$.

Then $(\mathcal{P}, \Gamma, L_{\delta})$ is a quasi-polarized monoid, where $\mathcal{P}$ de-
notes the chosen vertex of the pyramid \( \triangle \). Let us prove that there exists a polarized monoid \( (\mathcal{P}_e, \pi_e, \mathcal{M}_e) \) for which: \( \mathcal{P}_e \) is included in the interior of \( \Gamma \), \( \mathcal{M} = \mathcal{L}_k \), \( a \in \mathcal{L}_k \mathcal{M}(\mathcal{M}) \) and \( \alpha_k \in E_{\mathcal{M}}(R[\mathcal{M}]) \), where \( \mathcal{N}_k = \{ \mathcal{P}_e \} = \mathcal{M}_e \). The existence of a polarized monoid \( (\mathcal{P}_e, \mathcal{N}_k, \mathcal{M}_e) \), for which hold all these conditions, maybe except the condition \( \alpha_k \in E_{\mathcal{M}}(R[\mathcal{M}]) \), \( \alpha_k \) denotes the free generator of \( \mathcal{N}_k \), follows from the Approximation Theorem (B). Now consider the submonoid \( c^{-1} \mathcal{M}_e = \mathcal{L}_k \). We claim that \( \alpha_k \in E_{\mathcal{M}}(R[c^{-1} \mathcal{M}_e]) \).

Indeed, \( R[c^{-1} \mathcal{M}_e] = R[c^{-1} \mathcal{M}_e] \) according to the proposition 1.8 \( \mathcal{N}_k \mathcal{M}_e = \mathcal{Z} \oplus \bar{\mathcal{N}} \) for some normal \( \mathcal{Z} \) without nontrivial intertible elements (\( \text{rank}(\bar{\mathcal{N}}) = \text{rank}(\mathcal{M}_e) - 1 \)), hence \( R[c^{-1} \mathcal{M}_e] = R[c^{-1} \mathcal{M}_e] \). It's obvious that there exists a sufficiently large \( \bar{\mathcal{N}} \) for which \( \alpha_k \in E_{\mathcal{M}}(R[\mathcal{M}_e]) \).

Therefore, it can be assumed that \( \alpha_k \in \mathcal{L}_k \mathcal{M}(\mathcal{M}) \), \( a \in \mathcal{Z} \oplus \bar{\mathcal{N}} \), \( \mathcal{Z}_k \) being a variable. We have the inclusions \( R[\mathcal{M}_e] \subseteq R[c^{-1} \mathcal{M}_e] \subseteq R[c^{-1} \mathcal{M}_e] \), where \( \mathcal{Z}_k \subseteq \mathcal{Z} \). By the lemma 2.19 we obtain that \( \alpha_k \in E_{\mathcal{M}}(R[c^{-1} \mathcal{M}_e]) \). Now in place of \( (\mathcal{P}_e, \mathcal{M}_e) \) substitute \( \mathcal{P}_e \), \( \mathcal{M}_e \), \( \mathcal{M} = \{ c^{-1} \mathcal{M}_e \} \mathcal{M} = \mathcal{N}_k \) for sufficiently large \( \mathcal{N}_k \).

According to the proposition 2.14, there exists an element \( S = u + a_1 m_1 + \cdots + a_k m_k \), such that \( \alpha_k \in E_{\mathcal{M}}(R[\mathcal{M}_e]) \), where \( m_1, \ldots, m_k \in \mathcal{M}(\mathcal{P}_e) \), \( a_1, a_2, \ldots, a_k \in R \) and \( u \in R(R) \). It's obvious that we can assume \( u = 1 \) and (thus) \( S = 1 + a_1 m_1 + \cdots + a_k m_k \). Let

\[
\alpha_k = \sum_{k=1}^{\infty} \epsilon_k e_k^* (\delta_k) \quad \text{for some} \quad \delta_k \in R[\mathcal{M}]_k. \quad \text{Say} \quad \delta_k = c_k / S_k,
\]

where \( c_k \in R[\mathcal{M}] \) and \( S_k \in \{ 1, 2, 3, \ldots, 1 \} \). Since \( \mathcal{M} \) is polarized every element \( c_k \) admits the presentation \( c_k + \epsilon_1 k + c_k + \epsilon_2 k^2 + \cdots + c_k k^p \) (we don't expel the case \( c_k = 0 \)) for some \( c_{k_1}, c_{k_2}, \ldots, c_{k_p} \in R[\mathcal{M}(\mathcal{P}_e)] \). (Moreover, it can be achieved that the elements from \( \mathcal{M} \), involved in \( c_{k_1}, \ldots, c_{k_p} \), will be distributed on the positive faces of \( \mathcal{P}_e \) but we'll not use this more precise statement.) It should be noted that these presentations are not uniquely determined. For every \( k \) we fix such presentation. Let \( \xi_k \) be a variable and put \( \delta_k = \xi_k (1 + \epsilon_1 k) \).

Consider the matrix \( \beta = \alpha_k (\beta_k (1 + \epsilon_1 k))^{-1} \in E_{\mathcal{M}}(R[\mathcal{M}]_k) \). For some \( \beta_k \in R[\mathcal{M}]_k \) and \( \beta_k \in R[\mathcal{M}]_k \).

Thus \( \beta = \prod_{k=1}^{\infty} \beta_k e_k^* (\beta_k) (\beta_k \in \mathcal{M}_e) \). Since \( \mathcal{M}_e = \mathcal{L}_k \mathcal{M}(\mathcal{M}) \), we have \( \beta = \prod_{k=1}^{\infty} \alpha_k e_k^* (\beta_k) (\beta_k \in \mathcal{M}_e) \).

By the lemma 2.20 there exist a natural number \( n \) and matrices \( \gamma_k \in R[\mathcal{M}](\mathcal{M}) \), \( \gamma_k \in R[\mathcal{M}](\mathcal{M}) \), for which

\[
(\gamma_k)_k = A_k (\gamma_k (n_1 + \epsilon_1 k))^{-1}.
\]

Put

\[
\gamma_k = \prod_{k=1}^{\infty} \gamma_k e_k^* (\beta_k) (\beta_k \in \mathcal{M}_e) \quad \text{for some} \quad \beta_k \in R[\mathcal{M}]_k.
\]

Then

\[
\gamma_k = \prod_{k=1}^{\infty} \gamma_k e_k^* (\beta_k (n_1 + \epsilon_1 k) (\beta_k \in \mathcal{M}_e))^{-1}.
\]

and the elements \( \beta_k (n_1 + \epsilon_1 k) \) are obtained from the elements
\[ f_n((1+2)k) \] by the correspondence \( z \mapsto s^n k \). Since \( s \) is not a zero-divisor in \( R[\mu] \), we can identify the group \( E_{\varphi}(R[\mu][z]) \) with its canonical image in \( E_{\varphi}(R[\mu][z]) \). Therefore, we see that for all sufficiently large natural numbers \( h \)

\[ a_s((1+2k)k) = [a_s((1+3k)k)]^{-1} \in E_{\varphi}(R[\mu][z], z R[\mu][z]) \]

(here \( a_s \) is identified with \( a_s \)). Note that we could independently prove that for all sufficiently large natural numbers \( h \)

\[ a_s((1+3k)k) \in S_L^\varphi(R[\mu][z]) \].

Indeed

\[ a_s((1+3k)k) = \prod \left( e_{c_k, b_k} \left( f_k((1+3k)k) \right) \right) \frac{c_k}{b_k} = \prod \left( e_{c_k, b_k} \left( 1+3k \right) \frac{c_k}{b_k} \right) = \prod \left( e_{c_k, b_k} \left( 1+3k \right) \frac{c_k}{b_k} \right) \]

= \prod \left( e_{c_k, b_k} \left( 1+3k \right) \frac{c_k}{b_k} \right) \quad \text{for some } f_k \in R[\mu][z].

Consider the matrices \( E_{\varphi} = e_{c_k, b_k} \left( f_k((1+3k)k) \right) \) for every \( k \) by \( F_k \) denote the commutator \[ E_{\varphi} = e_{c_k, b_k} \left( f_k((1+3k)k) \right) \].

For every \( k \) by \( F_k \) denote the commutator \[ E_{\varphi} = e_{c_k, b_k} \left( f_k((1+3k)k) \right) \]

\[ \left( f_k((1+3k)k) \right) \frac{c_k}{b_k} \]

It's obvious that for arbitrary ring \( \Lambda \) and arbitrary \( b \in S_L^\varphi(\Lambda) \) the matrix \( e_{c_k, b_k}(x) \) belongs to \( GL_n(\Lambda, x) \) (here \( x \) is a variable). From this observation we obtain that for sufficiently large \( h \)

\[ f_h \in S_L^\varphi(R[\mu][z]) \].

Thus, for sufficiently large \( n \) we have \( a_s((1+3k)k) = \prod \left( f_k((1+3k)k) \right) \frac{c_k}{b_k} \left( 1+3k \right) \frac{c_k}{b_k} \), for \( s^n \) we have \( s^n = 1 + a'_q m'_q + \ldots + a'_q m'_q \). where \( a'_q, \ldots, a'_q \in R \) and \( m'_q, \ldots, m'_q \in \mu(\Gamma_\varphi) \). Let \( \ell \) be a sufficiently large natural number and consider the \( R[\mu] \)-homomorphism

\[ \Theta : R[\mu][\ell] \rightarrow R[\mu][\ell] \quad \text{for which} \]

\[ \ell \rightarrow \Theta \left[ -1 + (a'_q m'_q + \ldots + a'_q m'_q) \right] \in R[\mu][\ell] \]

then \( \Theta((1+3k)k) = \prod e_{c_k, b_k} \left( f_k((1+3k)k) \right) = e_{c_k, b_k} \left( f_k((1+3k)k) \right) \).

It's clear that when \( \ell \rightarrow \infty \), the images of the elements of \( \mu \), which are involved in the element \( (a'_q m'_q + \ldots + a'_q m'_q) \in R[\mu][\ell] \)

are distributed sufficiently close (in the sense of the euclidean metric) to the polytope \( \Gamma_\varphi \). Since \( \Gamma_\varphi \) is included in the interior of \( \mathbb{P} \), we obtain that for sufficiently large \( \ell \)

\[ \Theta((1+3k)k) \in \mathcal{S}_L^\varphi(R[\mu][\ell]) \].

Hence \( \Theta \) is the desired metric \( \mathcal{S}_L^\varphi \).

Lemma 2.2.1. Let \( c, \varphi, R \) and \( \Gamma_\varphi \) be as above. Then if \( B \rightarrow \mathcal{L} \) holds and \( R \) is local \( \mathcal{S}_L^\varphi(R[\mu][\ell]) = \mathcal{E}_{\varphi}(R[\mu][\ell]) \).

Proof. Let \( a \in \mathcal{S}_L^\varphi(R[\mu][\ell]) \). Let \( \ell \) be any vertex of the polytope \( \mathcal{P}(\varphi) \). Then we have an \( R \)-retraction \( R[\mu((Q_\varphi))] \rightleftharpoons R[\mu] \).

\[ \mathcal{P}(\ell) = \left\{ \ell : \ell \in Q_\varphi \right\} \]

where for any \( \ell \in \mathcal{L} \)

\[ \mathcal{P}(\ell) = \left\{ \ell : \ell \in Q_\varphi \right\} \]

\[ \mathcal{P}(\ell) = \left\{ \ell : \ell \in Q_\varphi \right\} \]
By induction hypothesis \( E_{\Sigma}(R[LU\{Q_1\}]) = \Sigma_{\Sigma}(R[LU\{Q_1\}]) \)
(here we use that \( B_{d-1} \Rightarrow B_{d-2} \Rightarrow \ldots \Rightarrow B_1 \)).
Thus there exists \( E_{\Sigma} \in E_{\Sigma}(R[LU\{Q_1\}]) \) for which the \( \Sigma \)-images of those elements from \( L \), which are involved in \( E_{\Sigma} \), belong to \( \Sigma(L) \setminus \{Q_1\} \).
Put \( L_1 = L(\Sigma(L)) \setminus \{Q_1\} \). Therefore \( E_{\Sigma} \in E_{\Sigma}(R[L_1]) \).
Now consider other vertex \( Q_1 \) of \( \Sigma(L) \) and repeat the same procedure.
We see that there exists \( E_{\Sigma} \in E_{\Sigma}(R[L_1]) \) for which \( E_{\Sigma} \in E_{\Sigma}(R[L_2]) \), where \( L_2 = L(\Sigma(L)) \setminus \{Q_2, Q_1\} \).
Continuing this procedure we obtain that for some \( E_{\Sigma} \in E_{\Sigma}(R[L_j]) \) the matrix \( E_{\Sigma} \) belongs to \( E_{\Sigma}(R[L_j]) \), where \( L_j = L(\Sigma(L)) \setminus \{Q_j\} \).
By \( F \) denote any edge (= 1-dimensional face) of \( \Sigma(L) \).
As it was above, we have an \( R \)-retraction \( R[L_j(\Sigma(F))] \rightarrow R[L_j] \).
By \( B_{E_{\Sigma}} \) \( E_{\Sigma}(R[L_j(\Sigma(F))] = E_{\Sigma}(R[L_j]) \).
By \( \Sigma_{\Sigma}(R[L_j(\Sigma(F))] \) can be represented as the union of the integrally closed \( \Sigma \)-divisible submonoids, embedded in each other, to whom correspond the closed segments). Therefore, we can "kill" these 1-dimensional faces as well. Then after we turn to the 2-dimensional faces, etc. Finally we obtain that there exists \( E_{\Sigma} \in E_{\Sigma}(R[L_j]) \) such that \( E_{\Sigma} \in E_{\Sigma}(R[L_j]) \).
But then, according to the statements 2.17 and 2.18, there exists \( E_{\Sigma} \in E_{\Sigma}(R[L_j]) \) for which \( E_{\Sigma} \in E_{\Sigma}(R[L_j]) \) where \( \Sigma \) is any preliminarily chosen subsimplex in \( \Sigma(L) \) of the same dimension.

By the Approximation Theorem (A) \( L(\Sigma) = \lim_{\rightarrow} Z^d \) (\( d = \operatorname{rank}(L) \)),
where the diagram is directed. To finish the proof we have to show that \( E_{\Sigma}(R[L_1, \ldots, L_j]) = E_{\Sigma}(R[L_1, \ldots, L_j]) \).

Step I. Let \( a \in GL(\Sigma) \) and \( a \in E_{\Sigma}(\Sigma) \) for some \( a \in A_0 \), Then there exists a natural \( \Sigma \) for which \( a(\Sigma(\Sigma)) = E_{\Sigma}(\Sigma(\Sigma)) \).
Whenever \( c = d \mod(a^d) \), \( c, d \in A_0 \).
Proof. Let $\beta(y, z) = a^*_o(y)(a^*_o(y + z))^{-1} \in E_{\phi}(A(y, z)) \in \mathcal{L}_{\phi}(A(y, z))$ (the meaning of the used notations is obvious). Then $\beta = \prod_{k=1}^{m} e_{\xi_k}(\xi_k (\delta_k + \epsilon_k))$ for some $\xi_k \in A(y, z)$, $\delta_k \in A(y, z)$, and $\epsilon_k \in A(y, z)$. We have $\beta = \prod_{k=1}^{m} \xi_k e_{\xi_k}(\xi_k^{-1})$, where $\gamma_k = \prod_{k=1}^{m} e_{\xi_k}(\xi_k^{-1})$. By the proposition 2.20 for sufficiently large natural $s_k$, each $\epsilon_k e_{\xi_k}(\xi_k^{-1}) \in E_{\phi}(A(y, z))$, but then $\beta(y, z) = \prod_{k=1}^{m} e_{\xi_k}(\xi_k^{-1})$ comes from $E_{\phi}(A(y, z)) \supseteq E_{\phi}(A(y, z))$, but then $\beta(y, z) = \prod_{k=1}^{m} e_{\xi_k}(\xi_k^{-1})$ comes from $E_{\phi}(A(y, z)) \supseteq E_{\phi}(A(y, z))$ as well. Now it suffices to put $S = S_k + S_\infty$ for sufficiently large $S_k$.

Step 2. Let $\alpha \in L_\phi(A,A^*)$, $\alpha \in A \subseteq A^*$, $\alpha \in A \subseteq A_\phi$, $\alpha \in E_{\phi}(A_\phi)$, then $\alpha \in E_{\phi}(A)$.

Proof. There exists $\epsilon \in \mathbb{N}$ for which $\alpha^*(c)(\alpha^*(d))^{-1} \in E_{\phi}(A)$ whenever $c \equiv d \mod (\alpha^*)$ and $d \in (\alpha^*)$. Since $\alpha^* \in \mathbb{N}$ we have $\alpha = \alpha^*(\alpha^*(\epsilon))^{-1} \in E_{\phi}(A)$.

Step 3. Let $\alpha \in L_\phi(A,A^*)$, then $I = \{ \alpha \in A \}$.

Proof. It suffices to show that $I$ is an additive subgroup in $A$. But if $c = \alpha + \epsilon$ for some $\alpha, \epsilon \in I$ then $\alpha \in I$ and by step 2 $c \in I$.

From the last statement follows the proposition 2.22.

Proof of the theorem 2.1. Let $c \in \mathbb{N}$, $c > 1$. Since $\phi(Q) = \lim_{d} Q_d$, where $d$ is any cardinal, $d_i$ are finite cardinals and the diagram is directed, we come to conclusion, that arbitrary $c$-divisible monoid $L$ can be represented as $\mathcal{E}_{\phi}(L)$, where the diagram is directed, $\text{rank}(L_i) < \infty$ for all $i$ and $L_i$ are $c$-divisible and seminormal (being the $\phi$-submonoids in $L_i$). Therefore the general case is reduced to the case of finite rank monoids. We carry out the proof by the induction on $d = \text{rank}(L)$. When $d = 1$ then either $L = \lim_{d} \mathbb{Z}$ or $L = \lim_{d} \mathbb{Z}$ (diagrams are directed). But since $R \in \mathbb{F}_\phi$, using aforementioned criteria for $\alpha \in E_{\phi}(R[\mathbb{Z}_1])$ and $\alpha \in E_{\phi}(R[\mathbb{Z}_1])$, concerning the localizations by the monic polynomials $(I)_{\phi}$, we have $\mathbb{Z}_1 \subseteq R[\mathbb{Z}_1] = \mathcal{E}_{\phi}(R[\mathbb{Z}_1])$ and $\mathbb{Z}_1 \subseteq R[\mathbb{Z}_1] = \mathcal{E}_{\phi}(R[\mathbb{Z}_1])$.

Now assume $d \geq 2$ holds and $L$ is a $c$-divisible integral closed monoid without nontrivial invertible elements, for which $\phi(L)$ is a finite closed polytope. Let $\alpha \in L_{\phi}(R[L])$. By the definition of $\mathbb{F}_\phi$ for any $\mu \in \mathbb{F}_\phi$ we have $R_\mu \in \mathbb{F}_\phi$. According to the lemma 2.1 holds $\alpha_\mu \in E_{\phi}(R_\mu[L])$. Hence for any $\mu \in \mathbb{F}_\phi$ there exists $\alpha_\mu \in R \setminus \mu$ such that $\alpha_\mu \in E_{\phi}(R_\mu[L])$. Since $X = \{ \alpha_\mu \mid \mu \in \mathbb{F}_\phi \}$ generates the unit ideal there exists a finite family $\{ \alpha_1, \ldots, \alpha_n \} = X$ for which $\alpha_1 \cdots \alpha_n \in E_{\phi}(R_\mu[L])$. Consequently, there exists a finitely generated submonoid $L_\mu \subseteq L$, such that $\alpha_1 \cdots \alpha_n \in E_{\phi}(R_\mu[L])$. Consider any enveloping rational simplex $\phi(L) \subseteq \mathbb{A}^d$ of the same dimension. Then $L_\mu = L_\mu \in \mathbb{Q}^d$ (see the proposition I.7 and I.15). Thus for sufficiently large $\ell \in \mathbb{N}$ $\mathbb{Z}_1 \subseteq \mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_1 \cong \mathbb{Z}_d^\infty$. We have obtained the inclusion $L_\mu \subseteq \mathbb{Z}_d^\infty$, which implies the grading $R[\mathbb{Z}^d_1] = R \oplus R \oplus R \oplus \cdots$ (inherited from the grading $R[\mathbb{Z}^d_1] \to \mathbb{Z}^d_1$), where $R \in \mathbb{F}_\phi$. Therefore $\alpha \in E_{\phi}(R[L_\mu])$ and $\alpha \in E_{\phi}(R[L_\mu])$ for any $\mu \in \mathbb{F}_\phi$. Since $\mathbb{Z}_1 \subseteq R[\mathbb{Z}_1] = \mathcal{E}_{\phi}(R[\mathbb{Z}_1])$ without the loss of generality we can assume that $\alpha \in E_{\phi}(R[L_\mu])$. By the proposition 2.22 $\alpha \in E_{\phi}(R[L])$. Now assume that $L$ is $c$-divisible, ...
2.5. The stable case. As it was above we assume $c > 1$.

Theorem 2.24. For any $c$-divisible monoid $L$ without nontrivial invertibles the groups $K_4(R)$ and $K_4(R, L')$ are naturally isomorphic, where $R$ is any regular ring.

Proof. By $F'$ denote the class of rings $R$ satisfying the conditions: $SK_4(R) = 0$ and $R' \in F'$ whenever there exists a finite sequence $R = R_1, R_2, \ldots, R_n = R'$ such that $R_i \in F'$ and either $R_i = \{(a), \exists \alpha \in (1, n) : R_{i+1} = R_i, \alpha \}$ for some $(a) \in \text{max}(R_i)$ or $R_{i+1} = R_i \otimes \mathbb{Z}[[\alpha]]$ (see 1, 1).

It is not hard to notice that our proof of the theorem 2.1 can be modified so that it will be applicable to the stable situation relatively to the class $F'$. Thus for all $c$-divisible monoids $L$ (may be with nontrivial $\mathcal{U}(L)$) the groups $K_4(R, L')$ are trivial ($R \in F'$). But every local ring $R$ belongs to $F'$. Indeed, we have to show that $SK_4(R, L') = 0$ ($d \in \mathbb{N}$) for arbitrary local regular $R$ (since the localizations, Laurent polynomial extensions preserve regularity). Since $K_4(R) = K_4(R, \mathbb{Z}) = \ldots = K_4(R, \mathbb{Z}[[\alpha]])$ ($\forall \alpha$), by the "main theorem" of K-theory (11) we obtain $K_4(R, L') = K_4(R) \otimes K(R') \otimes \cdots \otimes K(R) = K_4(R) \otimes \mathbb{Z}^d$ (since $R$ is local) $\implies K_4(R, L') = 0$, consequently $K_4(R, L') = 0$. Now assume $\mathcal{U}(L) = \{1\}$. Then the natural homomorphism $K_4(R) \rightarrow K_4(R, L')$ is an isomorphism, where $R'$ is any regular ring and $\mathcal{M}$ is its maximal ideal (since $K_4(R) = K_4(R, \mathcal{M})$; this equality follows from the fact that $R'$ is reduced). But then we can globalize the coefficient ring (as it was in the proof of the theorem 2.1) as follows: $K_4(R, L') = K_4(R) \otimes K(R, L') \otimes \mathbb{Z}^d$. Therefore $K_4(R') = K_4(R) \otimes \mathbb{Z}^d$. Q.E.D.

Note that if a monoid $L$ satisfies the conditions:

Theorem 2.11. Let $R$ be any PID with $SK_4(R) = 0$ (may an eucl.dom.) $L$ be any $c$-divisible monoid and $c > 1$. Then

$SL_4(R, L') = E_4(R, L')$.
for any regular $R$, then $L$ is necessarily seminormal and without nontrivial invertible elements. Indeed, $K_4(R[L][z]) = K_4(R[z][L]) = K_4(R[L]) = K_4(R) = K_4(R[L])$.

where $z$ is a variable (see [6]). Hence $R[L]$ is $K_4$-regular. It is well known that in this situation $R[L]$ is $K_4$-regular ([21]) and (hence) Pic-regular. According to [20] $R[L]$ is seminormal.

By the theorem I.2 $L$ is seminormal; finally, $U(L) = \{1\}$ because otherwise $U(R) \not\subset U(R[L])$.

Remark. In fact it can be shown, that the torsionfreeness (for any commutative, cancellative monoid $L$) itself follows from the isomorphisms $K_4(R) \overset{\cong}{\longrightarrow} K_4(R[L])$ at any regular $R$.

3. The isomorphism $K_4(R) \overset{\cong}{\longrightarrow} K_4(R[L])$ for $\Phi$-simplicial c-divisible monoids

All monoids under consideration have finite ranks. A monoid $L$ will be called $\Phi$-simplicial if $\Phi(L)$ is a finite closed simplex (or equivalently $Q_0 \otimes L \cong Q_i$).

Theorem 3.1. Let $L$ be any c-divisible ($c > 1$) $\Phi$-simplicial monoid and $R$ be any regular ring. Then $K_4(R) \overset{\cong}{\longrightarrow} K_4(R[L])$.

In the following $A$ and $B$ will denote integrally closed c-divisible nontrivial submonoids in $Q_0$. By $C$ denote the $\Phi$-submonoid $\{(a, \delta) \in A \bowtie B \mid \delta \neq 1 \} \cup \{(1,1)\} = A \bowtie B$ (the monoid structure is written multiplicatively).

Lemma 3.2. $C = \varprojlim \mathbb{Z}_+$, where the diagram is directed.

Proof. It is obvious that $A \overset{\cup}{\longleftarrow} \bigcup_{i \in \mathbb{Z}_+} A_i$ and $B \overset{\cup}{\longleftarrow} \bigcup_{i \in \mathbb{Z}_+} B_i$, where $A_i, B_i \subset \mathbb{Z}_+$ and for $i < j$ we have $A_i \subset A_j$ and $B_i \subset B_j$.

By $\delta_i$ (resp. $\delta_j$) denote the free generator of $A_i$ (resp. $B_j$). We have $C = \bigcup_{i=1}^{\infty} C_i$, where $C_i = \{(a, \delta) \in A \bowtie B \mid \delta \neq 1\} \cup \{(1,1)\}$.

Therefore, it suffices to show that $C_i = \varprojlim \mathbb{Z}_+$. Consider the submonoids $D_{\mathbb{Z}_+} = \{(\delta_i, \delta_j, \delta) \mid \tau, s \in \mathbb{Z}_+\} = C_i$.

It's obvious that $D_{\mathbb{Z}_+} = \mathbb{Z}_+$. Since $B_n = B_n$, we have $\delta_i = \delta_j$ for some $i < j$. Hence $(\delta_i, \delta_j, \delta) = (\delta_i, \delta_i) \subseteq (\delta_i, \delta_j, \delta) \subseteq D_{\mathbb{Z}_+}$.

Consequently, $D_{\mathbb{Z}_+} = D_{\mathbb{Z}_+}$. Let's show that $U(D_{\mathbb{Z}_+}) = C_i$. Consider any element $x \in (\delta_i, \delta) \subseteq C_i$. If $u = 0$ then for sufficiently large $k$ we have $\delta \in B_k$ and thus $x \subseteq D_{\mathbb{Z}_+}$. Now assume that $u \neq 0$ (in this case $\delta \neq 1$). Then there exists $u > u$ such that $\delta = \delta_k$ for some $k$. Hence $x = (\delta_i, \delta) = (\delta_i, \delta) \subseteq (\delta_i, \delta_k, \delta) \subseteq D_{\mathbb{Z}_+}$ (of course $\delta_k \neq 1$).

Q.E.D.

We'll use the following notation: for arbitrary homomorphism of rings $A \rightarrow A_x$ by $S_{k}(A)$ will be denoted the canonical image of $S_{k}(A_x)$ in the group $S_{k}(A_x)$.

Lemma 3.3. Let $A$, $B$, and $C$ be as above and $R$ be any $K_4$-regular ring with $S_{k}(R) = 0$. Then any element $u \in S_{k}(R) \bowtie B$ admits the presentation $u = uv$ where $u \in S_{k}(R[A])$ and $v \in S_{k}(R[C])$ (we fix the natural embeddings $R[A] \hookrightarrow R[A \bowtie B]$ and $R[C] \hookrightarrow R[A \bowtie B]$).

Proof. Consider the commutative square with horizontal embeddings.

$$
\begin{array}{ccc}
R[C] & \xrightarrow{\cong} & R[A \bowtie B] \\
\downarrow & & \downarrow \\
R & \xrightarrow{\cong} & R[A]
\end{array}
$$
where for any \( a \in A \) and \( b \in B \), \( \Theta(b) = a \) and \( \Theta(\epsilon) = \epsilon \). Since \( R \rightarrow R[A] \) is an R-retraction we can identify the subgroup \( St_\epsilon(\epsilon) \cong St(R[A]) \) with \( St(R) \). Thus we have the homomorphism
\[
St_\epsilon(\epsilon) \rightarrow St(R),
\]
which will be denoted by \( \Theta_\epsilon \).

Let's prove, that for any \( u \in St_\epsilon(\epsilon) \) and \( \epsilon \in ker(\Theta_\epsilon) \) there exist \( u \in St_\epsilon(\epsilon) \) and \( \epsilon \in ker(\Theta_\epsilon) \), such that \( u = u_1 \cdot u_2 \cdot u \cdot \epsilon \). Put \( u' = u_1 \cdot u_2 \cdot u \cdot \epsilon \in St(R[A@B]) \). Let \( u \in E(R[A@B]) \) denote the canonical images of \( u_1 \) and \( u' \) (resp.) in \( E(R[A@B]) \).

From the commutative square
\[
\begin{array}{ccc}
St_\epsilon(\epsilon) & \rightarrow & St(R) \\
\downarrow & & \downarrow \\
E(R[A@B]) & \rightarrow & E(R[A])
\end{array}
\]
we obtain, that \( \epsilon \in ker(E(\Theta)) \). Hence \( \epsilon \in ker(E(\Theta)) \). But in this case \( \epsilon \in ker(E(\Theta)) \). By the lemma 3.2 \( c = \lim Z_i \). Therefore, from the equality \( SK_i(\epsilon) = 0 \), the \( K_\epsilon \)-regularity of \( R \) (which in its turn is implied from the \( K_\epsilon \)-regularity of \( R \)) we obtain \( SL(R[C]) = E(R[C]) \). Consequently, there exists \( u' \in St_\epsilon(\epsilon) \) and \( \epsilon \in ker(\Theta_\epsilon) \) for which \( \epsilon = u_1 \cdot u_2 \cdot u \). Since \( R \) is \( K_\epsilon \)-regular and \( A@B = \lim Z_i \), we have \( \epsilon \in ker(\Theta_\epsilon) \). From the inclusion \( \epsilon \in ker(\Theta_\epsilon) \) follows the existence of \( \epsilon = u_1 \cdot u_2 \cdot u \), for which \( \epsilon = u_1 \cdot u_2 \cdot u \). We have \( u_1 \cdot u_2 \cdot u \cdot \epsilon = (u_1 \cdot u_2 \cdot u) \cdot \epsilon \). Put \( u_1 \cdot u_2 \cdot u \) and \( \epsilon = u_1 \cdot u_2 \cdot u \).

Now, to finish the proof of our lemma it just suffices to note that any element \( \epsilon \) from \( St(R[A@B]) \) can be represented as the product of the elements from \( St_\epsilon(\epsilon) \) and \( ker(\Theta_\epsilon) \). Indeed, for any \( \epsilon \in St(R[A]) \) and \( \epsilon \in ker(\Theta_\epsilon) \) we have \( \epsilon = \epsilon \cdot \epsilon \cdot \epsilon \), where \( \epsilon \in ker(\Theta_\epsilon) \) and \( \epsilon \in ker(\Theta_\epsilon) \), where \( \epsilon = \epsilon \cdot \epsilon \cdot \epsilon \).
index \( n' > n \) for which \( \ell_{n'} \in R[A] \) and \( \forall Q \in \mathcal{P}(L) \setminus \{ P \} \)
\[ \ell_{n'} \in R[L(Q)] \]. The lemma \( 3.3 \) guarantees that the transferred elements "leave behind them the traces lying in \( \mathcal{P}(L) \setminus \{ P \} \). 

Put \( \mathcal{L}' = L(L(\mathcal{P}) \setminus \{ P \}) \). We see, that \( u' = u'' \) for some \( u' \in S \mathcal{L}(R[A]) \). Since \( E(R[A]) \subseteq E(R[L]) = E(R) \) (this equality follows from \( A \mathcal{L} = \{ I \} \) there exists an element \( u'' \in S \mathcal{L}(R[L]) \), such that \( u = u'' u' \in K_2(R[L]) \) (just here we use that \( u \in K_2(R[L]) \), where \( K_2(R[L]) \) is identified with its natural isomorphic image in \( K_2(R[L]) \) via the retraction \( R[L] \to R[L] \) under which all elements from \( L' \) map into \( G \in R \). On the other hand \( R[L] = R[\mathcal{L}(Z_s) = \mathcal{L}(R[Z_s]) \), where the diagram is directed. Hence \( K_2(R[L]) = K_2(R) \) (\( R \) is \( K_2 \)-regular). Finally \( u'' = \left( u'' u' \right) v'' = v' \in S \mathcal{L}(R[L]) \).

Now we have to do the same relatively to another vertex \( P \neq P \) of the simplex \( \mathcal{P}(u) \) and so on. We come to the conclusion, that \( u \in \mathcal{L}(\mathcal{P}(u)) \), where \( \mathcal{L}' \subseteq \mathcal{L}(\mathcal{P}(u)) \). After this we turn to the interiors of the \( i \)-dimensional faces of the simplex \( \mathcal{P}(u) \). According to the approximation theorem \( A \) and proposition \( I.9 \) they define the \( \mathcal{P} \)-submonoids in \( \mathcal{L} \) which can be represented as the unions of embedded in each other free monoids (of rank \( 2 \)). Thus they can be "killed" as well. Then we "kill" the interiors of \( 2 \)-dimensional faces, etc. Finally we obtain \( \forall \in \mathcal{L}(\mathcal{P}(u_i)) \), where \( \mathcal{L}' \subseteq \mathcal{L}(\mathcal{P}(u)) \), \( 
( i = \text{rank}(\mathcal{L})) \). Hence \( K_2(R[L]) \approx K_2(R) \). The proposition is proved.

### Lemma 3.5

Let \( \Theta: H_1 \to H_2 \) be the homomorphism between the commutative monoids \( H_1 \) and \( H_2 \), which maybe are not cancellative (nor torsionfree), and \( \mathcal{L} = \mathcal{P} \mathcal{L}_{H_1} \) be the \( H_1 \)-graded ring (i.e. \( \mathcal{L}_k \mathcal{L}_k' = \mathcal{L}_{k+k'} \). If in \( H_2 \) are no nontrivial invertible elements then for arbitrary functor \( F: \text{Rings} \to \text{Ab. Groups} \) the following implication holds:

\[ F(\mathcal{L}) \Rightarrow F(\mathcal{L}_H) \Rightarrow F(\mathcal{L}_H H_2) \Rightarrow F(\mathcal{L}) \]

Proof. The case \( H_1 \leq H_2 = \mathcal{Z}_e, \) the identity map is well known (and very often used). The proof of our generalized version in essence is the same; we have to consider the maps:

\[ f: \mathcal{L} \to \mathcal{L}_H, \quad \epsilon_\mathcal{L}: \mathcal{L}_H \to \mathcal{L}, \quad \xi_\mathcal{L}: \mathcal{L}_H \to \mathcal{L} \]

where \( \mathcal{L}_H = \mathcal{L}(\mathcal{Z}_e) \to \mathcal{L} \), \( \xi_\mathcal{L} \), \( \mathcal{L}_H \to \mathcal{L} \to \mathcal{L} \)

Proof of the theorem 3.1. Let \( R \) be any regular ring and \( L \) be any \( c \)-ordinal \( (c > 1) \) \( \mathcal{L} \)-simplicial monoid. Then for any \( \mu \in \text{max}(R) \) the ring \( R[\mathcal{L}(Z_e)] \) satisfies the conditions from the proposition 3.4. Consequently \( K_2(R[\mathcal{L}(Z_e)]) = K_2(R[\mathcal{L})] \). Therefore, for any \( \mathcal{L} \subseteq K_2(R[\mathcal{L}(Z_e)]) \) there exists a finite family \( a_1, \ldots, a_\mu \in R \) such that \( \mathcal{L}_e \subseteq K_2(R[\mathcal{L})] \), where \( \mathcal{L}_e \) denotes the natural image of \( \mathcal{L} \) in \( K_2(R[\mathcal{L}(Z_e)]) \) and \( a_1 R \ldots + a_\mu R = R \). But in this situation for any \( \forall \in \text{max}(R[\mathcal{L})] \) the natural image \( \mathcal{L}_e \) in \( K_2(R[\mathcal{L}(Z_e)]) \) comes from \( K_2(R[\mathcal{L}(Z_e)]) \). Using the local-global technique for algebraic \( K \)-theory (see [21]) we obtain that \( \mathcal{L} \) itself comes from \( K_2(R[\mathcal{L})] \). Hence, the natural homomorphism \( K_2(R[\mathcal{L})] \to K_2(R[\mathcal{L}(Z_e)]) \) is an isomorphism. Then \( K_2(R[\mathcal{L})] \to K_2(R[\mathcal{L}(Z_e)]) \) is an isomorphism as well. The group \( K(\mathcal{L}) \) can be represented as \( \lim\mathcal{L}_k \), where \( \mathcal{L}_k \) are finitely generated subgroups in \( K(\mathcal{L}) \) and the diagram is directed. It is obvious that the monoids \( \mathcal{L}_k \subseteq \mathcal{L}_k \) are semiformal and \( \mathcal{L}_k = \lim\mathcal{L}_k \). Let us note, that the monoid al-
gebras $R[\underline{L}_k]$ admit the gradings $R[\underline{L}_k] = R \oplus R_{k_1} \oplus R_{k_2} \oplus \cdots$ (since each $L_k$ can be embedded in the free monoid $0 \oplus \mathbb{Z}_+ \oplus \cdots$). We have $\underline{L}_k = \frac{1}{\underline{k}} C^t L_k$, where $C^t L_k$ are c-divisible, $\Phi$-simplicial monoids. By the aforementioned remarks the natural homomorphism $K_2(R[C^t L_k]) \rightarrow K_2(R[C^t L_k][C^t \oplus \mathbb{Z}_+])$ is an isomorphism for every $k$. On the other hand $R[C^t L_k]$ admits the $C^t \oplus \mathbb{Z}_+$-grading $R[C^t L_k] = \bigoplus L_k \pi_j$ ($j \in C^t \oplus \mathbb{Z}_+$), where $L_k \pi_j \cong R$. By the lemma 3.5 we obtain the natural isomorphisms $K_2(R) \cong K_2(R[C^t L_k])$. Since $K_2(R[\underline{L}_1]) = \frac{1}{\underline{k}} K_2(R[C^t L_k])$, we have $K_2(R) \cong K_2(R[\underline{L}_1])$.

Now assume that for some $\Phi$-simplicial monoid $\underline{L}$ the natural homomorphism $K_2(R) \rightarrow K_2(R[\underline{L}_1])$ is an isomorphism whenever $R$ is regular. Since the $K_2$-regularity implies the $K_4$-regularity ([2I]), we can apply the same arguments, which were used in the proof of Th. 2.24, to show the seminormality of $\underline{L}$.

Q.E.D.

It should be noted that (since the $K_2$-regularity of some ring $\Lambda (\Lambda \in \mathbb{N})$ is preserved for any localization $S^{-1} \Lambda$ ([2I])) we could require only the $K_2$-regularity for the ring of the coefficients (instead of the regularity).

We close with some remarks:

Remark A. It is natural to ask: how nontrivial are $\Phi$-simplicial, integrally closed, c-divisible ($c > 1$) monoids? It turns out, that for any finite abelian group $G$ there exist $c > 1$ and a $\Phi$-simplicial, integrally closed, c-divisible monoid $\underline{L}$ such that $D_{\underline{L}}(\underline{L}) \cong G$, where $D_{\underline{L}}(\underline{L}) = D_{\underline{L}}(\underline{L})/\rho_{\underline{L}}(\underline{L})$ and $D_{\underline{L}}(\underline{L})$ denotes the subgroup in the group of all divisorial ideals of $\underline{L}$, consisting of the intersections of finite families of the principal divisorial ideals (for the definitions see [8]).

Remark B. All results obtained in this paper can be generalized to the monoids of the type $\underline{L} = \frac{1}{\underline{k}} M(\underline{L}_1, \underline{L}_2, \underline{L}_3, \underline{L}_4, \ldots)$, where $M$ is any seminormal monoid in some free (abelian) group and the natural numbers $c_i$ satisfy the inequality $c_i > 1$. In the case when $c_k = c_2 = \cdots$ we obtain the c-divisible monoid.

Remark C. All the isomorphisms of the type $F(R) \rightarrow F(R[\underline{L}_1])$, established in this paper, can be equivalently reformulated in terms of the isomorphisms $F(R[\underline{L}_1]) \cong F(R[\underline{L}_1][\lambda_1, \ldots, \lambda_n])$ where $\lambda_i$'s are the variables ($i \in \mathbb{N}$).

Question. Let $\underline{L}$ be any (commutative, cancellative, torsion-free) divisorial monoid (i.e. c-divisible for all $c \in \mathbb{N}$) without nontrivial invertible elements. Are the natural homomorphisms $K_n(R) \rightarrow K_n(R[\underline{L}_1])$ isomorphisms for all regular $R$ and all $n > 2$?

References

12. Gubeladze J., Anderson's conjecture and maximal class of monoids over which projective modules are free, Matematicheski Sbornik 135 (177), 169-185, 1989 (Russian).
In this appendix we prove that a (commutative, cancellative) monoid $\mathcal{M}$ must be torsionfree if all $R[\mathcal{M}]$ with (commutative) regular $R$ are $K_i$-regular for some $i > 0$. According to [21] we have $(K_i - \text{regularity}) \Rightarrow (K - \text{regularity})$ and since $(K - \text{regularity}) \Rightarrow (\text{Pic} - \text{regularity})$ it suffices to prove

**Theorem.** Let $\mathcal{M}$ be a (commutative, cancellative) monoid for which all $R[\mathcal{M}]$ with Dedekind domain $R$ are $\text{Pic} - \text{regular}$. Then $\mathcal{M}$ is torsionfree (i.e. there does not exist a torsion element in $K(\mathcal{M})$).

The monoid structure will be written multiplicatively.

We will use some results from [Rush D.E. Picard groups in abelian group rings, J.Pure Appl.Alg., v.26, 1982] = [R].

**Lemma 1 (R).** Let $A$ be a domain with char $A = 0$. Then $A[\mathcal{M}]$ is reduced.

**Lemma 2.** Let $A$ be a reduced ring and $G$ be a torsionfree (abelian) group. Then $(n \in \mathbb{N}, f \in A[G], f^n \in A) \Rightarrow (f \in A)$. 

**Proof.** Embedding $G$ into $\mathbb{Q} \otimes G$ we see that it can be assumed that $G = \mathbb{Q}^d$ for some $d < \infty$. By induction process on $d$ it suffices to consider the case $d = 1$. Since $A[\mathbb{Q}] = \mathbb{Q}[A[1]]$, where $t$ is a variable, the lemma follows from the following observation: for $g = \alpha_1 t^k + \alpha_2 t^{k+1} + \ldots + \alpha_d t^d \in \mathbb{Q}[A[1]]$ we have $g^n = \alpha_1 t^{kn} + \ldots + \alpha_d t^{dn}$ with $a_i = 0$.

**Lemma 3.** Let $A$ be a reduced ring and $x, y \in A$ for which $x^n = y^n$ and $x^d = y^d$, then $x - y = 0$.

**Proof.** $(x - y)^d = x^{2d} - 3x^n y^n + 3x^n y^n - y^{2d} = x^{2d} - 3x^n y^n + 3x^n y^n - y^{2d} = 0$. Hence $x - y = 0$.

**Lemma 4.** Let $D$ be the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{N}$ and $G = G_1 \subset G_2$ an extension of (abelian) torsion groups. Then $(\mathbb{Z} \in \mathbb{D}(G_2), \mathbb{Z} \in \mathbb{D}(G_1))$ implies $(\mathbb{Z} \in \mathbb{D}(G_1))$.

**Proof.** Fix finite subgroups $H_1 \subset H_2$ and $H_2 \subset G_2$ for which $H_1 \subset H_2$, $\mathbb{Z} \in \mathbb{D}(H_1)$ and $\mathbb{Z} \in \mathbb{D}(H_2)$. Let $\mathbb{P}$ be a prime number not dividing the order of $H_2$. Then by Maschke's theorem $\mathbb{D}(h_1, \mathbb{Z})$ and $\mathbb{D}(h_2, \mathbb{Z})$ are regular for any $\mathbb{P} \in \mathbb{D}(H_1)$ with $\mathbb{P} \in \mathbb{D}(\mu)$. Consequently, $\mathbb{D}(h_1, \mathbb{Z})$ and $\mathbb{D}(h_2, \mathbb{Z})$ are reduced and seminormal (in the sense of Swan, [20]). By the previous lemma $\mathbb{Z} \in \mathbb{D}(H_2)$ for some $\mathbb{P} \in \mathbb{D}(\mathbb{H}_{\mathbb{Z}})$. Considering the infinite number of primes $\mathbb{P}$ we obtain $\mathbb{Z} \in \mathbb{D}(H_2)$.

**Lemma 5.** For any ring $A$ and any cancellative monoid $\mathcal{M}$ ($f \in A[\mathcal{M}], m \in \mathcal{M}$, $m \not\in \mathcal{M}) \Rightarrow (f \not\in \mathcal{M})$.

**Proof is trivial.**

**Proof of the theorem.** Let $D$ be a Dedekind domain of algebraic integers in $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{N}$. By Lemma 1 $D[\mathcal{M}]$ is reduced, so by [20] the Pic $\text{-regularity of } D[\mathcal{M}]$ implies the seminormality of $D[\mathcal{M}]$ (in the sense of [20]). Let $\mathcal{F} \subset K(\mathcal{M})$ with $\mathcal{F} \subset \mathcal{M}$. Then $\mathcal{F} \subset D[\mathcal{M}]$ (Lemma 3). By Lemma 5 $\mathcal{F}$ is seminormal (may be with nontrivial torsion), in this situation the integral closure of a submonoid $\langle m \rangle \subset \mathcal{M}$ generated by some $m \in \mathcal{M}$ must have the form either $\mathbb{N} \times G$ or $\mathbb{N} \times G \setminus \{(1, g) \mid g \neq 1\}$ for some integrally closed submonoid $\mathbb{N} \subset \mathbb{Q}_+$ and some subgroup $G$ in the torsion subgroup $T$ of $K(\mathcal{M})$. In the both cases $D[G]$ is seminormal (in the sense of Swan). Indeed: let $\mathcal{F} \subset \mathcal{M}$ for arbitrary $u \in \mathcal{F}$ we have $u^2 x, u^2 y \in D[\mathcal{M}]$; therefore there exists $z \in D[\mathcal{M}]$ for which $z^2 = u^2 x$ and $z^2 = u^2 y$; choose a finitely generated submonoid $\mathcal{M}_0 \subset \mathcal{M}$ for which $x, z, u \in D[\mathcal{M}_0]$; let $K(\mathcal{M}_0) = T_0 \times F$ for some $T_0 \subset T$. 


and some free \( F \); we have \( u^{-1}z \in \mathcal{D}[T_0][F] \) and \( (u^{-1}z)^3 \in \mathcal{D}[T_0] \); since \( \mathcal{D}[T_0] \) is reduced (Lemma I) by Lemma 2 we obtain \( u^{-1}z \in \mathcal{D}[T_0] \); since \( (u^{-1}z)^2, (u^{-1}z)^3 \in \mathcal{D}[T_0 \cap G] \) by Lemma 4 \( u^{-1}z \in \mathcal{D}[T_0 \cap G] \subseteq \mathcal{D}[G] \); thus \( \mathcal{D}[G] \) is seminormal.

Now, let the torsion subgroup \( T \subset \mathcal{K}(\mathcal{M}) \) be nontrivial. Then for some \( m \in \mathcal{M} \) the corresponding subgroup \( G \subset T \) is nontrivial as well. Arbitrarily choose a finite nontrivial subgroup \( H \subset G \); let \( d \) be the order of \( H \) and \( \mathcal{D} \) be the ring of algebraic integers in \( \mathbb{Q}(\sqrt{d}) \). According to [R] \( \mathcal{D}[H] \) is not seminormal. On the other hand since \( \mathcal{D}[G] \) is seminormal by Lemma 4 \( \mathcal{D}[H] \) must be seminormal too, a contradiction. Q.E.D.

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