

A REGULARITY CRITERION FOR SEMIGROUP RINGS

WINFRIED BRUNS AND JOSEPH GUBELADZE*

ABSTRACT. An analogue of Kunz's Frobenius criterion for regularity of a local ring in positive characteristic is established for general commutative semigroup rings.

Let S be a commutative semigroup (we always assume that S contains a neutral element), and K a field. For every $m \in \mathbb{Z}_+$ the assignment $x \mapsto x^m$, $x \in S$, induces a K -endomorphism π_m of the semigroup ring $R = K[S]$. Therefore we can consider R as an R -algebra via π_m , and especially as an R -module. Let $R^{[m]}$ denote R with its R -module structure induced by π_m . If S is finitely generated, then $R^{[m]}$ is obviously a finitely generated R -module.

In this note we want to give a regularity criterion for S in terms of the homological properties of $R^{[m]}$ that is analogous to Kunz's [K] characterization of regular local rings of characteristic $p > 0$ in terms of the Frobenius functor. Our criterion, which generalizes a result of Gubeladze [G, 10.2], requires only a mild condition on S and we provide a 'pure commutative algebraic' proof. (In [G] the result was stated for seminormal simplicial affine semigroup rings and derived from the main result that K_1 -regularity implies regularity for such rings.)

Theorem 1. *Let S be a finitely generated semigroup, K a field, $R = K[S]$, and $m \in \mathbb{Z}_+$, $m > 0$. Suppose that S has no invertible element $\neq 1$ and is generated by irreducible elements. Then the following conditions are equivalent:*

- (a) $R^{[m]}$ has finite projective dimension;
- (b) $R^{[m]}$ is a free module;
- (c) S is free, in other words, $S \cong \mathbb{Z}_+^n$ for some $n \in \mathbb{Z}_+$.

Proof. It is obvious that (c) implies (b) and (b) implies (a). Now assume that (a) is satisfied. We first reduce the problem to a question of local algebra.

1991 *Mathematics Subject Classification.* Primary 13D05, 20M25; Secondary 13A35.

* Supported in part by Alexander von Humboldt Foundation and CRDF grant #GM1-115.

Set $T = S \setminus \{1\}$. The ideal $\mathfrak{m} = TR$ is a maximal ideal of R . In fact, $R/\mathfrak{m} \cong K$. Furthermore the only prime ideal of \mathfrak{q} of R such that $\mathfrak{m} = \pi_m^{-1}(\mathfrak{q})$ is \mathfrak{m} . Therefore $\pi_m \otimes R_{\mathfrak{m}}$ is an endomorphism of $R_{\mathfrak{m}}$ that makes $R_{\mathfrak{m}}$ a finitely generated $R_{\mathfrak{m}}$ -module of finite projective dimension. In particular $R_{\mathfrak{m}}$ has the same depth considered as an $R_{\mathfrak{m}}$ -module via $\pi_m \otimes R_{\mathfrak{m}}$ as it has in its natural $R_{\mathfrak{m}}$ -module structure (for example, see Bruns and Herzog [BH, 1.2.26]). The Auslander–Buchsbaum formula [BH, 1.3.3] thus implies that $R_{\mathfrak{m}}$ is a finite free module over itself via $\pi_m \otimes R_{\mathfrak{m}}$. The lemma below yields that $R_{\mathfrak{m}}$ is a regular local ring.

Let x_1, \dots, x_n be the irreducible elements of S . We claim that their images in $R_{\mathfrak{m}}$ form a minimal system of generators of the maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$. In fact, consider a presentation

$$R^r \xrightarrow{\varphi} R^n \xrightarrow{\psi} \mathfrak{m} \rightarrow 0,$$

where the i -th element e_i of the natural basis of R^n is mapped to x_i . We must show that all the entries of the matrix φ are in \mathfrak{m} . Suppose on the contrary that there is a relation

$$a_1x_1 + \dots + a_nx_n = 0$$

with, for example, $a_1 \notin \mathfrak{m}$. Then $a_1 = \alpha_1 + \alpha_2s_2 + \dots + \alpha_us_u$ with $\alpha_i \in K$, $\alpha_i \neq 0$, and $s_2, \dots, s_u \in T$. Writing a_2, \dots, a_m similarly, we see that there are only two possibilities, (i) $x_1 = s_ix_1$ for some i , or (ii) $x_1 = vx_j$ for some $v \in S$ and $j > 0$. Both cases are impossible because x_1 is irreducible.

However, $R_{\mathfrak{m}}$ is a regular local ring. Especially it is a factorial ring, in which the (images of the) x_i are pairwise non-associated prime elements. Therefore all the elements $x_1^{e_1} \dots x_n^{e_n}$, $e_1, \dots, e_n \in \mathbb{Z}_+$ are pairwise different, and it follows that $S \cong \mathbb{Z}_+^n$. \square

Remark 2. (a) If we omit the hypothesis that S be generated by irreducible elements, then the proof above shows just the following: the subsemigroup generated by $x_1, \dots, x_n \in S$ such that x_1, \dots, x_n form a minimal system of generators of the ideal $\mathfrak{m}R_{\mathfrak{m}}$ is free of rank n .

(b) One can weaken the hypothesis of the theorem by requiring only that the group S_0 of invertible elements of S is a free abelian group. Then $T = S \setminus S_0$ generates a prime ideal \mathfrak{p} in R , and part (c) of the theorem must be replaced by the condition that $S \cong \mathbb{Z}_+^n \times \mathbb{Z}^q$ for some $n, q \in \mathbb{Z}_+$.

The following lemma is just an abstract version of Herzog's argument [H] characterizing the modules of finite projective dimension in terms of the Frobenius functor.

Lemma 3. *Let R be a Noetherian local ring with maximal ideal \mathfrak{m} . If there exists an endomorphism π of R with $\pi(\mathfrak{m}) \subset \mathfrak{m}^2$ and such that R is a flat R -module via π , then R is a regular local ring.*

Proof. According to the criterion of Auslander–Buchsbaum–Serre [BH, 2.2.7] we must show that $k = R/\mathfrak{m}$ has finite projective dimension as an R -module. Write R' for R considered as an R -module via π , and let \mathcal{P} be the functor that takes an R -module M to $M \otimes R'$ considered as an R -module via the identification $R = R'$. We choose a minimal free resolution \mathcal{F} of k ,

$$\mathcal{F}: \cdots \rightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \rightarrow \cdots \rightarrow F_1 \rightarrow k \rightarrow 0.$$

One has $\mathcal{P}(R) = R$, $\mathcal{P}(F_i) = F_i$, and $\mathcal{P}(\mathcal{F})$ is the complex that we obtain from \mathcal{F} by replacing all entries in its matrices with their images under π . By hypothesis, $\mathcal{P}(\mathcal{F})$ is again exact, and the exactness is preserved by an e -fold iteration of this process. Especially, $\mathcal{P}^e(\mathcal{F})$ is a free resolution of $\mathcal{P}^e(k)$ for all $e > 0$.

Let $x_1, \dots, x_t \in \mathfrak{m}$ be a maximal R -sequence. Then $\bar{R} = R/(x_1, \dots, x_t)$ has projective dimension t , and so $\mathrm{Tor}_i^R(\bar{R}, \mathcal{P}^e(k)) = 0$ for all $i > t$ and $e > 0$. On the other hand, one can compute $\mathrm{Tor}_i^R(\bar{R}, \mathcal{P}^e(k))$ by tensoring $\mathcal{P}^e(\mathcal{F})$ with \bar{R} . Let B_i be the kernel of φ_i . Then for sufficiently large i and all $e > 0$ we have an exact sequence

$$0 \rightarrow \bar{R} \otimes \mathcal{P}^e(B_{i+1}) \rightarrow \bar{R} \otimes F_{i+1} \rightarrow \bar{R} \otimes \mathcal{P}^e(B_i) \rightarrow 0.$$

Since we have chosen a maximal R -sequence, $\mathrm{depth} \bar{R} \otimes F_{i+1} = 0$ if $F_{i+1} \neq 0$. On the other hand, for e sufficiently large, $\mathcal{P}^e(B_{i+1}) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_{i+1}$ and $\mathcal{P}^e(B_i) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_i$ have positive depth or are zero according to [H, Lemma 3.2]. This is a contradiction. \square

REFERENCES

- [BH] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge University Press, 1993.
- [G] J. Gubeladze, *Nontriviality of $SK_1(R[M])$* , J. Pure Appl. Algebra **104** (1995), 169–190.
- [H] J. Herzog, *Ringe der Charakteristik p und Frobeniusfunktoren*, Math. Z. **140** (1974), 67–78.
- [K] E. Kunz, *Characterizations of regular local rings of characteristic p* , Amer. J. Math. **91** (1969), 772–784.

UNIVERSITÄT OSNABRÜCK, FB MATHEMATIK/INFORMATIK, 49069 OSNABRÜCK,
GERMANY

E-mail address: `Winfried.Bruns@mathematik.uni-osnabrueck.de`

A. RAZMADZE MATHEMATICAL INSTITUTE, ALEXIDZE ST. 1, 380093 TBILISI,
GEORGIA

E-mail address: `gubel@imath.acnet.ge`