(1) Suppose $P$ and $Q$ are polyhedra. Then $P \times Q$ is a polyhedron. Moreover if $P$ and $Q$ are polytopes then $P \times Q$ is a polytope. The facets of $P \times Q$ are either $F \times Q$ where $F$ is a facet of $P$ or $P \times F$ where $F$ is a facet of $Q$. Finally, the faces of $P \times Q$ are $F \times G$ where $F$ is a face of $P$ and $G$ is a face of $Q$ (provided we include $P$ and $Q$ among the faces of $P$ and $Q$).

Proof. Suppose $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$. We will assume $d, e \geq 2$ and leave it to the reader to check the lower dimensional cases. There exist affine forms (linear functionals plus a constant) $H_i$ on $\mathbb{R}^d$, $1 \leq i \leq m$, such that $P = \cap_i H_i^+$ and $H_i^0 \cap P$ is a facet of $P$ for each $i$. This is a minimal representation of $P$ as an intersection of half-spaces. Similarly there exist affine forms $K_j$ on $\mathbb{R}^e$, $1 \leq j \leq n$, such that $Q = \cap_j K_j^+$ and the facets of $Q$ are the intersections $K_j^0 \cap Q$. Define affine forms $L_k$ on $\mathbb{R}^d \times \mathbb{R}^e = \mathbb{R}^{d+e}$, $1 \leq k \leq m+n$, as follows: $L_k(v, w) = H_k(v)$ if $1 \leq k \leq m$, and $L_k(v, w) = K_{k-m}(w)$ for $m + 1 \leq k \leq m+n$. Then $P \times Q = \cap_k L_k^+$, so $P \times Q$ is a polyhedron. If $P$ and $Q$ are polytopes (bounded polyhedra), then $P \times Q$ is also bounded, so $P \times Q$ is a polytope as well.

The sets $L_k^0 \cap (P \times Q)$ are either $(H_k^0 \cap P) \times Q$ or $P \times (K_k^0 \cap Q)$. In either case $P \times Q \subset L_k^+$ and these sets are dimension one less than the dimension of $P \times Q$, so they are facets of $P \times Q$. It remains to show only that every facet of $P \times Q$ has this form.

Suppose $F$ is a facet of $P \times Q$ with vertices $(v_0, w_0), \ldots, (v_t, w_t)$. Since $\dim(P \times Q) \geq 4$, $t \geq 3$. We will show that all the $v_i$ lie on the same facet of $P$ or all the $w_i$ lie on the same facet of $Q$, thus proving that the facet has the form $F \times Q$ or $P \times F$. Otherwise either there are two vertices $(v_0, w_0)$ and $(v_1, w_1)$ where neither $v_i$ are on the same facet of $P$ nor $w_i$ are on the same facet of $Q$, in which case the line between them goes through the interior of $P \times Q$ and does not lie on a facet, or there are three points $(v_0, w_0), (v_1, w_1)$ and $(v_2, w_2)$ such that the $v_i$ are not all on the same facet of $P$ nor are the $w_i$ on the same facet of $Q$. A convex combination of the three vertices will then be in the interior of $P \times Q$, showing that the three points are not on a facet of $P \times Q$.

Since every face of a polytope is an intersection of facets (perhaps the empty intersection for the face which is the polytope itself), it follows that every face of $P \times Q$ is product of a face of $P$ with a face of $Q$. \hfill \Box

(2) Suppose $P \subset \mathbb{R}^3$ is a polytope. If every pair of vertices of $P$ is joined by an edge then $P$ is a simplex.
Proof. The only one-dimensional polytope is a line segment, the one-dimensional simplex.

If \( P \) is two-dimensional, then it is a polygon in a plane. If the polygon has more than three vertices, choose four: \( A, B, C, D \). Since two of the edges \( AB, AC \) and \( AD \) cannot be coincident, any two of the three vectors \( B - A, C - A \) and \( D - A \) are linearly independent. Since the three vectors are in a plane, they are linearly dependent and there exist non-zero constants \( a, b, c \) such that \( a(B - D) + b(C - A) + c(D - A) = 0 \). Suppose the scalars \( a, b, c \) are all the same sign. We may assume \( a, b, c > 0 \) and \( a + b + c = 1 \). Then \( aB + bC + cD = A \). But \( aB + bC + cD \) is an interior point of \( P \), so it cannot be a vertex \( A \).

Now suppose the signs of \( a, b, c \) are not all the same. We may assume \( a, b > 0, c < 0 \) and \( a + b - c = 1 \). Then \(-cA + aB + bC = (a + b)A - cD\). Thus a point on the edge \( AD \) is interior to \( P \), a contradiction. \( P \) has only three vertices and is a triangle (a two-dimensional simplex).

If \( P \) is three-dimensional, then we can use an argument similar to the two-dimensional argument. Suppose \( P \) has five vertices \( A, B, C, D \) and \( E \). The four vectors \( B - A, C - A, D - A \) and \( E - A \) are linearly dependent, but any three are independent. Thus there are non-zero constants \( a, b, c, d \) such that \( a(B - A) + b(C - A) + c(D - A) + d(E - A) = 0 \). We may assume \(|a| + |b| + |c| + |d| = 0 \). If \( a, b, c, d \) have the same sign, we may assume all are positive. Then \( aB + bC + cD + dE = A \), which is not possible since a vertex cannot be a convex combination of other vertices.

If one sign of \( a, b, c, d \) is different from the others, we may assume \( a, b, c > 0 \) and \( d < 0 \). Then (as above) \(-dA + aB + bC + cD = (a + b + c)A - dE\). Thus a point on the edge \( AE \) is interior to the quadrilateral \( ABCD \), which again is not possible.

Finally, if two signs of \( a, b, c, d \) are positive and two negative, we may assume \( a, b > 0 \) and \( c, d < 0 \). We will also assume \( a + b + c + d \geq 0 \). Then

\[
\frac{a}{a+b} + \frac{b}{a+b} C = \frac{a+b+c+d}{a+b} A - \frac{c}{a+b} D - \frac{d}{a+b} E
\]

Thus a point in the interior of the edge \( BC \) is either in the interior of the triangle \( ADE \) or in the interior of the edge \( DE \), either being impossible. Thus a three-dimensional polytope in \( \mathbb{R}^3 \) with all edges connected cannot have five or more vertices and so must be a simplex with four vertices. \( \Box \)

The proof we have just given does not generalize to polytopes of dimension greater than three. In fact we can construct a four-dimensional polytope with any number of vertices greater than five for which any two vertices are joined by an edge. The following example is taken from Grünbaum, *Convex Polytopes*, Springer, 2003 (second ed.), pp. 61-63. This example is slightly different than that suggested by Gubeladze; his example is discussed on page 67 of Grünbaum. The example can also be found in Ziegler, *Lectures on Polytopes*, Springer, 1995, pp. 14-15.

The moment curve \( M(t) \) in \( \mathbb{R}^d \) is the curve parametrized by the function \( M : \mathbb{R} \rightarrow \mathbb{R}^d, M(t) = (t, t^2, \ldots, t^d) \). We will show that if at least \( d + 1 \) points are selected on this curve in \( \mathbb{R}^d, d \geq 4 \), then the points are the vertices of a simplicial polytope, we will give a criterion for \( d \) points to lie
on a facet (that's the maximum number since the polytope is simplicial),
and we will show that any two vertices are joined by an edge.

We recall the formula, the Vandermonde determinant. Let \( k \) be a field.

In the polynomial ring \( k[S_0, \ldots, S_d] \) we have:

\[
V_d = \det \begin{bmatrix}
1 & S_0 & \cdots & S_0^d \\
\vdots & \vdots & \ddots & \vdots \\
1 & S_d & \cdots & S_d^d
\end{bmatrix} = \prod_{0 \leq i < j \leq d} (S_j - S_i)
\]

In particular, this formula holds with real numbers replacing the \( S_i \). To
see why this holds, note that the determinant and the product are both
homogeneous polynomial in the \( S_i \) of degree \( 1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \).
Since the determinant vanishes when \( S_i = S_j \), the product must divide the
determinant. Since both are the same degree, they are constant multiples of
each other. The coefficient of \( S_n^d \) in the determinant is \( V_{n-1} \); in the product
it is \( \prod_{0 \leq i < j \leq n-1} (S_j - S_i) \). By induction these coefficients are equal, so the
constant multiplier is 1.

One more general fact: a set of vectors \( v_0, \ldots, v_d \in \mathbb{R}^d \) is affinely inde-
pendent if and only if

\[
\det \begin{bmatrix}
1 & v_0 \\
\vdots & \vdots \\
1 & v_d
\end{bmatrix} \neq 0
\]

Suppose \( t_0 < \cdots < t_n \in \mathbb{R} \) and let \( v_i = M(t_i) \). If \( 0 \leq i_0 < \cdots < i_d \leq n \)
then the Vandermonde determinant show that the points \( v_{i_0}, \ldots, v_{i_d} \) are
affinely independent. Thus any set of points \( W = \{v_{i_0}, \ldots, v_{i_d}\} \) spans a
hyperplane in \( \mathbb{R}^d \). We will show that either there are vertices \( v_j \) both sides
of this hyperplane, so that \( W \) does not span a facet, or all the remaining
points are on same side of the hyperplane and \( W \) is the set of vertices for
a facet. Thus every facet is a simplex. We will also show that every pair of
points is contained in at least one facet, which will prove that every pair of
points is connected by an edge (because a facet is a simplex) and all points
\( v_j \) lie on a facet, the points are vertices of a polytope.

The affine equation defining the affine set spanned by \( W \) is:

\[
0 = H(x) = H(x_1, \ldots, x_d) = a_0 + a_1 x_1 + \cdots + a_d x_d = \det \begin{bmatrix}
1 & t_{i_1} & \cdots & t_{i_1}^d \\
\vdots & \vdots & \ddots & \vdots \\
1 & t_{i_d} & \cdots & t_{i_d}^d \\
1 & x_1 & \cdots & x_d
\end{bmatrix}
\]

If \( j \notin W \) then \( H(v_j) \neq 0 \), so a facet can contain at most \( d \) vertices.

Now when are all the points \( v_j \) on the same side of \( H(x) = 0 \)? When
for all \( j \notin W \) do all the values \( H(v_j) \) have the same sign?
(a) If \( j, j+1 \notin W \) then \( H(v_j) \) and \( H(v_{j+1}) \) have the same sign, as shown
by the product expansion of the Vandermonde determinant.
(b) If \( i \in W \) and \( i-1, i+1 \notin W \) then \( H(v_{i-1}) \) and \( H(v_{i+1}) \) have opposite
signs.
(c) If \( i, i+1 \in W \) then \( H(v_{i-1}) \) and \( H(v_{i+2}) \) have the same sign.
Thus a set of points $v_1, \ldots, v_d$ lie on a facet if and only if the indices can be divided into adjacent pairs with any left-over index being the first or last index 0 or $n$. Examples if $n = 7$ and $d = 4$: (1245), (0134), (0347); examples if $n = 7$ and $d = 5$: (02356), (23567), (01457). If $d \geq 4$ then it is possible to include any two points in a facet. This is not possible for $d \leq 3$.

(3) Every polytope is the intersection of a simplex with an affine set.

Proof. We begin with a new and useful way to describe polytopes imbedded in $\mathbb{R}^d$. Suppose a polytope $P$ is the intersection of half-spaces defined by linear functionals $L_i(x)$ and scalars $c_i$, $1 \leq i \leq n$. That is, $P = \bigcap_i \{x : L_i(x) \geq c_i\}$. If we define a matrix $M$ with rows $L_i$ so that

$$Mx = \begin{bmatrix} L_1(x) \\
\vdots \\
L_n(x) \end{bmatrix}$$

and define $c = \begin{bmatrix} c_1 \\
\vdots \\
c_n \end{bmatrix} \in \mathbb{R}^n$, then $P$ is the set of solution to a matrix inequality:

$$P = \{x : Mx \geq c\}$$

The matrix $M$ is injective because $P$ is bounded and if $x \in P$ then $x + \ker(M) \subset P$. Thus $\ker(M) = 0$ and $Q = \{Mx : x \in P\}$ is a subset of $\mathbb{R}^n$ that is isomorphic to $P$.

In $\mathbb{R}^n$, $Q \subset \{y : y \geq c\}$. In fact if $H = \text{im}(M) \subset \mathbb{R}^n$, then $H$ is an affine set (actually a subspace, but we don’t care about translations here), and $Q = H \bigcap \{y : y \geq c\}$. Let $o = [1 \cdots 1]$. Since $Q$ is bounded, there exists a real number $K$ such that $y \in Q \implies oy \leq K$. The set of points $\{y : y \geq c \text{ and } oy \leq K\}$ is a simplex $S \subset \mathbb{R}^n$. Finally, $Q = H \bigcap S$, showing that every polytope is (isomorphic to) the intersection of an affine set and a simplex. □

(4) Any pointed cone is the intersection of a simplicial cone and a linear subspace.

Proof. If $C$ is a pointed cone, the $C = C(P)$ for some polytope $P$ not containing $0$. By the previous problem we may imbed $P$ in a space where $P = H \bigcap S$, $S$ a simplex and $H$ an affine set. Choose coordinates so $0 \notin S$ but $0 \in H$, so $H$ is a subspace. $C(S)$ is a simplicial cone and

$$C = C(P) = C(S \cap H) = C(S) \cap H$$

□

(5) Any two vertices on a polytope are connected by a sequence of edges.

Proof. The result is obvious if $\text{dim}(P) \leq 1$, so let the dimension of the polytope be $d \geq 2$. It suffices to show that two facets of the dual polytope can be connected by a path that does not cross any faces of dimension less than $d - 1$.

Let $P$ be the dual polytope. Assume $P \subset \mathbb{R}^d$ and $0$ is in the interior of $P$. Rays from the origin intersect the boundary of $P$ (denoted $\partial P$) in exactly one point. If we choose a sphere $S^{d-1}$ around the origin in $\mathbb{R}^d$, then the rays from the origin map $\partial P$ homeomorphically onto $S^{d-1}$. Moreover
faces of $P$ of dimension $e$ are mapped to subsets of $S^{d-1}$ of dimension $e$. If we remove from $S^{d-1}$ all the images of faces of dimension less than $d - 1$, what remains is still path connected. Let $T$ denote what remains.

Suppose $F_1$ and $F_2$ are facets of $P$. Choose points $p_1 \in \text{int}(F_1)$ and $p_2 \in \text{int}(F_2)$ and connect the images of $p_1$ and $p_2$ in $T$ by a path. The pre-image of the path on $\partial P$ connects $F_1$ to $F_2$ and does not cross any face of dimension less than $d - 1$. □

(6) In the category of polytopes, every polytope is the image of a simplex.

Proof. Suppose $P \subset \mathbb{R}^n$ is a polytope with vertices $v_0, \ldots, v_n$. There exists a unique affine map $T : \mathbb{R}^n \to \mathbb{R}^d$ such that $T(0) = v_0$ and $T(e_i) = v_i$, where $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$. The convex hull $S$ of $0, e_1, \ldots, e_n$ is a simplex, and $T$ maps $S$ onto $P$, the convex hull of $v_0, \ldots, v_n$. □

(7) Every centrally symmetric polytope is the image of a cross polytope.

Proof. Let $P$ be a centrally symmetric polytope with vertices $\pm v_1, \ldots, \pm v_n$. As in the problem above, $P$ is the image of the cross polytope in $\mathbb{R}^n$ spanned by $\pm e_1, \ldots, \pm e_n$. The map is the linear map that sends $e_i$ to $v_i$. □

(8) Two polytopes, $P$ and $Q$, are projectively equivalent if there exists a cone $D$ and hyperplanes $H$ and $J$ such that $P \cong D \cap H$ and $Q \cong D \cap J$. Suppose $P$ and $Q$ are full-dimensional polytopes containing $0$ in their interiors. Then $P$ and $Q$ are projectively equivalent if and only if the dual polytopes $P^*$ and $Q^*$ are projectively equivalent.

Our proof has three parts. We will use $P \sim Q$ to mean $P$ is projectively equivalent to $Q$.

(a) $P \sim Q$ if and only if $C(P) \cong C(Q)$

Proof. Suppose $P \subset \mathbb{R}^d$. The cone $C(P) = \{(a, av) \in \mathbb{R}^{d+1} : a \geq 0, v \in P\}$. If $P \sim Q$ then, in the definition above, $C(D \cap H) = D = C(D \cap J)$ because $P$ and $Q$ are bounded so $D$ is pointed and $H$ and $J$ meet every ray in $D$. Thus $C(P) \cong C(D \cap H) = C(D \cap J) \cong C(Q)$. Conversely let $T : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ define an isomorphism $C(P) \cong C(Q)$. Since $P$ and $Q$ are full dimensional, $T$ is an isomorphism. Let $H = \{(1,v) : v \in \mathbb{R}^d\}$. $H$ is an affine subset of $\mathbb{R}^{d+1}$, and $P = C(P) \cap H$. Also $Q = C(Q) \cap H$, and $P \cong T(P) = T(C(P) \cap H) = C(Q) \cap T(H)$. Since $T(H)$ is an affine subset, $P \sim Q$. □

(b) If $0 \in \text{int}(P)$ then $C(P^*) = C(P)^*$.

Proof. Both $C(P^*)$ and $C(P)^*$ are contained in $\mathbb{R}^{d+1*}$, which we will identify with $\mathbb{R}^{d+1}$ via the dot product. If $v \in \mathbb{R}^{d+1}$, we will denote the first coordinate by $v' \in \mathbb{R}$ and the remaining coordinates by $v'' \in \mathbb{R}^d$.

The cone $C(P^*) = \{v \in \mathbb{R}^{d+1} : v \cdot w \geq 0 \ \forall \ w \in C(P)\}$. The fact that $0$ is in the interior of $P$ leads to two key conclusions:

(i) If $v \in C(P^*)$ then $v' \geq 0$, since $v' = v \cdot (1, 0)$ and $(1, 0) \in C(P)$.

(ii) If $v \in C(P^*)$ and $v' = 0$ then $v = 0$, because $(0, v'') \cdot (1, x) \geq 0$ for all $x \in P$. But if $v'' \neq 0$ then $v'' \cdot x$ takes both positive and negative values on an neighborhood of $0$ in $P$. 


Now we can prove the desired equality.

\[ \mathbf{v} \in C(P)^* \iff (v', v'') \cdot (a, ax) \geq 0 \quad \forall a \geq 0, \forall x \in P \]

\[ \iff v' + v'' \cdot x \geq 0 \quad \forall x \in P \]

\[ \iff v = 0 \text{ or } \frac{1}{v'}v'' \cdot x \leq 1 \quad \forall x \in P \]

\[ \iff v = 0 \text{ or } \frac{1}{v'}v'' \in P^* \]

\[ \iff v = (a, ax) \quad \text{for some } a \geq 0 \text{ and } x \in P^* \]

\[ \iff v \in C(P^*) \]

(c) Now we are ready to prove the main result.

\[ P \sim Q \iff C(P) \cong C(Q) \]

\[ \iff C(P)^* \cong C(Q)^* \quad \text{since } C(P)^{**} = C(P) \]

\[ \iff C(P^*) \cong C(Q^*) \]

\[ \iff P^* \sim Q^* \]

Department of Mathematics
San Francisco State University
San Francisco, CA 94132
E-mail address: meredith@sfsu.edu
URL: http://online.sfsu.edu/~meredith