Piecewise isometries – an emerging area of dynamical systems

Arek Goetz

Abstract. We present several examples of piecewise isometric systems that give rise to complex structures of their coding partitions. We also list and comment on current open questions in the area that pertain to fractal-like structure of cells. Piecewise isometries are two and higher dimensional generalizations of interval exchanges and interval translations. The interest in the dynamical systems of piecewise isometries is partially catalyzed by potential applications and the fact that simple geometric constructions give rise to rich phenomena and amazing fractal graphics. Piecewise isometric systems appear in dual billiards, Hamiltonian systems, and digital filters.

The goal of this article is to illustrate the beauty and complexity of two dimensional piecewise isometries. We do not attempt here to survey all recent work on the subject. Our objective is merely to intrigue and widen interest in systems $T : X \to X \subset R^2$ such that T is a local Euclidean isometry. We do so by first briefly describing examples and then listing chosen open questions.

1. Introduction

A simple yet intriguing example of a piecewise isometric systems is illustrated in Figure 1. The space is the union of two isosceles triangles P_0 and P_1 (called *atoms*). The generating map T is a rotation on each of the two triangles. In each case T rotates each triangle in such a way that atoms not overlap after the application of T.

Figure 1 illustrates the action of the map $T : X \to X$ as well as the partition of the space X into domains that always follow the same pattern of visits to both atoms.

Two points in one domain exhibit the same long term behavior. Understanding the shape and structure of domains in thus crucial and it central in research of piecewise isometries.

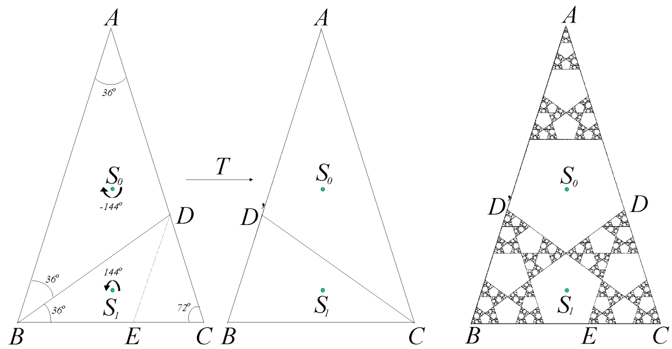

Figure 1. A simple example of a piecewise rotation with two atoms that are isosceles triangles. The angle $\alpha = \pi/5$. The right figure illustrates the partition into domains (cells) that follow the same pattern of visits to P_0 and P_1. The open sets in this partition are periodic pentagons. Its structure is self-similar and it resembles the Sierpiǹski gasket.

The key idea in understanding the self-similarity of the gasket of pentagons is the observation that $T : P_0 \cup P_1$ is conjugated to the action of the first return map $T_\triangle$ to the bottom atom. The reader may check that the conjugacy is a similarity Γ. The map Γ is the composition of the contraction at a with contraction ratio $\lambda = 2e^{3\pi i/5}\cos(\pi/5)$ and the reflection with respect to the line perpendicular to ac passing through a.

While in Example 1 the fractal structure is justified rigorously [20], there are very few other examples of invertible piecewise isometries for which the structure of cells is understood.

Figure 2 illustrates another invertible example with two triangular atoms. This newly discovered system is believed to be the only other example in the class of noninvertible piecewise isometries with two triangular atoms. The mosaic of cells is extremely complicated, yet it appears to exhibit self-similar structures that can be found with an aid of a high resolution computer graphics. This indicates, for example, that the number of periodic cells is infinite.

Potential applications. Piecewise rotations are natural two dimensional generalizations of well studied interval exchange transformations defined in [25, 26, 31, 40] and studied in [2, 3, 11, 10, 24, 34, 27, 29, 43, 45, 44] and interval translation maps [12, 42]. Invertible piecewise rotations [1, 6, 7, 22, 21, 23, 37] (those that preserve Lebesgue measure) are closely related to the theory of dual billiards [41] and Hamiltonian systems [39].

A somewhat unusual application is found outside of mathematics, in electrical engineering, in particular in the theory of digital filters [4, 5, 15, 14, 16, 30, 36]. Digital filters are algorithms widely incorporated in electronic components in contemporary electronics devices such as cellular phones, radio devices and voice and image recognition systems.

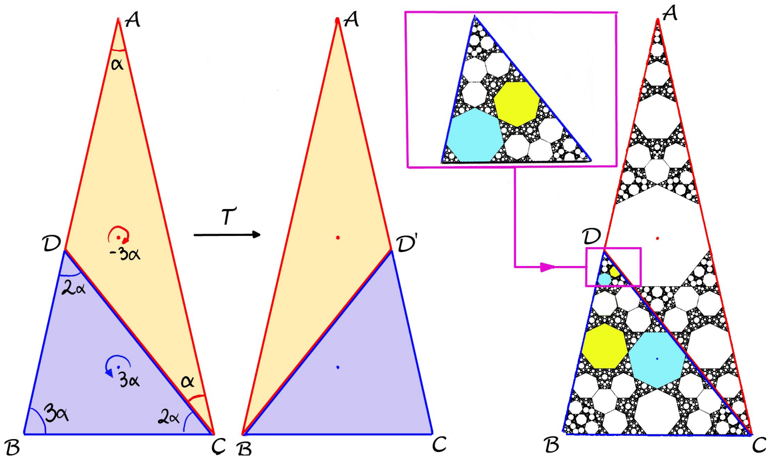

Figure 2. A newly discovered remaining example of an invertible piecewise rotation with two triangles ($\alpha = \frac{\pi}{7}$). The white heptagons in the right triangle are the interiors of "cells," maximal regions that follow the same pattern of visits to $\triangle ACD$ and $\triangle BCD$. The structure of cells appears to be fractal, at least in some areas. However, this mosaic is not well understastood and the size of the "chaotic set" (the black region) is unknown. One of the most tantalizing computer observations and questions is whether all rational and irrational piecewise rotations give rise to self-similar structures of periodic cells. The measure of the collection nonperiodic cells is not known. Currently, no results exist except in two very special cases.

2. Definitions and a brief survey of open questions

Before we briefly illustrate some open questions that pertain to the fractal structure of cells, we first formally define generating systems and cells.

Definition. Let X be a subset of R^2 (or R^n) and $P = \{P_0, \ldots, P_{r-1}\}$ be a finite partition ($r > 1$) of X, that is $\bigcup_{0 \le i < r} P_i = X$, and P_i disjoint with P_j for $i \ne j$.

A *piecewise isometry* is a pair (T, P), where $T\colon\ X \to X$ is a map such that its restriction to each *atom* $P_i, i = 0, \ldots, r-1$ is a Euclidean isometry. The sets $\{P_0, \ldots, P_{r-1}\}$ are called *atoms.*

In the above definition we assume that the partition P is minimal in the sense that T is not an isometry on the union of two distinct elements of P.

In the literature domains that follow the same pattern of visits to atoms are called *cells* (see Figure 1, 2, and 3) and they are defined via the itinerary map $i : X \to \{0, \ldots, r-1\}$. The map i encodes the forward orbit of a point by recording the indices of atoms visited by the orbit, that is $i(x) = w_0 w_1 \ldots$, where $T^k x \in P_{w_k}$.

Definition. A *cell* is the set of all points with the same itinerary.
Assuming that the generating partition consists of convex atoms of total finite area, cells of positive measure are either discs, or they are polygons. The latter case occurs when the action of T on each atom is a rotation through an angle that is a rational multiple of π.

Definition. Cells whose itineraries are eventually periodic are called *rational cells.*

In all the figures in this article the white regions are rational cells. The complement of rational cells (represented by black regions in Figure 1, 2, and 4) is the set of points with irrational itineraries, sometimes called the *exceptional set* E. In Figure 1, the set E is nonempty, has zero two dimensional measure, but a fractional (between one and two) Hausdorff dimension.

The fundamental Problem. The central unsolved question in the area of piecewise isometries is to determine the size of the set E. In all known cases for which the Lebesgue measure of E can be computed, it is zero.

Since there is no general theorem describing the measure of E, researchers try to study E by analyzing in details many examples of rational and irrational piecewise isometries. During this investigation a number of related questions were posed some of which we list below.

A few open problems. Even though two-dimensional Euclidean Piecewise Rotations are the most basic generalizations of one dimensional maps, their dynamics is very intriguing and complex. This has already been observed in [1] for the case of a piecewise affine maps on the torus. Such maps can be conjugated with two-dimensional piecewise rotations on the rhombus, and they give rise to very interesting orbit structures. We have also observed the complexity of piecewise rotations in [17] where our motivation was to study the following natural piecewise rotation on the flat torus. Let $T = [0,1] \times [0,1] \to [0,1] \times [0,1]$, $Tx = Mx \ mod \ \mathbf{Z}^2$ where $M \in SO(\mathbf{R}, 2)$ (the entries of M are not integers, hence T is not an Anosov automorphism). Iterations of T resulted in many fascinating and intriguing orbit behaviors that resemble parallelism with **(a)** the dynamics of rational functions, especially as seen from the beautiful computer graphics, **(b)** interval exchange transformations. The dynamics of T and other piecewise isometric planar maps is still poorly understood.

The existence of a system with an infinite number of periodic domains implies results about the invariant ergodic measures for piecewise rotations. The problem of determining invariant measures of interval exchanges has been fundamental to many researchers [26, 34, 45, 38, 28]. Experimental results indicate that two-dimensional systems, unlike one-dimensional interval exchanges, have an infinite number of ergodic invariant measures concentrated on the invariant set of periodic domains.

Problem 1. *Determine all invariant non-atomic probability Borel measures for two dimensional piecewise rotations.*

Since all invariant measures are always concentrated on the set $\Omega = \cap_{n>0} T^n X$, it is natural to study the structure of the attractor Ω. This was first done in [18]. Other known attractor structures include Miguel Mendes [35], and Sierpinski type gasket (Figure 1 and [20]). There are examples of piecewise rotations T with two half planes as atoms for which the attractor contains the structure from Figure 2, or the attractors Ω are rhombi (Figure 3). The dynamics of the rhombus is conjugate to the dynamics of piecewise affine maps on the torus studied in electrical engineering.

Problem 2. *Classify completely all piecewise isometric attractors in two dimensional systems.*

Let $f : [0,1] \times [0,1] \to [0,1] \times [0,1]$, $f(x,y) = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ mod \ Z^2$. If $a \in (0,2)$, then f is an *invertible piecewise affine map* of the torus and it can be conjugated to a piecewise rotation on the unit rhombus [1] by the angle $\theta = \cos^{-1} \frac{a}{2}$ via the affine matrix $\begin{pmatrix} 1 & -\cos\theta \\ 0 & \sin\theta \end{pmatrix}$.

One of the most tantalizing phenomena present in piecewise affine maps on the torus for a rational choice of the parameter is a complicated self-similar structure of the periodic domains. In [1], Adler, Kitchens, and

Tresser were able to describe in detail two examples whose induced piecewise rotation is the rotation of the unit rhombus about its center (see also [23]). Also [20] showed that Sierpiński-type gasket of pentagons appear as a return map in specific examples of more complicated maps.

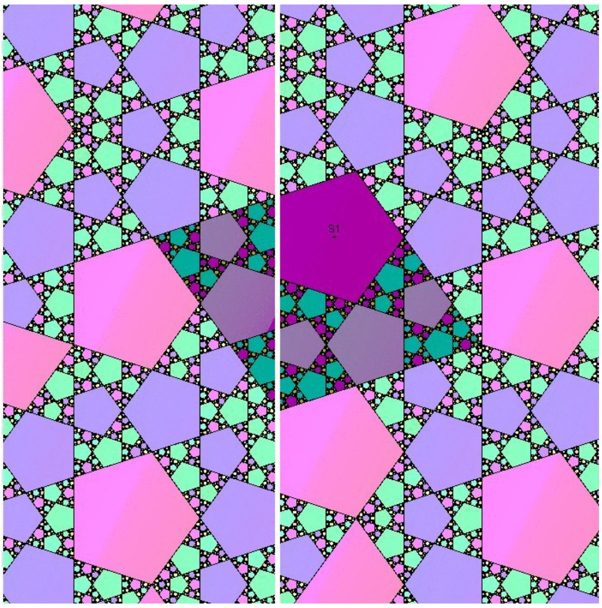

Figure 3. An example of piecewise rotations with two atoms as half-planes. All preperiodic cells are eventually trapped in the attractors in the shape of a rhombus. The dynamics on the attractor is almost everywhere invertible. It is conjugated to piecewise affine maps on the torus, which are transformations that emerge in digital filters. The map restricted to the darker rhombus, is conjugated a piecewise affine map on the torus with the parameter parameter $a = 2\cos\frac{2\pi}{5}$ The partitions inside the attractor contains self-similar gasket of pentagons from Figure 1. The examples of attractor in the shape of a rhombus can be also constructed for angles that are not rational multiple of π.

The centrally located dark rhombus in Figure 3 illustrates the structure of cells for the map $f : [0,1]\times[0,1] \to [0,1]\times[0,1]$, where the parameter $a = 2\cos\frac{2\pi}{5}$.

By direct inspection [32, 33, 20], it can be shown that up to Lebesgue measure zero, the dynamics of the first return map to a suitably chosen triangle in the attractor in Figure 3 (left) is an elementary transformation of the Sierpiński-like triangle (Figure 1) that was described in [20] . It is remarkable that only for a very few values of a, is it possible to isolate and describe a fractal structure of periodic sets under the action of f [1, 20, 32].

Problem 3. *Do there exist an infinite number of different periodic sets for a generic value of a?*

Some insight into the problem above can be obtained by the study of induced maps on a subset of the torus. In some cases this induced map is the piecewise rotation with two atoms (perturbed 8-attractor map).

Our recent numerical and symbolic (with Guillaume Poggiaspalla) computations indicate that the structure of cells is self-similar for a few other rational angles (Figure 4). The self-similarity of cells will follow from affirmation the following central conjecture:

Problem 4. Self-similar conjecture for piecewise rotations. *Let $\overline{T}$ denote the first return map to an n-cell. An n-cell is a maximal set whose first n iterates follow the same pattern of visits to the atoms. For all rational choices of the angles of rotation, up to rescaling, there are only a finite number of induced transformations $\overline{T}$.*

The affirmative answer to this conjecture would answer a number of open questions in the theory of dual billiards and piecewise affine maps on the torus (overflow maps). Some of these questions were listed in [1]. For example, it will allow us to show that the Lebesgue measure of the closure of the set of points that will hit discontinuities is zero. This will confirm numerical results obtained by [5].

Finally, the above conjecture will allow us to conclude a number of currently open questions about polygonal dual billiards [41]. For example, it would follow that rational polygonal dual billiards have an infinite number of distinct symbolic periodic orbits. It will allow us to conclude that almost all points are periodic.

It is interesting to note that [9] proved a similar statement to the above conjecture for one dimensional interval exchange transformations.

Symbolic growth. An important aspect in the investigation of piecewise rotations is to study the symbolic word growth of all finite words realizable by the piecewise isometry $T : X \to X$. A finite sequence w of indices is realizable if there exists a point $x \in X$ such that its itinerary begins with w. Let W_n denote the number of sequences of length n realizable by T. The growth of W_n measures the complexity of the dynamics of piecewise isometric dynamical systems. Moreover, this symbolic growth is related to the topological entropy of $\hat{T} : \hat{X} \to \hat{X}$, the continuous extension of T on the graph of the itinerary map. In particular, the topological entropy of the shift map on the coding image gives a lower bound on the topological entropy of $\hat{T}$.

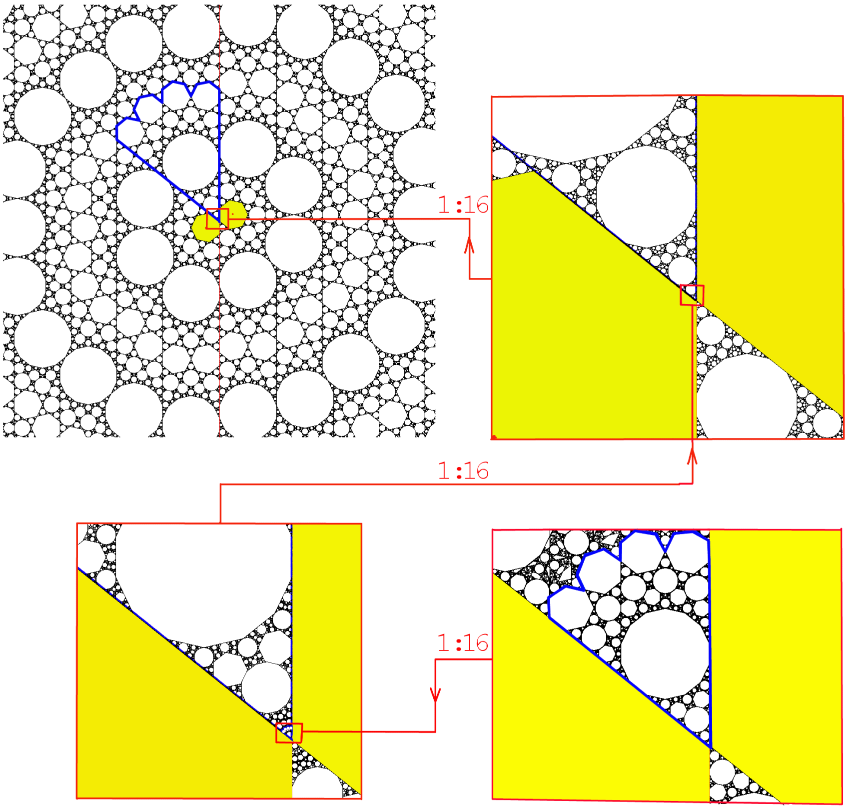

Figure 4. Strong numerical evidence that invertible piecewise rotations exhibit self-similar structure of cells. The angle of rotation is $\frac{2\pi}{7}$. The only two known cases in the literature are based on the angle of rotation $\frac{2\pi}{5}$.

In [19], we constructed a two-dimensional example with maximal induced symbolic word growth. This example, however, featured a "Cantor-like" partition. A natural and important question is,

Problem 5. *Determine the maximal symbolic growth for a partition with natural boundaries.*

Recently, [13] showed that piecewise isometries whose atoms are polytopes have zero entropy.

Parallelism with the theory of rational maps. Another tantalizing theme is to investigate the potential analogies of the dynamics of piecewise isometric dynamical systems with the dynamics of rational functions. There is some experimental and theoretical evidence [17] that the dynamics of piecewise isometries in $\mathbf{R}^n$ for $n > 1$ is closer in nature to the complex dynamics of rational maps [8], then to one-dimensional interval exchanges. Below, we have tabulated some of the observed parallelism.

Dynamics of Piecewise Isometries	Dynamics of Rational Functions
Irrationally coded sets	The Julia sets
Rational cells in Σ	The components of the Fatou set
Convexity of the cells	Simple connectivity of the Fatou components
Periodic cells contain at most one periodic point (piecewise irrational rotations only).	Components of the Fatou set contain at most one periodic point.
Periodic behavior of cells of positive Lebesgue measure	No Wandering Domains Theorem
8-attractor	Siegel Disks

Problem 6. *Study the parallelisms of higher-dimensional piecewise isometric systems with the dynamics of rational maps, with a particular focus on the "piecewise isometric" version of the No Wandering Domains Theorem, and the many parallelisms between the number-theoretic implications on the dynamics of the Siegel disks and the number-theoretic implications on 8-attractors.*

References

[1] Roy Adler, Bruce Kitchens, and Charles Tresser. Dynamics of piecewise affine maps of the torus. *Ergodic Theory and Dynamical Systems*, 21(4):959–999, 2001.

[2] P. Arnoux. Echanges d'intervalles at flots sur less surfaces. Theorie Erodique, Monographie n. 29 de l'Ensegnement Mathematique, Geneve.

[3] Pierre Arnoux, Donald S. Ornstein, and Benjamin Weiss. Cutting and stacking, interval exchanges and geometric models. *Israel J. Math.*, 50(1-2):160–168, 1985.

[4] P. Ashwin. Non-smooth invariant circles in digital overflow oscillations. *Proceedings of NDE96: fourth international workshop on Nonlinear Dynamics of Electronic Systems, Seville, Spain*, 1996.

[5] P. Ashwin, W. Chambers, and G. Petrov. Lossless digital filter overflow oscillations; approximation of invariant fractals. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 7:2603–2610, 1997.

[6] Peter Ashwin and Xin-Chu Fu. Tangencies in invariant disk packings for certain planar piecewise isometries are rare. *Dynamical Systems*, 16:333–345, 2001.

[7] Peter Ashwin and Xin-Chu Fu. On the geometry of orientation preserving planar piecewise isometries. *to appear in the Journal of Nonlinear Science*, 2002.

[8] Alan F. Beardon. *Iteration of Rational Functions*. Springer-Verlag, New York, 1991.

[9] M. D. Boshernitzan and C. R. Carroll. An extension of Lagrange's theorem to interval exchange transformations over quadratic fields. *J. Anal. Math.*, 72:21–44, 1997.

[10] Michael Boshernitzan. A condition for minimal interval exchange maps to be uniquely ergodic. *Duke Mathematical Journal*, 52:723–752, 1985.

[11] Michael Boshernitzan. Rank two interval exchange transformations. *Ergodic Theory and Dynamical Systems*, 8:379–394, 1988.

[12] Michael Boshernitzan and Isaac Kornfeld. Interval translation mappings. *Ergodic Theory and Dynamical Systems*, 15:821–831, 1995.

[13] Jéróme Buzzi. Piecewise isometries have zero topological entropy. *Ergodic Theory and Dynamical Systems*, 21(5):1371–1377, 2001.

[14] L. O. Chua and T. Lin. Chaos and fractals from third-order digital filters. *Internat. J. Circuit Theory Appl.*, 18(3):241–255, 1990.

[15] Leon O. Chua and Tao Lin. Chaos in digital filters. *IEEE Trans. Circuits and Systems*, 35(6):648–658, 1988.

[16] A.C. Davies. Geometrical analysis of digital filters overflow oscillation. *Proc. IEEE Symp. Circuits and Systems, San Diego*, pages 256–259, 1992.

[17] Arek Goetz. *Dynamics of Piecewise Isometries*. PhD thesis, University of Illinois at Chicago, 1996.

[18] Arek Goetz. Dynamics of a piecewise rotation. *Continuous and Discrete Dynamical Systems*, 4(4):593–608, 1998.

[19] Arek Goetz. Sofic subshifts and piecewise isometric systems. *Ergodic Theory and Dynamical Systems*, 19:1485–1501, 1999.

[20] Arek Goetz. A self-similar example of a piecewise isometric attractor. In *Dynamical systems (Luminy-Marseille, 1998)*, pages 248–258. World Sci. Publishing, River Edge, NJ, 2000.

[21] Eugene Gutkin and Nicolai Haydn. Topological entropy of generalized polygon exchanges. *Ergodic Theory and Dynamical Systems*, 17:849–867, 1997.

[22] Hans Haller. Rectangle exchange transformations. *Monatsh. Math.*, 91(3):215–232, 1981.

[23] Byungik Kahgng. Dynamics of symplectic piecewise affine maps of tori. *Ergodic Theory and Dynamical Systems, to appear.*

[24] Anatole Katok. Interval exchange transformations and some special flows are not mixing. *Israel J. Math*, 35:301–310, 1980.

[25] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory in Dynamical Systems*. Cambridge University Press, 1995.

[26] Michael Keane. Interval exchange transformations. *Math. Z.*, 141:25–31, 1975.

[27] Michael Keane. Non-ergodic interval exchange transformation. *Isr. L. Math.*, 26:188–196, 1977.

[28] S. P. Kerckoff. Simplicial systems for interval exchange maps and measured foliations. *Ergodic Theory and Dynamical Systems*, 5:257–271, 1985.

[29] H. Keynes and D. Newton. A minimal, non-uniquely ergodic interval exchange transformation. *Math. Z.*, 148:101–105, 1976.

[30] Lj. Kocarev, C.W. Wu, and L.O. Chua. Complex behavior in digital filters with overflow nonlinearity: analytical results. *IEEE Trans. Circuits and Systems - II: Analog and digital signal processing*, (43):234–246, 1996.

[31] I. P. Kornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic Theory*. Springer-Verlag, 1982.

[32] J. H. Lowenstein, S. Hatjispyros, and F. Vivaldi. Quasi-periodicity, global stability and scaling in a model of hamiltonian round-off. *Chaos*, 7:49–66, 1997.

[33] J. H. Lowenstein and F. Vivaldi. Anomalous transport in a model of hamiltonian round-off. *Nonlinearity*, 11:1321–1350, 1998.

[34] Howard Masur. Interval exchange transformations and measured foliations. *Annals of Mathematics*, 115:169–200, 1982.

[35] Miguel Mendes. *Piecewise rotations and new concepts of Symmetry and Invariance*. PhD thesis, University of Surrey, England, 2001.

[36] Maciej J. Ogorzałek. Complex behavior in digital filters. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2(1):11–29, 1992.

[37] Guillaume Poggiaspalla. *thesis in progress*. PhD thesis, University of Marseille, 2002.

[38] M. Rees. An alternative approach to the ergodic theory of measured foliations on surfaces. *Ergodic Theory and Dynamical Systems*, 1:461–488, 1981.

[39] A.J. Scott, C.A. Holmes, and G.J. Milburn. Hamiltonian mappings and circle packing phase spaces. *Physica D*, 155:34–50, 2001.

[40] Ya. G. Sinai. *Introduction to ergodic theory*. Princeton University Press, 1976.

[41] Serge Tabachnikov. Billiards. *Panor. Synth.*, (1):vi –142, 1995.

[42] S. Troubetzkoy and J. Schmeling. Interval translation maps. *World Scientific J.M. Gambaudo et.al eds*, pages 291–302, 2000.

[43] W. Veech. Boshernitzan's critetion for unique ergodicity of an interval exchange transformation. *Ergodic Theory Dyn. Sys.*, 7:149–153, 1987.

[44] William Veech. Interval exchange transformations. *J. D'Analyse Math.*, 33:222–272, 1978.

[45] William Veech. Gauss measures for transformations on the space of interval exchange maps. *Annals of Mathematics*, 115:201–242, 1982.

Department of Mathematics,
San Francisco State University,
1600 Holloway Avenue
San Francisco, CA 94703, USA

E-mail address: goetz@sfsu.edu
http://math.sfsu.edu/goetz/