Complejos de Bergman

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The plan to follow: (or not to follow)

1. Amoebas and the Bergman complex.
2. Review of matroids.
3. The Bergman complex of a matroid.
4. The space of phylogenetic trees.

(with Carly Klivans)

5. The positive Bergman complex of an oriented matroid.

(with Carly Klivans and Lauren Williams)

6. The Bergman complex of a root system.

(with Vic Reiner and Lauren Williams)

1. Amoebas and the Bergman complex

Consider a variety $X \subset \mathbb{C}^n$, described by a system of polynomial equations in $\mathbb{C}[z_1, \ldots, z_n]$:

$$f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0.$$

Question: Given $r_1, \ldots, r_n > 0$, is there a solution $z \in X$ with

$$|z_1| = r_1, \ldots, |z_n| = r_n?$$

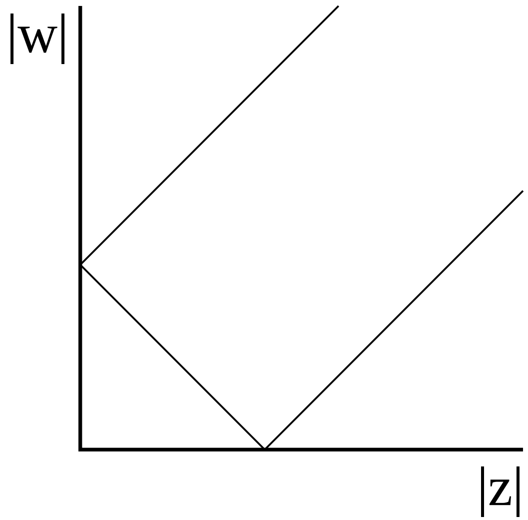
The amoeba of X is

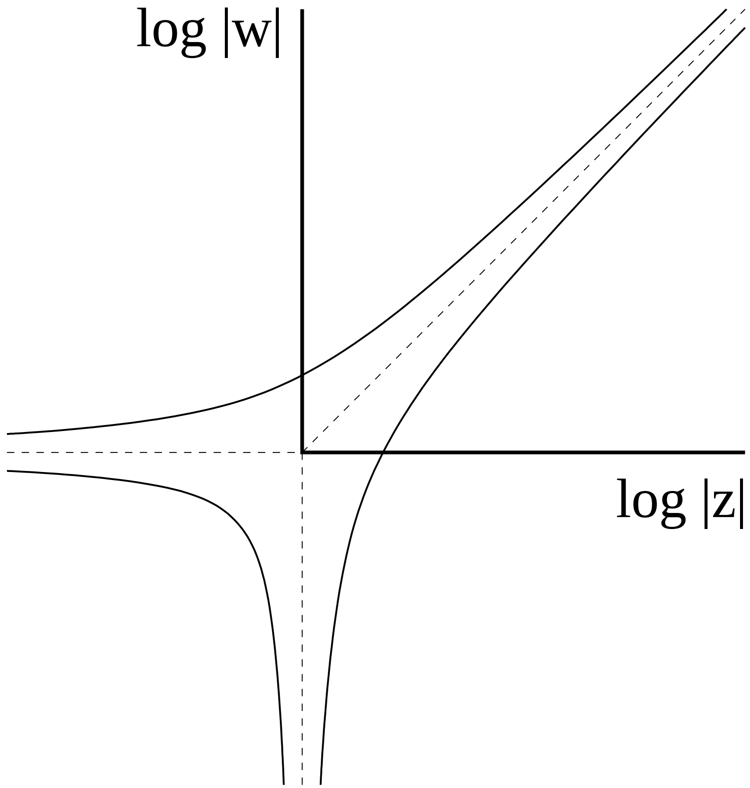
$$\mathcal{A}(X) = \text{Log } X = \{(\log|z_1|, \ldots, \log|z_n|) \;:\; z \in X \cap (\mathbb{C}^*)^n\}.$$

Example: $X = \{(w,z) \in \mathbb{C}^2 \mid 1 + w + z = 0\}$

There is a solution with given $|w|$ and $|z|$ if and only if

$$1 \leq |w| + |z|, \quad |w| \leq 1 + |z|, \quad |z| \leq 1 + |w|.$$





In general, amoebas are very difficult to describe. Their "tentacles" are simpler:

The Bergman complex of X, $\mathcal{B}(X)$, is a subset of the sphere S^{n-1}. It is (roughly) the set of directions where $\mathcal{A}(X)$ goes to infinity.

The Bergman fan or tropical variety, $\widetilde{\mathcal{B}}(X)$, is the fan over the Bergman complex.

Some appearances: real algebraic geometry, dynamical systems, measure theory.

Theorem. (Bergman, '71; Bieri and Groves, '84)

If X is d-dimensional and irreducible, then $\mathcal{B}(X)$ is a pure $(d-1)$-dimensional polyhedral complex.

Let I be the ideal of X.

Let $\text{in}_\omega(I)$ be the initial ideal w.r.t. $\omega \in \mathbb{R}^n$:

$$\text{in}_{(0,2,1)}(2xy^2 - x^3z + 3z^4) = 2xy^2 + 3z^4$$

0+4 0+1 4

$$\text{in}_\omega(I) = < \text{in}_\omega(f) \,|\, f \in I >$$

Theorem. (Kapranov, Sturmfels, '02)
$\mathcal{B}(X) = \{\omega \in S^{n-1} \mid \text{in}_\omega(I) \text{ contains no monomials}\}$

Corollary.
If V is a linear subspace, then $\omega \in \mathcal{B}(V)$ if and only if:

for each equation $a_1 x_{i_1} + \cdots + a_k x_{i_k} = 0$ satisfied by V,
the maximum weight of $\{x_{i_1}, \ldots, x_{i_k}\}$ is achieved twice.

2. Review of matroids

What combinatorial properties should a "good" notion of independence have?

Here is an answer, in terms of minimal dependent sets:

A matroid M on a finite ground set E is a collection $\mathcal{C}$ of circuits (subsets of E) such that:

C0. $\emptyset$ is not a circuit.

C1. No circuit properly contains another.

C2. If C_1 and C_2 are circuits and $x \in C_1 \cap C_2$, then $C_1 \cup C_2 - x$ contains a circuit.

Example:

$E = \{1, 2, 3, 4, 5, 6\}$

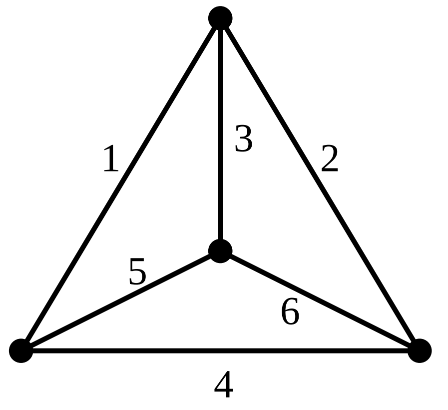
$\mathcal{C} = \{124, 135, 236, 456, 1256, 1346, 2345\}$

Four sources of matroids:

- A graph.

  E - the edges of a graph G.

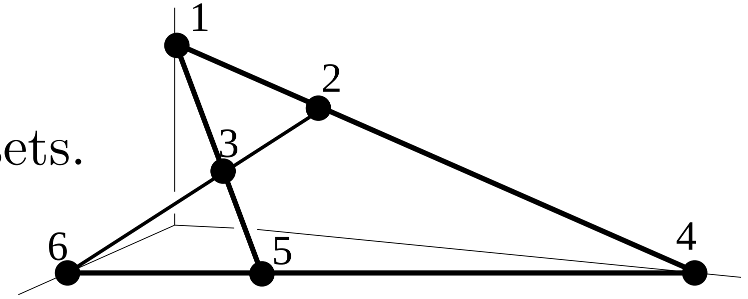
  $\mathcal{C}$ - the cycles of G.



- A collection of vectors.

  E - the set of vectors.

  $\mathcal{C}$ - the minimal dependent subsets.



$E = \{1, 2, 3, 4, 5, 6\}$

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- A subspace V of $\mathbb{C}^n$ (with a given basis).

  E - $\{1, \ldots, n\}$.

  $\mathcal{C}$ - the minimal $\{i, \ldots, k\}$ such that an equation of the form

  $a_i x_i + \cdots + a_k x_k = 0$ holds in V.

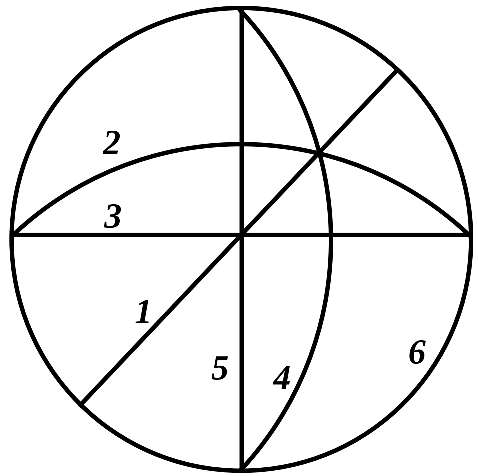
  $V = \{(a-b, a-c, a-d, b-c, b-d, c-d) : a, b, c, d \in \mathbb{C}\}$

- A hyperplane arrangement.

  E - the hyperplanes.

  $\mathcal{C}$ - the minimal sets of k hyperplanes

  intersecting in codimension $< k$.



$E = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{C} = \{124, 135, 236, 456, 1256, 1346, 2345\}$

3. The Bergman complex of a matroid

Definition. The Bergman complex of a matroid M on E is $\{\omega \in S^{E-2}$: every circuit achieves its ω-max more than once.$\}$

(Here $S^{E-2} = \{\omega \in \mathbb{R}^E \mid \sum \omega_i^2 = 1, \sum \omega_i = 0\}$.)

Example. $E = [6]$, $\mathcal{C} = \{124, 135, 236, 456, 1256, 1346, 2345\}$

- $\omega = (0, 9, 7, 9, 7, 7)$ is **not** in $\mathcal{B}(M)$. (Problem: ω-max in 456.)
- $\omega = (0, 9, 7, 9, 7, 9)$ **is** in $\mathcal{B}(M)$.

Problem. (Sturmfels)

Describe $\mathcal{B}(M)$ topologically and combinatorially.

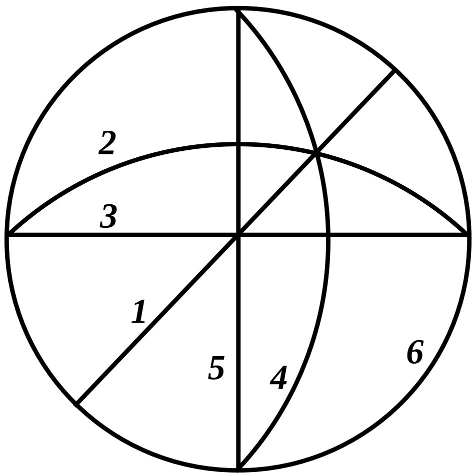
(Is it connected? Pure-dimensional? What is its homology?)

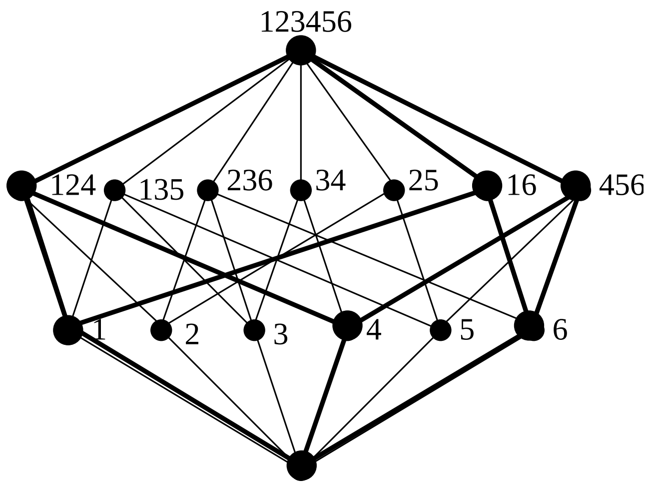
We need some definitions.

A subset $F \subseteq E$ is a flat (or closed) if $|F - C| \neq 1$ for all circuits C.

(In a hyperplane arrangement, the flats are the intersections.)

The lattice of flats L_M is the poset of flats ordered by containment. It is a lattice. Let $\bar{L}_M = L_M \backslash \{\hat{0}, \hat{1}\}$.





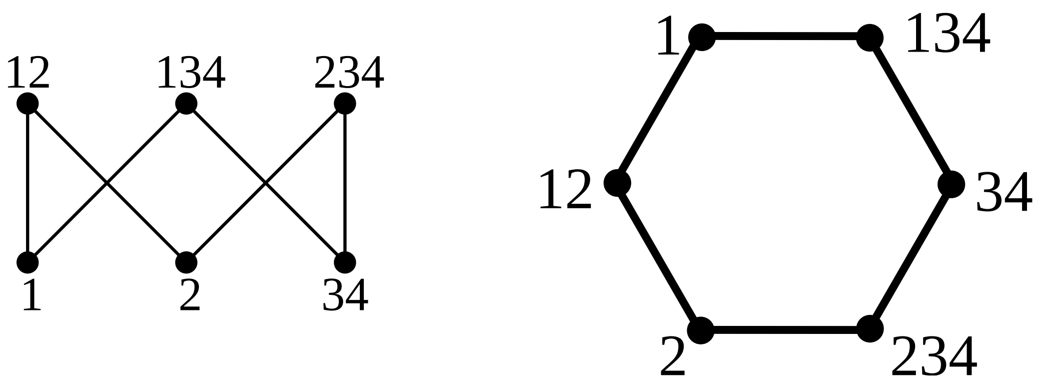
$E = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{C} = \{124, 135, 236, 456, 1256, 1346, 2345\}$

The order complex $\Delta(\bar{L}_M)$ of $\bar{L}_M$ is the following simplicial complex:

- vertices: elements of $\bar{L}_M$
- faces: chains of $\bar{L}_M$

An example of a poset and its order complex:



Theorem. (Björner, 1980)

$\Delta(\bar{L}_M)$ is a pure, shellable simplicial complex. It is homotopy equivalent to a wedge of $\hat{\mu}(L_M)$ $(r-2)$-dimensional spheres.

Theorem. (Ardila and Klivans, 2003)

Let M be a loopless matroid.
(A natural subdivision of) the Bergman complex of M is (a geometric realization of) $\Delta(\bar{L}_M)$.

Corollary.

The Bergman complex of M is a connected, pure-dimensional polyhedral complex. It is homotopy equivalent to a wedge of $\hat{\mu}(L_M)$ $(r-2)$-dimensional spheres.

Sketch of proof:

Two vectors $u, v \in S^{n-2}$ have the same order type when they have the same relative order.

An order type: $\omega_1 = \omega_4 < \omega_2 < \omega_3 = \omega_5$.

We denote it $\omega_{\emptyset \subset 14 \subset 124 \subset 12345}$.

- The condition $\omega \in \mathcal{B}(M)$ depends only on the order type of ω.
- The Bergman complex $\mathcal{B}(M)$ is a union of order types.
- The order type $\omega_{\emptyset \subset F_1 \subset \cdots \subset F_k \subset E}$ is in $\mathcal{B}(M)$ if and only if $F_1, \ldots, F_k$ are flats.

4. The space of phylogenetic trees.

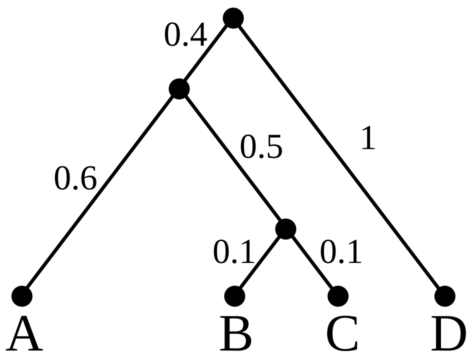
Consider $\mathbb{R}^{\binom{n}{2}} = \{(x_{12}, x_{13}, x_{23}, \ldots, x_{n-1,n}) : x_{ij} \in \mathbb{R}\}$.

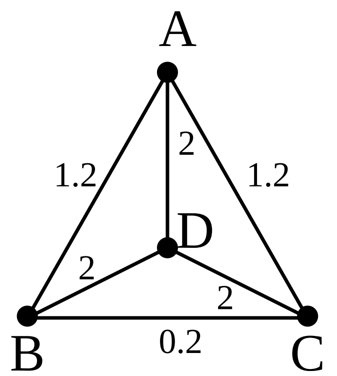
An ultrametric is a vector $\omega \in \mathbb{R}^{\binom{n}{2}}$ such that $\max\{\omega_{ij}, \omega_{jk}, \omega_{ik}\}$ is achieved twice for all i, j, k. (A weighting of the edges of K_n where in each triangle, the two heaviest edges have the same weight.)

One source of ultrametrics:

T = equidistant n- tree (rooted metric tree with n labelled leaves, where all the distances from the root to the leaves are equal.)

d_{ij} = distance between leaves i and j





In fact, that is the only source of ultrametrics!

Theorem. (Semple and Steel, 2003)
A vector $\delta \in \mathbb{R}^{\binom{n}{2}}$ is an ultrametric if and only if it is the distance function of an equidistant n-tree.

Theorem. (Ardila and Klivans, 2003)
A vector $\delta \in \mathbb{R}^{\binom{n}{2}}$ is an ultrametric if and only if it is in the Bergman fan $\widetilde{\mathcal{B}}(K_n)$.

(The two heaviest edges of each triangle are equal if and only if the two heaviest edges of each cycle are equal.)

Therefore, we can think of the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ as a space of phylogenetic trees.

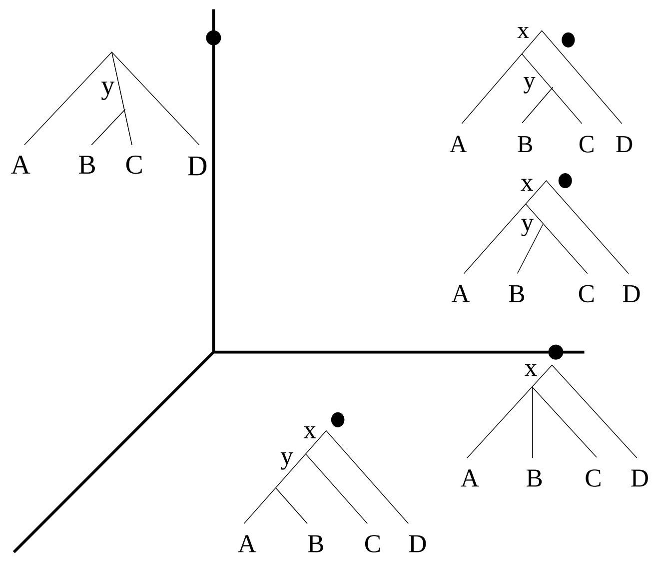
The space of phylogenetic trees $\mathcal{T}_n$.

(Vogtmann, 1990; Whitehouse, 1996; Billera, Holmes, V., 2001)

A binary n-tree T has $n-2$ internal edges. An orthant $\mathbb{R}^{n-2}_{\geq 0}$ parameterizes the possible equidistant n-trees of shape T.

When some edge lengths are 0, we get "degenerate" non-binary trees, which could come from different binary trees.

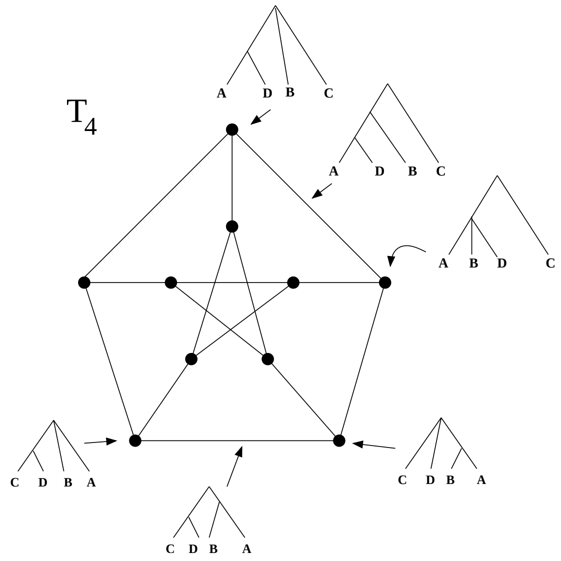
Glue the $(n-2)$-dimensional orthants where they agree.



T_n = link of the origin in $\mathcal{T}_n$ – Whitehouse complex

Other appearances:

- homotopy theory
- moduli space of curves $\overline{M}_{0,n}$
- WDVV eqs. of string theory



Theorem. (Vogtmann, 1990; Robinson and Whitehouse, 1996; Trappmann and Ziegler, 1998; Wachs, 1998; Sundaram, 1999)

T_n is a simplicial complex, homotopy equivalent to a wedge of $(n-1)!$ $(n-3)$-dimensional spheres.

Same homotopy type as $\Delta(\bar{\Pi}_n)$, where Π_n is the partition lattice of $[n]$ - the S_n-representations on their homology are also isomorphic!

What can we say about T_n?

We have two different parameterizations of equidistant n-trees.

- $\mathcal{T}_n$: combinatorial type, internal edge lengths.
- $\widetilde{\mathcal{B}}(K_n)$: distances between leaves.

We get a bijection $f : \mathcal{T}_n \to \widetilde{\mathcal{B}}(K_n)$.

Theorem. (Ardila and Klivans, 2003)

The map f is a piecewise linear homeomorphism between the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ and the space of phylogenetic trees $\mathcal{T}_n$.

Since $\Delta(\bar{\Pi}_n)$ is the subdivision of $\mathcal{B}(K_n)$ into order types,

Corollary.

The Whitehouse complex T_n is **homeomorphic** to $\Delta(\bar{\Pi}_n)$.

5. The positive part of $\mathcal{B}(M)$

Suppose M is an oriented matroid. We now define and study $\mathcal{B}^+(M)$, the positive part of $\mathcal{B}(M)$.

Motivation: (Speyer, Sturmfels, Williams)

Bergman complex $\mathcal{B}(M) \leftrightarrow$ tropical variety.

Positive part $\mathcal{B}^+(M) \leftrightarrow$ totally positive part.

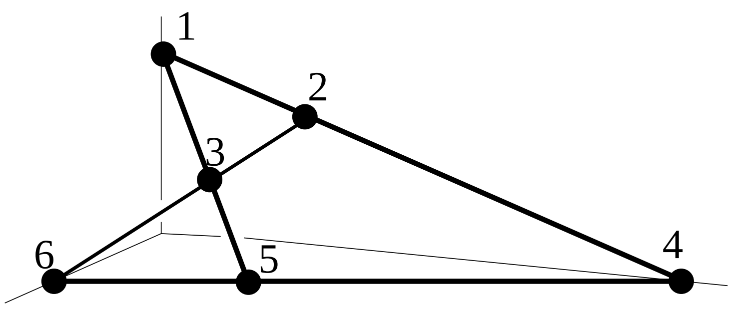
Goal: To prove analogous results for $\mathcal{B}^+(M)$.

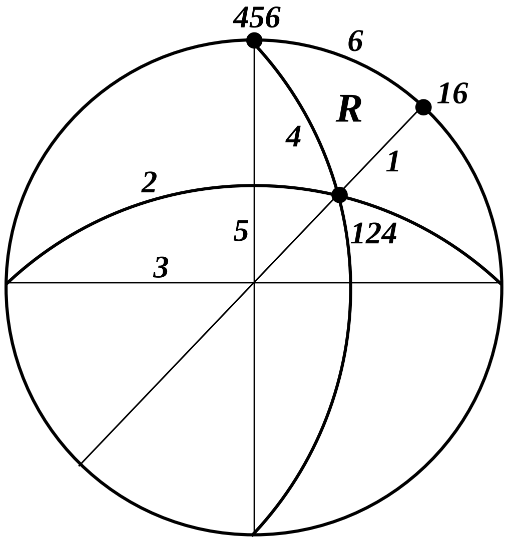
An oriented matroid is a collection of signed circuits (satisfying certain axioms).

Example.

$E = [6]$

$$\mathcal{C} = \{1\bar{2}4, 1\bar{3}5, 2\bar{3}6, 4\bar{5}6, 1\bar{2}5\bar{6}, 1\bar{3}46, 2\bar{3}\bar{4}5,$$
$$\bar{1}2\bar{4}, \bar{1}3\bar{5}, \bar{2}3\bar{6}, \bar{4}5\bar{6}, \bar{1}2\bar{5}6, \bar{1}3\bar{4}\bar{6}, \bar{2}34\bar{5}\}$$





Let M be an oriented matroid on $[n]$, with signed circuits $\mathcal{C}$.

Definition.

The positive Bergman complex $\mathcal{B}^+(M)$ is

$\{\omega \in S^{n-2} :$ each $C \in \mathcal{C}$ achieves its max weight in C^+ **and** $C^-\}$.

(Forgetting signs, each C achieves its max weight at least twice
Therefore $\mathcal{B}^+(M) \subseteq \mathcal{B}(M)$.)

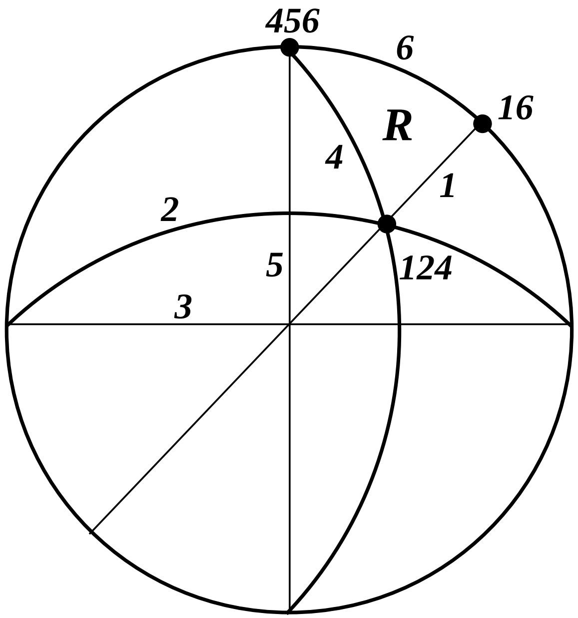
Example. $E = [6]$, $\mathcal{C} = \{1\bar{2}4, 1\bar{3}5, 2\bar{3}6, 4\bar{5}6, 1\bar{2}5\bar{6}, 1\bar{3}46, 2\bar{3}\bar{4}5, \ldots\}$

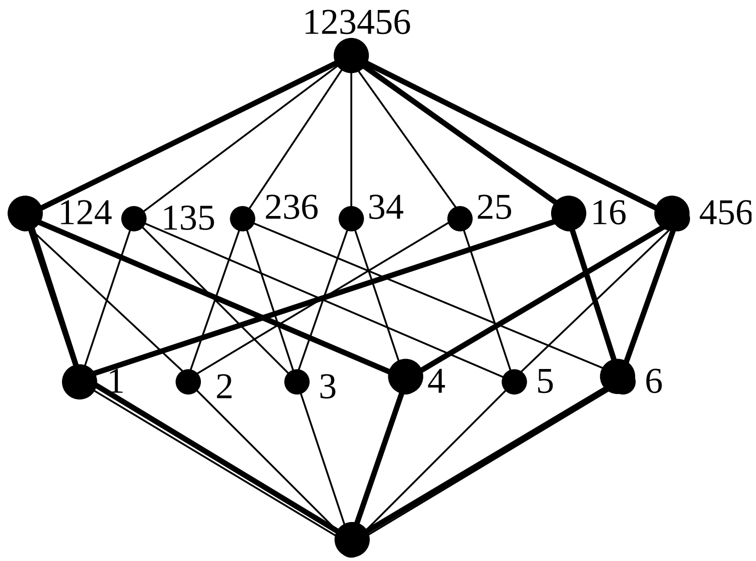
- $\omega = (0, 9, 7, 9, 7, 9)$ is **not** in $\mathcal{B}(M)$. (Problem: ω-max in 456.)
- $\omega = (9, 9, 9, 7, 7, 0)$ **is** in $\mathcal{B}(M)$.

Let M be an acyclic oriented matroid. The Las Vergnas face lattice $\mathcal{F}_{\ell v}(M)$ is the lattice of "positive" flats of M, ordered by inclusion.

Example. $E = [6]$, $\mathcal{C} = \{1\bar{2}4, 1\bar{3}5, 2\bar{3}6, 4\bar{5}6, 1\bar{2}5\bar{6}, 1\bar{3}46, 2\bar{3}\bar{4}5, \ldots\}$

Positive flats: $\emptyset, 1, 4, 6, 124, 16, 456, 123456$.





The topology of $\Delta(\bar{\mathcal{F}}_{\ell v}(M))$, the order complex of (the proper part of) the Las Vergnas face lattice of M, is also known:

Theorem.

$\Delta(\bar{\mathcal{F}}_{\ell v}(M))$ is homotopy equivalent to a sphere.

We have:

Theorem. (Ardila, Klivans, Williams, 2004)

Let M be an acyclic oriented matroid.

(A subdivision of) the positive Bergman complex of M is

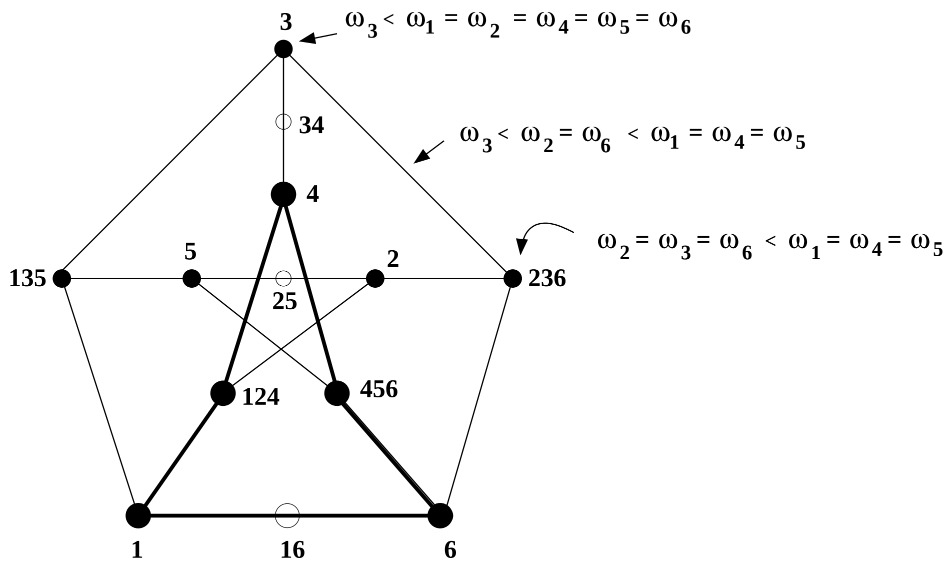
(a realization of) $\Delta(\bar{\mathcal{F}}_{\ell v}(M))$.

Therefore, $\mathcal{B}^+(M)$ is one of the spheres in $\mathcal{B}(M)$.

For the oriented matroid of the complete graph K_n, we have:

Theorem. (Ardila, Klivans, Williams, 2004)

$\mathcal{B}^+(K_n)$ is dual to the associahedron A_{n-2}.



The graph K_n has $n!$ different acyclic orientations, which give rise to $n!$ (almost) different "positive parts" of $\mathcal{B}(K_n)$.

In this way, we recover the known covering of the Whitehouse complex T_n with $n!$ dual associahedra.

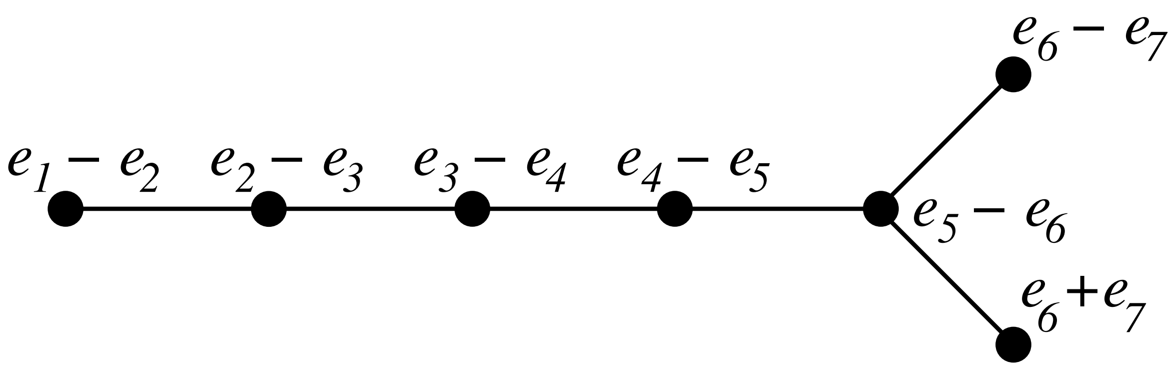
6. $\mathcal{B}(M)$ for Coxeter arrangements.

Let Φ be the root system of a Coxeter system (W, S).

Let M_Φ be the corresponding matroid.

Goal: To describe $\mathcal{B}^+(M_\Phi), \mathcal{B}(M_\Phi)$ combinatorially.

Key ingredient: The Dynkin diagram of Φ.



First we describe $\mathcal{B}^+(M_\Phi)$; we need an acyclic orientation of M_Φ.

We have a correspondence:

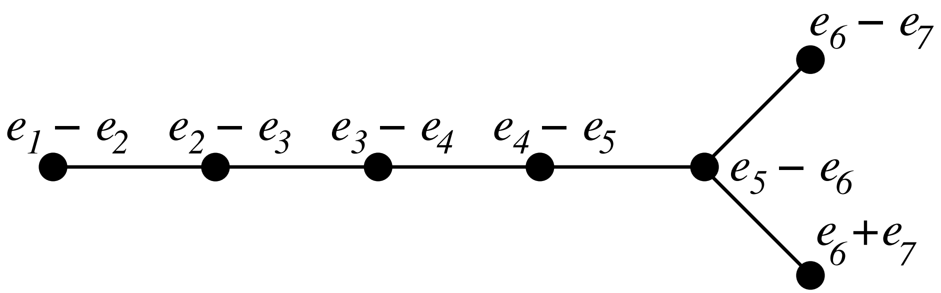
acyclic orientations of the matroid M_Φ

$\updownarrow$

regions of the real arrangement $\mathcal{A}_\Phi$

$\updownarrow$

choices of simple roots of Φ.

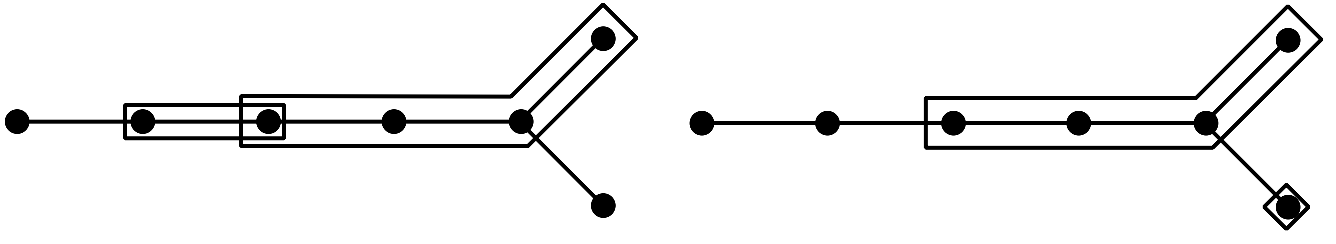


A tube in a Dynkin diagram is a connected subgraph.

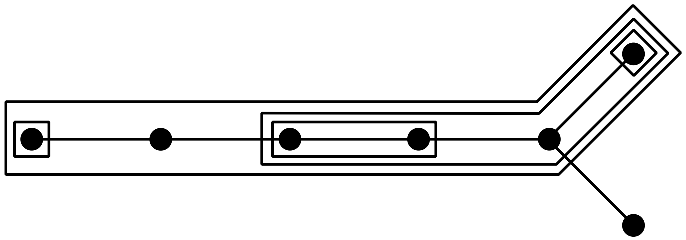
Two tubes t_1 and t_2 are compatible if

- $t_1 \subset t_2$, or
- t_1 and t_2 are disjoint and non-adjacent.

The two kinds of incompatible tubes:



A tubing is a collection of compatible tubes.



Poset of tubings:

Say $T_1 < T_2$ if T_2 can be obtained from T_1 by adding tubes.

The graph associahedron of type Φ is a polytope whose face poset is the poset of tubings of the Dynkin diagram of Φ.

It was discovered independently by Carr and Devadoss, Davis and Januszkiewicz, and Postnikov.

Theorem. (Ardila, Reiner, Williams, 2004)
$\mathcal{B}^+(M_\Phi)$ is dual to the graph associahedron of type Φ.

The wonderful model of a hyperplane arrangement $\mathcal{A}$ blows up the non-normal crossings of $\mathcal{A}$ without changing the topology of the complement. (de Concini and Procesi, 1995)

The nested set complex of $\mathcal{A}$ encodes the combinatorics of this wonderful model.

Theorem. (Ardila, Reiner, Williams, 2004)
$\mathcal{B}(M_\Phi)$ equals the nested set complex of the Coxeter arrangement of type Φ.

This recovers Carr and Devadoss's tiling of the minimal blowup of a Coxeter complex by graph associahedra.

Thank you.

The first two preprints and an extended abstract of the third are available at:

- www.math.washington.edu/∼federico
- arxiv:math.CO/0311370,0406116