Positroids, non-crossing partitions, and a conjecture of da Silva

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Joint work with:
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1. Positroids and non-crossing partitions.  
   Trans. Amer. Math Soc., to appear  
   http://arxiv.org/abs/...

2. Positively oriented matroids are representable.  
   J. European Math. Soc., to appear  
   http://arxiv.org/abs/...
1. Matroids

A matroid $M$ on $[n] := \{1, \ldots, n\}$ is a collection $\mathcal{B}$ of subsets of $[n]$ (called bases) satisfying the basis exchange axiom:

- If $A, B$ are bases and $a \in A \setminus B$,
  there exists $b \in B \setminus A$ such that $A \setminus a \cup b$ is a basis.

All elements of $\mathcal{B}$ have the same size, called the rank of $M$.

Motivating example. If $K$ is any field and $A \in K^{m \times n}$ has rank $m$, the collection

$$\mathcal{B} := \{ B \subset [n] \mid \text{the submatrix } A_B \text{ is invertible} \}$$

is a matroid $M(A)$ of rank $m$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \leadsto M(A) = \{12, 13, 14, 23, 24\}$$
Axiom systems for matroids

There are many equivalent ways of defining matroids:
- simplicial complex (independent sets)
- submodular function (rank function)
- closure operator (span)
- lattice (flats)
- polytope (bases) (My favorite.)

Given a matroid $\mathcal{B}$ of subsets of $[n]$, the matroid polytope is

$$P_{\mathcal{B}} := \text{convex} \left\{ \sum_{i \in B} e_i \mid B \in \mathcal{B} \right\}.$$ 

$\mathcal{B} = \{12, 13, 14, 23, 24\}$
Matroid polytopes

Given a matroid \( \mathcal{B} \) (or any collection of \( d \)-subsets) on \([n]\), let

\[
P_\mathcal{B} := \text{convex} \left\{ \sum_{i \in B} e_i \mid B \in \mathcal{B} \right\}.
\]

\[\mathcal{B} = \{12, 13, 14, 23, 24\} \implies \]

**Theorem** (Edmonds, Gelfand-Goresky-MacPherson-Serganova)

\( \mathcal{B} \) is a matroid \( \iff \) all edges of \( P_\mathcal{B} \) have the form \( e_i - e_j \).

**Remark:**

basis exchanges in \( \mathcal{B} \) \( \iff \) edges of \( P_\mathcal{B} \)
2. Positroids

If $A \in \mathbb{R}^{m \times n}$ is a rank $m$ totally nonnegative matrix (i.e., all its maximal minors are nonnegative) then $M(A)$ is called a positroid.

$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \mapsto \{12, 13, 14, 23, 24\}$ is a positroid.

$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & <0 & 0 \\ 0 & 1 & 0 & >0 \end{pmatrix} \mapsto \{12, 14, 23, 34\}$ is not a positroid.

Positroids have a rich, beautiful geometric and combinatorial structure:

A. Postnikov: totally nonnegative Grassmannian

They have very interesting applications in algebra:

J. Scott: cluster algebras

and physics:

N. Arkani-Hamed et. al.: scattering amplitudes

Y. Kodama and L. Williams: KP-solitons
2. Positroids

If \( A \in \mathbb{R}^{m \times n} \) is a rank \( m \) totally nonnegative matrix (i.e., all its maximal minors are nonnegative) then \( M(A) \) is called a **positroid**.

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A = \begin{pmatrix}
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Indexing positroids

Positroids have several axiom systems of their own:

\[(2\ 3\ 6\ 8,\ 2\ 3\ 6\ 8,\ 3\ 5\ 6\ 8,\ 4\ 5\ 6\ 8,\ 5\ 6\ 8\ 9,\ 6\ 7\ 8\ 9,\ 6\ 7\ 8\ 9,\ 2\ 6\ 8\ 9,\ 2\ 6\ 9\ 10,\ 2\ 3\ 6\ 10)\]

Grassmann necklaces

\[
\begin{array}{|c|c|c|c|}
\hline
0 & + & 0 & + \\
+ & + & + & + \\
0 & 0 & 0 & \\
+ & + & \\
\hline
\end{array}
\]

Le-diagrams

Plabic graphs

Decorated permutations
Positroid polytopes

\[ B = \{12, 13, 14, 23, 24\} \quad \rightsquigarrow \]

A key result:

**Theorem.** *(Gelfand-Serganova ’87)*

\( B \) is a matroid \iff all edges of \( P_B \) have the form \( e_i - e_j \).

**Theorem.** *(Lam-Postnikov, A.-Reiner-Williams ’13)*

\( B \) is a *positroid* \iff additionally, all facets of \( P_B \) have the form \( \sum_{i \in S} x_i \leq a_S \) with \( S \) a *cyclic interval*.

Sketch of \( \implies \).

- Define \( Q \) by all ineqs \( \sum_{i \in S} x_i \leq a_S \) (\( S \) cyclic interval) sat. by \( P_B \).
- Matrix of \( Q \) is totally unimodular \( \Rightarrow \mathbb{Z}^n \) vertices \( \Rightarrow 0/1 \) vertices.
- Check \( P_B \) and \( Q \) have the same 0/1 vertices. “Just combinatorics”, using Grassmann necklaces.
Positroid polytopes

\[ \mathcal{B} = \{12, 13, 14, 23, 24\} \]

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3. Connectivity for matroids

A matroid $M$ is **disconnected** if it can be written as

$$M = M_1 \oplus M_2 := \{ B_1 \sqcup B_2 \mid B_1 \in M_1 \text{ and } B_2 \in M_2 \}.$$

Any matroid $M$ can be written uniquely as

$$M = M_1 \oplus \cdots \oplus M_k$$

with all the $M_i$ connected (called its **connected components**).

**Fact.** $M$ is connected $\iff P_M$ is (almost) full-dimensional.

**Conjecture.** Almost every matroid is connected.

**Theorem.** At least $1/2$ of matroids are connected.
Enumerating connected matroids

Let

\[ m(n) = \# \text{ matroids on } [n], \quad m_{\text{conn}}(n) = \# \text{ connected matroids on } [n]. \]

\[ M(x) = \sum_{n \geq 0} m(n) \frac{x^n}{n!}, \quad M_{\text{conn}}(x) = \sum_{n \geq 0} m_{\text{conn}}(n) \frac{x^n}{n!}. \]

Then if \( \Pi_n \) is the collection of set partitions of \([n]\),

\[ m(n) = \sum_{\{S_1, \ldots, S_k\} \in \Pi_n} m_{\text{conn}}(|S_1|) \cdots m_{\text{conn}}(|S_k|) \]

and the Exponential Formula gives

\[ M(x) = e^{M_{\text{conn}}(x)}. \]

This is nice, but gives no useful bounds for \( m_{\text{conn}}(n)/m(n) \).
Connectivity for positroids.

For positroids, connected components look quite different.

**Theorem.** (A. - Rincón - Williams, Ford ’13) The connected components of a positroid are the “connected components” of its decorated permutation. They form a non-crossing partition of $[n]$. 

![Diagram](image_url)
Enumerating connected positroids

\[ p(n) = \# \text{ positroids on } [n], \quad \text{\( p_{\text{conn}}(n) = \# \) connected positroids on } [n]. \]

\[ P(x) = \sum_{n \geq 0} p(n)x^n, \quad P_{\text{conn}}(x) = \sum_{n \geq 0} p_{\text{conn}}(n)x^n \]

Then if \( NC_n \) is the set of non-crossing partitions of \([n]\),

\[ p(n) = \sum_{\{S_1, \ldots, S_k\} \in NC_n} p_{\text{conn}}(|S_1|) \cdots p_{\text{conn}}(|S_k|) \]

We get

\[ xP(x) = \left( \frac{x}{P_{\text{conn}}(x)} \right)^{-1} \quad \text{(Beissinger '85, Speicher '94).} \]

This brings us to free probability.
## Detour: Free Probability

A non-commutative probability theory. *(Voiculescu ’92)*

(Operator algebras, random matrix theory, representation theory, ...)

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**Theorem:** *(A. - Rincón - Williams ’13)* For $Y \sim 1 + \text{Exp}(1)$,
- moments $m_n(Y) = \#$ positroids on $[n]$
- free cumulants $k_n(Y) = \#$ connected positroids on $[n]$
# Detour: Free Probability

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Enumerating positroids

\( p(n) = \# \) positroids on \([n]\), \( p_{\text{conn}}(n) = \# \) connected positroids on \([n]\).

No bound for \( p_{\text{conn}}(n)/p(n) \) from \( xP(x) = \left( \frac{x}{P_{\text{conn}}(x)} \right)^\langle -1 \rangle \) (∗).

**Theorem.** (A. - Rincón - Williams ’13, Postnikov ’06)

\[
p(n) = n! \cdot \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right) \sim n! \cdot e.
\]

**Proof.** Not hard, just count “decorated permutations”.

Now we can hope:

(∗) → Sage → OEIS + veladora → \( p_{\text{conn}}(n) = \) something good
Enumerating connected positroids

\[ p(n) = \# \text{ positroids on } [n], \quad p_{\text{conn}}(n) = \# \text{ connected positroids on } [n]. \]

\[(*) \rightarrow \text{Sage} \rightarrow \text{OEIS } + \text{ veladora} \rightarrow \text{A075834} \]

**Theorem.** (A. - Rincón - Williams ’13)

\[ p_{\text{conn}}(n) = \# \text{ of permutations on } [n] \text{ with no fixed intervals} \]

(Callan ’04, Salvatore-Tauraso ’09)

\[ \sim \# \text{ of permutations on } [n] \text{ with no fixed points} \sim \frac{n!}{e}. \]

**Proof.** Not so easy, requires more subtle estimates.
Enumerating positroids vs. connected positroids

Since $p(n) \sim n! \cdot e$ and $p_{conn}(n) \sim n!/e$, we get:

**Theorem.** (A.-Rincón-Williams ’13) A positroid is connected with probability

$$1/e^2 = 0.1353\ldots$$

Compare with

**Conjecture** (Mayhew-Newman-Welsh-Whittle ’11) Almost every matroid is connected. (**Theorem.** At least $1/2$ of them are.)

This is not evidence against MNWW’s conjecture.

It is evidence that positroids and matroids are very different.
4. Realizability for matroids

**BIG Question.** Which matroids are realizable by a matrix?

**Conjecture.** *(Brylawski-Kelly ’80)* Almost no matroid is realizable.

(“Exercise”. The proof didn’t fit in the margin.)

**Good news:**

**Theorem** *(Geelan-Gerards-Whittle ’16)* Rota’s Conjecture, ’71
Over $\mathbb{F}_q$, finitely many obstructions to being realizable. (Any $q$.)

**Bad news:**

**Theorem** *(Vámos ’78, Mayhew-Newman-Whittle ’12, ’14)*
“The missing axiom of matroid theory is lost forever”.
Over infinite fields, the realizability question is very difficult.
4. Realizability for oriented matroids

An **oriented matroid** is a matroid where bases have signs, and if $Sac$ and $Sbd$ have the same sign, then

- $Sab$ and $Scd$ have the same sign, or
- $Sad$ and $Sbc$ have the same sign.

Here $\text{sign}(\ldots x \ldots y \ldots) = - \text{sign}(\ldots y \ldots x \ldots)$.

**Motivating example.** A **real** matrix $A \in \mathbb{R}^{m \times n}$ gives an oriented matroid, where a basis $I$ is given the sign of the minor $\Delta_I(A)$.

\[
\Delta_{Sac}\Delta_{Sbd} = \Delta_{Sab}\Delta_{Scd} + \Delta_{Sad}\Delta_{Sbc}. \quad \text{(Plücker)}
\]

**BIG Question.** Which matroids are realizable by a matrix?

(Probably) **very difficult:**

**Theorem (Sturmfels ’87)** The following are **equivalent**:

- There’s an algorithm to determine if any oriented matroid is realizable over $\mathbb{Q}$.
- There’s an alg. to decide solvability of any system of Diophantine eqs over $\mathbb{Q}$.
- There’s an algorithm to decide if any lattice is the face lattice of a $\mathbb{Q}$-polytope.
A **positively oriented matroid** is an oriented matroid whose bases are all positive. *(da Silva - Las Vergnas ’87)*

Goal: generalize combinatorics of **cyclic polytope**. Da Silva did find several elegant combinatorial properties.

**Conjecture.** *(da Silva, 1987)*

Every positively oriented matroid is realizable.

- Are there any antecedent results for realizability of OMs?
- Remember, we believe almost no matroid is realizable. This conjecture seems rather optimistic.
Realizability of positively oriented matroids

A **positively oriented matroid** is an oriented matroid whose bases are all positive. (da Silva - Las Vergnas ’87)

**Theorem.** (A. -Rincón -Williams 13) (da Silva’s Conjecture)
Every positively oriented matroid is realizable over \( \mathbb{Q} \).

**Idea of the proof.** Use matroid polytopes!

\[
M \text{ is a positroid} \iff \text{facet dirs. of } P_M \text{ are cyclic intervals.}
\]

\[
M \text{ is positively oriented} \iff \text{facet dirs. of } P_M \text{ are cyclic intervals.}
\]

⇒: If \( P_M \) has a facet which is **not** a cyclic interval, play with the chirotope to contradict the combinatorial Plücker relations.
- First do it for full-dim polytopes (connected positroids)
- Then do it in general, via the non-crossing partition structure.
If $\chi$ and $\chi'$ are oriented matroids, we say $\chi$ specializes to $\chi'$ if

$$\chi(I) \neq \chi'(I) \implies \chi'(I) = 0.$$ 

The **MacPhersonian** (or combinatorial Grassmannian) $\text{MacP}(m, n)$ is the poset of rank $m$ OMs on $[n]$ ordered by (reverse) specialization.


- For $m \in \{1, 2, n - 2, n - 1\}$, $\text{MacP}(m, n)$ and $\text{Gr}_\mathbb{R}(m, n)$ are homotopy equivalent. (MacPherson ’93, Babson ’93).
- Some info on $\mathbb{Z}_2$-cohomology and homotopy groups. (Anderson-Davis ’02)
- “Otherwise, the topology of $\text{MacP}(m, n)$ is a mystery”.

**Open question:** Is $\text{MacP}(m, n)$ homotopy equivalent to $\text{Gr}_\mathbb{R}(m, n)$?
The **positive MacPhersonian** $\text{MacP}^+(m, n)$ is the poset of rank $m$ positively oriented matroids on $[n]$ ordered by (reverse) specialization.

The **positive Grassmannian** $\text{Gr}^+(m, n)$ is the subset of $\text{Gr}(m, n)$ where all Plücker coordinates are nonnegative.

The **positroid stratification** of $\text{Gr}^+(m, n)$ makes it a $CW$ complex. (Postnikov-Speyer-Williams ’09). Is it regular?

**Theorem.** (A.-Rincón-Williams 2013) $\text{MacP}^+(m, n)$ is **homeomorphic to a ball**, and thus homotopy equiv. to $\text{Gr}^+(m, n)$ [Rietsch-Williams ’10].
many thanks

The papers and slides are at:
http://math.sfsu.edu/federico

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