# Positroids, non-crossing partitions, and a conjecture of da Silva

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# Joint work with:

Felipe Rincón (Warwick / Los Andes) and Lauren Williams (Berkeley)

1. Positroids and non-crossing partitions.
   *Trans. Amer. Math Soc., to appear*
   `http://arxiv.org/abs/1308.2698`
2. Positively oriented matroids are representable.
   *J. European Math. Soc., to appear*
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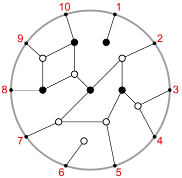


# 1. Matroids

A **matroid** $M$ on $[n] := \{1, \dots, n\}$ is a collection $\mathcal{B}$ of subsets of $[n]$ (called **bases**) satisfying the **basis exchange axiom**:

- If $A, B$ are bases and $a \in A - B$,
  there exists $b \in B - A$ such that $A - a \cup b$ is a basis.

All elements of $\mathcal{B}$ have the same size, called the **rank** of $M$.

**Motivating example.** If $\mathbb{K}$ is any field and $A \in \mathbb{K}^{m \times n}$ has rank $m$, the collection

$$\mathcal{B} := \{B \subset [n] \mid \text{the submatrix } A_B \text{ is invertible}\}$$

is a matroid $M(A)$ of rank $m$.

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{array} \quad \rightsquigarrow \quad M(A) = \{12, 13, 14, 23, 24\}$$

# Axiom systems for matroids

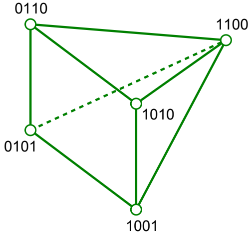
There are many equivalent ways of defining matroids:

- simplicial complex (independent sets)
- submodular function (rank function)
- closure operator (span)
- lattice (flats)
- polytope (bases) (My favorite.)

Given a matroid $\mathcal{B}$ of subsets of $[n]$, the **matroid polytope** is

$$P_{\mathcal{B}} := \text{convex}\left\{\sum_{i\in B} e_i \;\middle|\; B \in \mathcal{B}\right\}.$$

$\mathcal{B} = \{12, 13, 14, 23, 24\} \quad \rightsquigarrow$

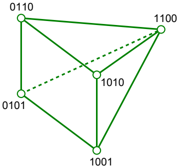


# Matroid polytopes

Given a matroid $\mathcal{B}$ (or any collection of $d$-subsets) on $[n]$, let

$$P_{\mathcal{B}} := \text{convex}\left\{\sum_{i\in B} e_i \;\middle|\; B \in \mathcal{B}\right\}.$$

$\mathcal{B} = \{12, 13, 14, 23, 24\}$ $\rightsquigarrow$



**Theorem.** (Gelfand-Goresky-MacPherson-Serganova '87)
$\mathcal{B}$ is a matroid $\iff$ all edges of $P_{\mathcal{B}}$ have the form $e_i - e_j$.

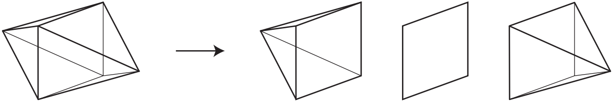
Remark:

basis exchanges in $\mathcal{B} \iff$ edges of $P_{\mathcal{B}}$

# Matroid polytopes

A few reasons I like matroid polytopes:

- simp. cx. is a matroid $\Longleftrightarrow$ greedy algorithm works (Edmonds)
- Torus orbits in Grassmannian $\Longrightarrow$ matroid polytope (GGMS)
- There are Coxeter matroids for any root system. (GS)



Subdivisions of a matroid polytope into matroid polytopes.

- Tropical linear spaces → matroid subdivisions (Speyer - Sturmfels)
- If $P_M$ has no nontrivial matroid subdivisions, $M$ has finitely many realizations. (Lafforgue)
- Matroid valuations → Derksen-Fink invariant is the universal valuation. (Billera-Jia-Reiner, Speyer, A.-Fink-Rincón, Derksen-Fink)

# 2. Positroids

If $A \in \mathbb{R}^{m \times n}$ is a rank $m$ **totally nonnegative** matrix (i.e., all its maximal minors are nonnegative) then $M(A)$ is called a **positroid.**

$$A = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \left(\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} -1 \\ 1 \end{array} & \left.\begin{array}{c} -2 \\ 2 \end{array}\right) \end{array} \rightsquigarrow \{12, 13, 14, 23, 24\} \text{ is a positroid.}$$

$$A = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \left(\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} < 0 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ > 0 \end{array}\right) \end{array} \rightsquigarrow \{12, 14, 23, 24\} \text{ is not a positroid.}$$

Positroids have a rich, beautiful geometric and combinatorial structure:

A. Postnikov: **totally nonnegative Grassmannian**

They have very interesting applications in algebra:

J. Scott: **cluster algebras**

and physics:

N. Arkani-Hamed et. al.: **scattering amplitudes**

Y. Kodama and L. Williams: **KP-solitons**

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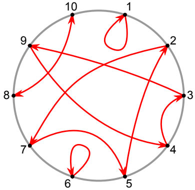
# Indexing positroids

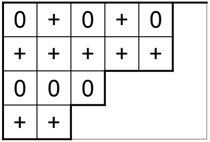
Positroids have several axiom systems of their own:

(2368, 2368, 3568, 4568, 5689, 6789, 6789, 2689, 26910, 23610)

Grassmann necklaces

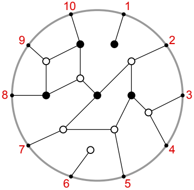


Decorated permutations



| | | | | |
|---|---|---|---|---|
| 0 | + | 0 | + | 0 |
| + | + | + | + | + |
| 0 | 0 | 0 | | |
| + | + | | | |

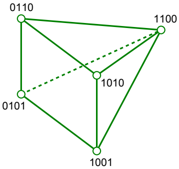
Le-diagrams



Plabic graphs

# Positroid polytopes

$\mathcal{B} = \{12, 13, 14, 23, 24\} \quad \rightsquigarrow$



A key result:

**Theorem.** (Gelfand-Serganova '87)
$\mathcal{B}$ is a matroid $\Longleftrightarrow$ all edges of $P_{\mathcal{B}}$ have the form $e_i - e_j$.

**Theorem.** (Lam-Postnikov, A.-Reiner-Williams '13)
$\mathcal{B}$ is a **positroid** $\Longleftrightarrow$ additionally, all facets of $P_{\mathcal{B}}$ have the form $\sum_{i \in S} x_i \leq a_S$ with $S$ a **cyclic interval.**

**Sketch of $\Longrightarrow$.**

- Define $Q$ by all ineqs $\sum_{i \in S} x_i \leq a_S$ ($S$ **cyclic interval**) sat. by $P_{\mathcal{B}}$.
- Matrix of $Q$ is totally unimodular $\Rightarrow \mathbb{Z}^n$ vertices $\Rightarrow$ 0/1 vertices
- Check $P_{\mathcal{B}}$ and $Q$ have the same $0/1$ vertices. “Just combinatorics"!

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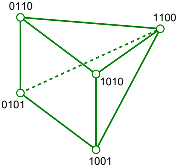

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- Matrix of $Q$ is totally unimodular $\Rightarrow \mathbb{Z}^n$ vertices $\Rightarrow$ 0/1 vertices
- Check $P_{\mathcal{B}}$ and $Q$ have the same $0/1$ vertices. “Just combinatorics"!

# 3. Connectivity for matroids

A matroid $M$ is **disconnected** if it can be written as

$$M = M_1 \oplus M_2 := \{B_1 \sqcup B_2 \mid B_1 \in M_1 \text{ and } B_2 \in M_2\}.$$

Any matroid $M$ can be written uniquely as

$$M = M_1 \oplus \cdots \oplus M_k$$

with all the $M_i$ connected (called its **connected components**).

**Fact.** $M$ is connected $\Longleftrightarrow$ $P_M$ is (almost) full-dimensional.

Mayhew - Newman - Welsh - Whittle ’11

**Conjecture.** Almost every matroid is connected.

**Theorem.** At least $1/2$ of matroids are connected.

# Enumerating connected matroids

Let

$m(n)$ = # matroids on $[n]$, $m_{conn}(n)$ = # **connected** matroids on $[n]$.

$$M(x) = \sum_{n \geq 0} m(n) \frac{x^n}{n!}, \qquad M_{conn}(x) = \sum_{n \geq 0} m_{conn}(n) \frac{x^n}{n!}.$$

Then if $\Pi_n$ is the collection of set partitions of $[n]$,

$$m(n) = \sum_{\{S_1, \ldots, S_k\} \in \Pi_n} m_{conn}(|S_1|) \cdots m_{conn}(|S_k|)$$

and the Exponential Formula gives

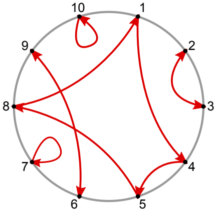
$$M(x) = e^{M_{conn}(x)}.$$

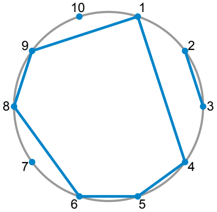
This is nice, but gives no useful bounds for $m_{conn}(n)/m(n)$

# Connectivity for positroids.

For positroids, connected components look quite different.

**Theorem.** (A. - Rincón - Williams, Ford '13) The connected components of a positroid are the "connected components" of its decorated permutation. They form a **non-crossing partition** of $[n]$.





# Enumerating connected positroids

$p(n) =$ # positroids on $[n]$, $p_{conn}(n) =$ # **connected** positroids on $[n]$.

$$P(x) = \sum_{n \geq 0} p(n) x^n, \qquad P_{conn}(x) = \sum_{n \geq 0} p_{conn}(n) x^n$$

Then if $NC_n$ is the set of **non-crossing** partitions of $[n]$,

$$p(n) = \sum_{\{S_1,\ldots,S_k\} \in NC_n} p_{conn}(|S_1|) \cdots p_{conn}(|S_k|)$$

We get

$$xP(x) = \left( \frac{x}{P_{conn}(x)} \right)^{\langle -1 \rangle}$$ (Beissinger '85, Speicher '94).

This brings us to free probability.

# Detour: Free Probability

A non-commutative probability theory. (Voiculescu '92)

(Operator algebras, random matrix theory, representation theory,...)

| (Normal) probability | Free probability |
|---|---|
| independence | freeness |
| moments: $E(e^{tX}) = \sum_{n\geq 0} m_n(X)\frac{t^n}{n!}$ | moments: $E(e^{tX}) = \sum_{n\geq 0} m_n(X)\frac{t^n}{n!}$ |
| cumulants:<br>$\displaystyle m_n = \sum_{\{S_1,\ldots,S_k\}\in\Pi_n} c_{\lvert S_1\rvert}\cdots c_{\lvert S_k\rvert}$ | free cumulants:<br>$\displaystyle m_n = \sum_{\{S_1,\ldots,S_k\}\in NC_n} k_{\lvert S_1\rvert}\cdots k_{\lvert S_k\rvert}$ |
| $X, Y$ independent $\Rightarrow$<br>$c_n(X+Y) = c_n(X) + c_n(Y)$ | $X, Y$ free $\Rightarrow$<br>$k_n(X+Y) = k_n(X) + k_n(Y)$ |

**Theorem**: (A. - Rincón - Williams '13) For $Y \sim 1 + \text{Exp}(1)$,

- moments $m_n(Y) = \#$ positroids on $[n]$
- free cumulants $k_n(Y) = \#$ connected positroids on $[n]$

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| $X, Y$ independent $\Rightarrow$ $c_n(X+Y) = c_n(X) + c_n(Y)$ | $X, Y$ free $\Rightarrow$ $k_n(X+Y) = k_n(X) + k_n(Y)$ |

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- moments $m_n(Y) = \#$ positroids on $[n]$
- free cumulants $k_n(Y) = \#$ connected positroids on $[n]$

# Enumerating positroids

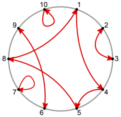
$p(n)$ = # positroids on $[n]$, $p_{conn}(n)$ = # **connected** positroids on $[n]$.

No bound for $p_{conn}(n)/p(n)$ from $xP(x) = \left(\frac{x}{P_{conn}(x)}\right)^{\langle -1 \rangle}$ $(*)$.

**Theorem.** (A. - Rincón - Williams '13, Postnikov '06)

$$p(n) = n! \cdot \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \sim n! \cdot e.$$

**Proof.** Not hard, just count "decorated permutations".



Now we can hope:

$(*) \rightarrow$ Sage $\rightarrow$ OEIS + veladora $\rightarrow$ $p_{conn}(n)$ = something good

# Enumerating connected positroids

$p(n)$ = # positroids on $[n]$, $p_{conn}(n)$ = # **connected** positroids on $[n]$.

$(*) \rightarrow$ Sage $\rightarrow$ OEIS + veladora $\rightarrow$ A075834

**Theorem.** (A. - Rincón - Williams '13)

$p_{conn}(n)$ = # of permutations on $[n]$ with no fixed intervals

(Callan '04, Salvatore-Tauraso '09)

$\sim$ # of permutations on $[n]$ with no fixed points $\sim \dfrac{n!}{e}$.

**Proof.** Not so easy, requires more subtle estimates.

# Enumerating positroids vs. connected positroids

Since $p(n) \sim n! \cdot e$ and $p_{conn}(n) \sim n!/e$, we get:

**Theorem.** (A.-Rincón-Williams '13) A positroid is connected with probability

$$1/e^2 = 0.1353\ldots$$

Compare with

**Conjecture** (Mayhew-Newman-Welsh-Whittle '11) Almost every matroid is connected. (**Theorem.** At least 1/2 of them are.)

This is not evidence against MNWW's conjeture.

It is evidence that **positroids and matroids are very different.**

# 4. Realizability for matroids

**BIG Question.** Which matroids are realizable by a matrix?

**Conjecture.** (Brylawski-Kelly ’80) Almost no matroid is realizable.

Good news:

**Theorem** (Geelan-Gerards-Whittle ’16) **Rota’s Conjecture, ’71**
Over $\mathbb{F}_q$, finitely many obstructions to being realizable. (Any $q$.)

Bad news:

**Theorem** (Vámos ’78, Mayhew-Newman-Whittle ’12, ’14)
“The missing axiom of matroid theory is lost forever”.
Over infinite fields, the realizability question is **very difficult.**

# 4. Realizability for oriented matroids

An **oriented matroid** is a matroid where bases have signs, and

If *Sac* and *Sbd* have the same sign, then

- *Sab* and *Scd* have the same sign, or
- *Sad* and *Sbc* have the same sign.

**Motivating example.** A **real** matrix $A \in \mathbb{R}^{m \times n}$ gives an oriented matroid, where a basis $I$ is given the sign of the minor $\Delta_I(A)$.

$$\Delta_{Sac}\Delta_{Sbd} = \Delta_{Sab}\Delta_{Scd} + \Delta_{Sad}\Delta_{Sbc}. \qquad \text{(Plücker)}$$

**BIG Question.** Which matroids are realizable by a matrix?

(Probably) very bad news:

**Theorem** (Sturmfels '87) The following are **equivalent**:

- There's an algorithm to determine if any oriented matroid is realizable over $\mathbb{Q}$.
- There's an alg. to decide solvability of any system of Diophantine eqs over $\mathbb{Q}$.
- There's an algorithm to decide if any lattice is the face lattice of a $\mathbb{Q}$-polytope.

# Positively oriented matroids

A **positively oriented matroid** is an oriented matroid whose bases are all positive. (da Silva - Las Vergnas ’87)

Goal: generalize combinatorics of **cyclic polytope**.
Da Silva did find several elegant combinatorial properties.

**Conjecture.** (da Silva, 1987)
Every positively oriented matroid is realizable.

- Are there any antecedent results for realizability of OMs?
- Remember, we believe almost no matroid is realizable.

This conjecture seems rather optimistic.

# Realizability of positively oriented matroids

A **positively oriented matroid** is an oriented matroid whose bases are all positive. (da Silva - Las Vergnas '87)

**Theorem.** (A. -Rincón -Williams 13) (da Silva's Conjecture)
Every positively oriented matroid is realizable over $\mathbb{Q}$.

**Idea of the proof.** Use matroid polytopes!

$M$ is a positroid $\iff$ facet dirs. of $P_M$ are cyclic intervals.
$M$ is positively oriented $\iff$ facet dirs. of $P_M$ are cyclic intervals.

$\Rightarrow$: If $P_M$ has a facet which is **not** a cyclic interval, play with the chirotope to contradict the combinatorial Plücker relations.

- First do it for full-dim polytopes (connected positroids)
- Then do it in general, via the non-crossing partition structure.

# Topology: The MacPhersonian

If $\chi$ and $\chi'$ are oriented matroids, we say $\chi$ **specializes** to $\chi'$ if

$$\chi(I) \neq \chi'(I) \quad \Longrightarrow \quad \chi'(I) = 0.$$

The **MacPhersonian** (or **combinatorial Grassmannian**) $\mathrm{MacP}(m, n)$ is the poset of rank $m$ OMs on $[n]$ ordered by (reverse) specialization.

The combinatorial Grassmannian serves as a classifying space for *matroid bundles* of *combinatorial differential manifolds*.

- For $m \in \{1, 2, n-2, n-1\}$, $\mathrm{MacP}(m, n)$ and $\mathrm{Gr}_{\mathbb{R}}(m, n)$ are homotopy equivalent. (MacPherson '93, Babson '93).
- Some info on $\mathbb{Z}_2$-cohomology and homotopy groups. (Anderson-Davis '02)
- "Otherwise, the topology of $\mathrm{MacP}(m, n)$ is a mystery".

**Open question:** Is $\mathrm{MacP}(m, n)$ homotopy equivalent to $\mathrm{Gr}_{\mathbb{R}}(m, n)$?

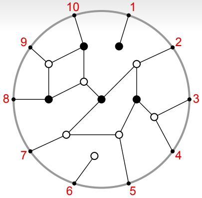
# Topology: the positive MacPhersonian

The **positive MacPhersonian** $\mathrm{MacP}^+(m,n)$ is the poset of rank $m$ **positively** oriented matroids on $[n]$ ordered by (reverse) specialization.

The **positive Grassmannian** $\mathrm{Gr}^+(m,n)$ is the subset of $\mathrm{Gr}(m,n)$ where all Plücker coordinates are nonnegative.

The **positroid stratification** of $\mathrm{Gr}^+(m,n)$ makes it a *CW* complex. (Postnikov-Speyer-Williams '09). Is it regular?

**Theorem.** (A.-Rincón-Williams 2013) $\mathrm{MacP}^+(m,n)$ is **homeomorphic to a ball**, and thus homotopy equiv. to $\mathrm{Gr}^+(m,n)$ [Rietsch-Williams '10].



# many thanks

The papers and slides are at:
`http://math.sfsu.edu/federico`

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