linearity
in the
tropics
linearity in the tropics

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(3,-2,0)
Summary. Tropical varieties are not simple objects; even tropical linear spaces have a very rich and interesting combinatorial structure which we only partially understand.
Tropical geometry: a general philosophy

*Tropicalisation* is a very useful general technique:

\[ \text{algebraic variety} \quad \mapsto \quad \text{tropical variety} \]

\[ V \quad \mapsto \quad \text{Trop}(V). \]

Idea: Obtain information about \( V \) from \( \text{Trop}(V) \).

- \( \text{Trop}(V) \) is simpler, but contains some information about \( V \).
- \( \text{Trop}(V) \) is a polyhedral complex, where we can do combinatorics.

Similar to: toric variety \( \mapsto \) polyhedral fan
Tropicalisation.

The field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series:

$$f(t) = \alpha_1 t^{r_1} + \alpha_2 t^{r_2} + \cdots, \quad \alpha_i \in \mathbb{C}, \{r_1 < r_2 < \cdots\} \subset \mathbb{Q}.$$

has valuation $\text{deg} : K \to \mathbb{R} \cup \{\infty\} =: \overline{\mathbb{R}}$ where $\text{deg}(f) = r_1$.

**Tropicalising points:** $\text{deg} : K^n \to \overline{\mathbb{R}}^n$

$A = (A_1, \ldots, A_n) \mapsto a = (\text{deg} A_1, \ldots, \text{deg} A_n)$

$$\begin{align*}
(t^2 + 3t^3 + t^4 + \cdots, t^{1.5} + 2t^2) &\mapsto (2, 1.5)
\end{align*}$$

**Tropicalising polynomials:** $\text{Trop} : K[X_1, \ldots, X_n] \to \{f : \mathbb{R}^n \to \mathbb{R}\}$

$A \mapsto \text{deg} A \quad X + Y \mapsto \min(x, y) \quad X \cdot Y \mapsto x + y$

$$\begin{align*}
(t^{1.5} + t^3)X^2 + 2YZ &\mapsto \min(1.5 + 2x, y + z)
\end{align*}$$
Fundamental Theorem of Tropical Geometry.

**Theorem/Defn.** (Einsiedler-Lind-Kapranov, Speyer-Sturmfels)

Let $I$ be an ideal in $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and let

\[
V = V(I) = \{ A \in (K^*)^n \mid F(A) = 0 \text{ for } F \in I \}
\]

The **tropical variety** $\text{Trop}(V)$ is

\[
\text{Trop}(V) := \{ a \in \mathbb{R}^n \mid (\text{Trop } F)(a) \text{ is achieved twice for } F \in I \}
\]

\[= \text{cl} (\text{deg } A \mid A \in V)\]

Informally,

\[\text{Trop}(V) := \text{Solutions of tropical equations}
\]

\[= \text{cl} (\text{Tropicalisation of the solutions}).\]
Trop(\(V\)) := Solutions of tropical equations 
= cl (Tropicalisation of the solutions).

Ex. \(V = \{(X, Y, Z) \in (K^*)^3 | (t^{-3} + 2)X + (t + 5t^{1.5})Y + Z = 0\}\)

1. Tropicalise equations:
\[
\text{Trop} V = \{(x, y, z) \in \mathbb{R}^3 | \min(x - 3, y + 1, z) \text{ att. twice}\}.
\]

2. Tropicalise solutions:
\[
Trop(\(V\)) = cl \{(\deg X, \deg Y, \deg Z) | (X, Y, Z) \in V\}
\]

(2 \subseteq 1): Exercise.
(1 \subseteq 2): Harder.
Tropicalisation:

algebraic variety $\mapsto$ tropical variety

$V \mapsto \text{Trop}(V)$.

To apply this technique, we ask two questions:

1. **What does** $\text{Trop}(V)$ **know about** $V$?
   Find the right questions in alg. geom. to “tropicalise”.
   - Gromov-Witten invariants $\mathcal{N}_{g,d}^C$ of $\mathbb{CP}^2$ (Mikhalkin)
   - Double Hurwitz numbers. (Cavalieri-Johnson-Markwig)

2. **What do we know about** $\text{Trop}(V)$? Not very much!
   - $(V$ irred.$)$ Pure, connected in codimension 1. (Bieri-Groves).
   - $(V$ Schön$)$ Links have only top homology. (Hacking)

Tropical varieties are ‘simpler’, not ‘simple’. Study them!
Examples of tropical varieties

**Example 1.** Tropical hyperplanes in $\mathbb{T}\mathbb{P}^{n-1}$.

$$A_1 X_1 + \ldots + A_n X_n = 0 \mapsto \min(x_1 + a_1, \ldots, x_n + a_n) \text{ ach. twice}$$

$\mathbb{T}\mathbb{P}^2$: $\min(x - 3, y + 2, z)$ twice $\mathbb{T}\mathbb{P}^3$: $\min(x_1, x_2, x_3, x_4)$ twice

**Tropical projective plane** $\mathbb{T}\mathbb{P}^2$: $(a, b, c) \sim (a - c, b - c, 0)$

**Polar fan of the simplex centered at** $-(a_1, \ldots, a_n)$. 
Example 2. Tropical conics in $\mathbb{T}\mathbb{P}^2$:

$$AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = 0 \mapsto \min(a + 2x, b + 2y, \ldots, e + x + z, f + y + z) \text{ achieved twice.}$$

Two tropical conics:

In principle, could have up to $\binom{6}{2} = 15$ edges.
In fact, they all have 4 vertices and 9 edges (3 bounded).
Example 3. A tropical line in $\mathbb{T}^3$.

$$L = \text{rowspace} \begin{bmatrix} 1 & t & t^2 & t^3 \\ t^3 & t^2 & t & 1 \end{bmatrix}$$

Trop $L$: The following are attained twice:

$$\min(x_1 + 2, x_2 + 1, x_3 + 2), \ \min(x_1 + 1, x_2, x_4 + 2),$$
$$\min(x_1 + 2, x_3, x_4 + 1), \ \min(x_2 + 2, x_3 + 1, x_4 + 2)$$
The goal of this talk:
To summarize what we know about tropical linear spaces.
Tropical linear spaces, part 1: constant coefficients.

**Goal.** If $V$ is a linear subspace, describe $\text{Trop} V$.

(Part 1: Assume that all coefficients are in $\mathbb{C}$.)

$$w \in \text{Trop} V \iff \text{for each circuit (equation)} a_1 X_{i_1} + \cdots + a_k X_{i_k} = 0 \text{ of } V, \min(w_{i_1}, \ldots, w_{i_k}) \text{ is achieved twice.}$$

**Example.** $L =$ rowspace $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$.

$X_1 - X_2 + X_3 = 0, \ X_4 = 2X_3$ \hspace{1cm} Circuits: 123, 34, 124.

$\text{Trop} L: \min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4) \text{ att. twice.}$
L = rowspace \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}. \text{Circuits: 123, 34, 124.}

Trop L: \min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4) \text{ att. twice.}

w_1 = w_2 < w_5 = w_3 = w_4 \text{ ok} \quad w_1 = w_3 = w_5 < w_s = w_4 \text{ no}

Note.

- w_5 is irrelevant.
- Order of w_1, w_2, w_3, w_4 is either
  - w_1 > w_2 = w_3 = w_4,
  - w_2 > w_1 = w_3 = w_4, or
  - w_3 = w_4 > w_1 = w_2.
So Trop $V$ only depends on the matroid (set of circuits) of $V$.

For any matroid $M$ (set of circuits) we define

$$\text{Trop } M := \{ w \in \mathbb{R}^E \mid \min_{c \in C} w_c \text{ is achieved twice for all circuits } C. \}$$

(sometimes called the Bergman fan of $M$.)

This calls for a crash course in matroid theory.
Matroid theory, v1: circuits.

Matroid theory: An abstract theory of **independence**.

(Instances: linear, algebraic, graph independence.)

The key properties of (minimal) dependence:

A matroid $M$ on a finite ground set $E$ is a collection $C$ of circuits (subsets of $E$) such that:

1. $\emptyset$ is not a circuit.
2. No circuit properly contains another.
3. If $C_1$ and $C_2$ are circuits and $x \in C_1 \cap C_2$, then $C_1 \cup C_2 - x$ contains a circuit.

Ex: The matroid of a vector space / config. $L = \text{row}(E)$

(circuits) $\leftrightarrow$ (minl eqns. of $L$) $\leftrightarrow$ (minl linear deps on cols of $E$)
Why matroids?

- They are general, applicable, and well-developed. Example: Every matroid has a well-defined rank function.
  - Dimension of vector spaces
  - Transcendence degree of a field extension
  - The spanning trees of a graph have the same size.
- Many different (but equivalent) points of view.
  - Matroid polytopes. We need it.
  - Lattice of flats. We need it.
  - Optimization (greedy algorithms). We need it.
- (Our main reason today.) Loosely speaking:
  
  algebraic geometry $\mapsto$ tropical geometry
  specialises to
  linear algebra $\mapsto$ matroid theory.
Matroid theory, v2: lattices of flats.

$E$: set of vectors
- **flat**: (the vectors of $E$ in) $\text{span}(A)$ for $A \subseteq E$.
- **lattice of flats** $L_M$: the poset of flats ordered by containment.
- **order complex** $\Delta(L_M)$: the simplicial complex of chains of $L_M$.

$(\text{vertices} = \text{flats}, \text{faces} = \text{flags}; L_M = L_M - \{\hat{0}, \hat{1}\})$.

$L = \text{rowspace} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}, C = \{123, 124, 34\}$.
- **Flats**: $\mathcal{F} = \{\emptyset, 1, 2, 34, 5, 1234, 15, 25, 345, 12345\}$.

**Theorem.** (Björner, 1980) $\Delta(L_M)$ is a pure, shellable simplicial complex. It has the homotopy type of a wedge of $|\mu(L_M)|$ $(r - 2)$-dimensional spheres.
The main theorem.
Let $\text{Trop} M = \text{Trop} M \cap \text{(unit sphere)}$.

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Theorem. (.f. - Klivans)
$\text{Trop}'(M) \quad \text{“} = \text{”} \quad \Delta(\bar{L}_M)$.
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More precisely, $\Delta(\bar{L}_M)$ is a subdivision of $\text{Trop}'(M)$.

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Corollary. (.f. - Klivans) In constant coefficients, tropical linear spaces are cones over wedges of $(r - 2)$-spheres. The number of spheres is computable combinatorially.
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Key observation:

$w_{a_1} = \cdots = w_{a_k} \geq w_{b_1} = \cdots = w_{b_l} \geq \cdots$ is in $\text{Trop}(M)$ if and only if $A, A \cup B, A \cup B \cup C, \ldots$ are flats of $M$. 
Some interesting special cases.

1. $A_{n-1} = \{e_i - e_j | 1 \leq i < j \leq n\}$
   - $\text{Trop } A_{n-1}$ is the space of phylogenetic trees $T_n$. (f. - Klivans)
     ($T_n$ also appears naturally in homotopy theory and in $\overline{M}_{0,n}$.)
   - $T_n$ has homotopy type $\bigvee (n-1)! S^{n-3}$. (Vogtmann)
   - (Chepoi-F. tree reconstruction alg.) = (tropical projection) (f.)

2. $\Phi = \text{root system of a finite Coxeter system } (W, S)$
   - $\text{Trop'} \Phi = \text{(nested set complex of } \Phi)$, which encodes De Concini and Procesi’s “wonderful compactification" of $\mathbb{C}^n - A_{\Phi}$.
   - $\text{Trop } \Phi$ can be described combinatorially as a space of “phylogenetic trees of type } W$, which come from tubings of the Dynkin diagram. (f. - Reiner - Williams)
Matroid theory, v3: matroid polytopes

A basis of $M$ is a maxl. indept. set. The matroid polytope is

$$P_M = \text{conv}(e_{b_1} + \cdots + e_{b_r} \mid \{b_1, \ldots, b_r\} \text{ is a basis}).$$

$L =$ rowspace \[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 0
\end{bmatrix},
\]
$C = \{123, 124, 34\}.$

- Bases: $B = \{125, 135, 145, 235, 245\}$
- $P_M = \text{conv}(11001, 10101, 10011, 01101, 01011).$

Interpretations:
- linear programming and greedy algorithms
- moment polytope of the closure of a torus orbit in $\text{Gr}(d, n)$

**Theorem. (GGMS)** A 0-1 polytope is a matroid polytope if and only if all its edges are of the form $e_i - e_j.$
A matroid is \textbf{loopless} if every element is in some basis.

\begin{center}
\textbf{Proposition. (Sturmfels)}
\end{center}
\begin{align*}
\text{Trop } M \text{ is the fan dual to the loopless faces of } P_M: \\
\text{Trop } M = \{ w \in \mathbb{R}^E \mid \text{The } w\text{-max face of } P_M \text{ is loopless.} \}
\end{align*}
(from constant to arbitrary coeffs) Let $L$ be a linear space with arbitrary coeffs and $u \in \text{Trop } L$. The local cone at $u$ is
\[
\text{cone}_u \text{Trop } L = \text{Trop } L_u
\]
for a linear space $L_u$ with constant coefficients.

\[L = \text{rowspace} \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}, \text{Trop } L = \]

\[
\begin{align*}
1 & : (1,2,1,0) \\
2 & : (1,3,1,0) \\
3 & : (-1,0,1,0) \\
4 & : (-3,-2,-1,0) \\
1 & : (3,2,1,0) \\
2 & : (1,2,1,0) \\
3 & : (-1,0,2,0) \\
4 & : (-3,-2,-1,0)
\end{align*}
\]
Each local cone is dual to (loopless part of) a matroid polytope. The matroid polytopes give a subdivision of the hypersimplex

$$\Delta(n, d) = \text{conv}(e_{i_1} + \cdots + e_{i_d} \mid \{i_1, \ldots, i_d\} \subseteq [n])$$

(which is the matroid polytope of a generic vector space.)
Theorem. (Speyer) A $d$-dimensional tropical linear space in $n$-space is dual to a matroid subdivision: a subdivision of $\Delta(n, d)$ into matroid polytopes.

Tropical linear spaces:

- constant coeffs. $\rightarrow$ matroids
- arbitrary coeffs. $\leftrightarrow$ matroid subdivisions
### Tropical linear spaces:

<table>
<thead>
<tr>
<th>constant</th>
<th>→</th>
<th>matroids</th>
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<tr>
<td>arbitrary</td>
<td>→</td>
<td>matroid subdivs.</td>
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Other occurrences of matroid subdivisions:

- Kapranov’s generalized Lie complexes.
  - Chow quot. $Gr(d, n)//\mathbb{T}$ - limits of torus orbit closures in $Gr(d, n)$
- Hacking, Keel, and Tevelev’s very stable pairs.
  - generalized hyperplane arrangements.
- Lafforgue’s compactif of fine Schubert cells in Grassmannian.

**Lafforgue**: $P_M$ indecomposable $\rightarrow M$ has finitely many realizations.

Mnëv: Realization spaces of $M$s can have arbitrarily bad singularities.
Matroid subdivisions

How can a matroid polytope can be divided into smaller matroid polytopes?

(Constructor? Verify? Prove impossibility?)

One approach:
Find “measures” of a matroid $M$ that behave like valuations on $P_M$.

A function $f : \text{Matroids} \rightarrow G$ is a matroid valuation if for any subdivision of $P_M$ into $P_{M_1}, \ldots, P_{M_m}$ we have

$$f(M) = \sum_{i=1}^{m} (-1)^{\dim P_M - \dim P_{M_i}} f(M_i)$$  \hspace{1cm} (1)
Some matroid valuations:

- \( \text{Vol}(P_M) \) (f.-Benedetti-Doker) (Lam-Postnikov, Stanley)
- \( |P_M \cap \mathbb{Z}^n| = \text{number of bases of } M \)
- Ehrhart polynomial \( E_{P_M}(t) = |tP_M \cap \mathbb{Z}^n| \). (f. - Doker)
- Tutte polynomial \( T_M(x, y) \) (Speyer)
  (the mother of all (del.-contr.) matroid invariants)
- Quasisym function \( Q_M(x_1, \ldots, x_n) \) (Billera-Jia-Reiner)
- Invariants coming from \( K \)-theory of \( \text{Gr}(d, n) \) (Speyer)

**Theorem. (Speyer)** A \( d \)-dimensional tropical linear space in \( n \)-space has \( \leq \binom{n-i-1}{d-i} \binom{2n-d-1}{i-1} \) \( i \)-dimensional faces.

He uses a mysterious invariant \( g_M(t) \) from \( K \)-theory. What does it mean combinatorially? If we knew, we could prove:

**Conjecture.** This bound holds for any matroid subdivision.
A very general matroid valuation.

Define $V : \text{Matroids} \rightarrow G$ by:

$$V(M) = \sum_{\pi \in S_n} (\pi, r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n))$$

where $G$ is the free abelian group generated by such symbols.

For $L = \text{rowspace} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$,

$$V(M) = (1234, 1, 2, 2, 2) + \cdots + (3421, 1, 1, 2, 2) + \cdots.$$
\[ V(M) = \sum_{\pi \in S_n} (\pi, (r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n)) \]

**Theorem.** (.f. - Fink - Rincón, Derksen) \( V \) is a matroid valuation.

**Example.** For the subdivision of \( \Delta(6, 3) \)

\[ V(M) = V(M_1) + V(M_2) + V(M_3) \]
\[ -V(M_{12}) - V(M_{13}) - V(M_{23}) + V(M_{123}) \]

The summands with \( \pi = 132456 \) give

(writing \( 132456, 1, 2, 3, 3, 3, 3 \) → \( 1, 2, 3, 3 \))

\( (1, 2, 3, 3) = (1, 2, 3, 3) + (1, 2, 2, 3) + (1, 2, 2, 2) \)
\( -(1, 2, 2, 3) - (1, 2, 2, 3) - (1, 2, 2, 2) + (1, 2, 2, 2) \)

**Idea of proof.** Interpret each term like

\( (1, 2, 2, 2) - (1, 2, 2, 2) - (1, 2, 2, 2) + (1, 2, 2, 2) = 0 \)

as a reduced Euler characteristic of a contractible space.
All matroid valuations.

\[ V(M) = \sum_{\pi \in S_n} (\pi, (r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n)) \]

**Theorem.** (Derksen - Fink)

\( V \) is a **universal** matroid valuation.

**Theorem.** (Derksen - Fink)

Let \( v(n, r) \) be the rank of the abelian group of valuations on matroids of \( n \) elements and rank \( r \). Then

\[
\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} v(n, r) \frac{x^{n-r} y^r}{n!} = \frac{x - y}{xe^{-x} - ye^{-y}}.
\]

So in principle we know how far we can push this approach. In practice there is more to do.
summary

• We do not understand tropical varieties very well yet.
• We understand tropical linear spaces to some extent.
  • Locally, they “are” matroids.
  • Globally, they “are” matroid subdivisions.
  • We know many things about matroids, and a few things about matroid subdivisions.

some future directions

• Understand matroid subdivisions better. Systematic construction? Mixed subdivisions? Secondary polytope?
• Generalize this story to subdivisions of Coxeter matroids and tropical homogeneous spaces (under certain hypotheses, to be determined). (.f. - Rincón - Velasco)
• What about general tropical varieties?
many thanks !!!

Papers available at:

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