Double Hurwitz numbers as splines

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The plan.

1. Motivation: Kostant partition functions.
2. Hurwitz numbers.
3. Tropical Hurwitz numbers and polytope measures.
4. Remarkable spaces.
5. Remarkable spaces and double Hurwitz numbers.
6. Questions.

The point.

Double Hurwitz numbers are piecewise polynomial functions. De Concini-Procesi-Vergne’s machinery of remarkable spaces allows us to understand this piecewise polynomial structure.
1. Motivation: Partition functions. (discr. truncated powers)

Let $\Lambda_n = \{ x \in \mathbb{Z}^n | \sum x_i = 0 \}$ and $A_{n-1}^+ = \{ e_i - e_j | 1 \leq i < j \leq n \}$.

The partition function $\Pi_{A_{n-1}^+} : \Lambda_n \to \mathbb{Z}$ is

$$\Pi_{A_{n-1}^+}(b) = \# \text{ of sols. to } b = \sum_{i<j} a_{ij}(e_i - e_j) \text{ with } a_{ij} \in \mathbb{N}.$$  

$\Pi_{A_{n-1}^+}(b)$ is the number of lattice points in the flow polytope:

$$P_{K_n}(b) = \{ a \in \mathbb{R}^n \choose 2 | b = \sum_{i<j} a_{ij}(e_i - e_j), \ a_{ij} \geq 0 \},$$

which is the polytope of $b$-excessive flows on the graph $K_n$.  
(Think: $b = \text{ vector of "leaks" on vertices}.$)

(Kostant, 1958): This is crucial in representation theory of $\mathfrak{sl}_n$.  

1. **Motivation: Partition functions.** (discr. truncated powers)

**Theorem.** The partition function $\Pi_{A_n^+}(b)$ is piecewise polynomial.

- Domains of polynomiality: (ess) the **chamber complex** $\text{Ch}(A)$, determined by all hyperplanes spanned by subsets of $A$.
- Many “structural” results. (Baldoni, Beck, Brion, Dahmen, Jia, Micchelli, Paradan, Stanley, Sturmfels, Szenes, Vergne, ...)
2. Hurwitz numbers.

- $g, d$: non-negative integers, and
- $\mu = (\mu_1, \ldots, \mu_a)$ and $\nu = (\nu_1, \ldots, \nu_b)$: partitions of $d$.

The double Hurwitz number $H^g(\mu, \nu)$ is the "number" of distinct connected ramified covers of $\mathbb{CP}^1$ which have
- degree $d$ and genus $g$,
- ramification profiles $\mu$ at 0 and $\nu$ at 1, and
- simple ramification elsewhere.

By the Riemann-Hurwitz formula, there are exactly $r = 2g - 2 + a + b$ points with simple ramification.

**Theorem. (Hurwitz)** The double Hurwitz number $H^g(\mu, \nu)$ is also the "number" of ways of writing $1 = \pi(\tau_1 \tau_2 \cdots \tau_r)\pi' \in S_d$ where
- $\pi$ and $\pi'$ have cycle types $\mu$ and $\nu$, respectively,
- $\tau_1, \ldots, \tau_r$ are transpositions, and
- the subgroup $\langle \pi_1, \tau_1, \ldots, \tau_r, \pi_2 \rangle$ acts transitively on $[d]$. 
2. Hurwitz numbers.

Recall \( a = \ell(\mu), b = \ell(\nu) \).

**Theorem.** (Goulden-Jackson-Vakil, 2005) For fixed \( g, a, b \), \( \mathcal{H}^g(\mu, \nu) \) is a piecewise polynomial function in \( \mu \) and \( \nu \).

In fact, we can just fix \( a + b = n \).

Let \( \Lambda_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum x_i = 0\} \).

Define the Hurwitz function \( \mathcal{H}_r : \Lambda_n \to \mathbb{Z} \) by

\[
\mathcal{H}_r(2, -3, 1, 5, 0, 2, -3, -4) = \mathcal{H}^g(433, 5221)
\]

(where \( n = a + b \) and \( r = 2g - 2 + n \))

**Theorem.** (Cavalieri-Johnson-Markwig, 2009) For fixed \( r, n \), the Hurwitz function \( \mathcal{H}_r : \Lambda_n \to \mathbb{Z} \) is piecewise polynomial.
2. Hurwitz numbers.

**Theorem.** (Cavalieri-Johnson-Markwig, 2009) For fixed $r, n$, the function $H_r : \Lambda_n \to \mathbb{Z}$ is piecewise polynomial.

- Domains of polynomiality are the chamber complex $\text{Ch}(A^+_{n-1})$, determined by all hyperplanes spanned by subsets of $A^+_{n-1}$, and
- “Wall-crossing” formula. (Cavalieri-Johnson-Markwig, 2009)
3. Tropical Hurwitz numbers and polytope measures.

Tropical geometry can sometimes solve enumerative geometry problems by *tropicalizing* them into enum. discrete-geometric problems. Cavalieri et. al. do this for double Hurwitz numbers:

An \((n, r)\)-Hurwitz graph \(G\) is a connected directed graph with:
- \(n + r\) vertices: \(\{1, \ldots, r\}\) trivalent and \(\{1’, \ldots, n’\}\) leaves.
- internal edges oriented from smaller to larger vertex.

**Theorem. (CJM, 2009)** The double Hurwitz number \(\mathcal{H}_r(x)\) is

\[
\mathcal{H}_r(x_1, \ldots, x_n) = \sum_{(n,r)-\text{Hurwitz } G} \frac{(-1)^{|E(G)|}}{|\text{Aut } G|} Q_G(x, 0),
\]

where

\[
Q_G(x, 0) = \sum_{(x, 0)\text{-excessive } \mathbb{N}\text{-flows } f} \prod_e f_e = \sum_{f \in P_G(x, 0) \cap \mathbb{Z}^E} \prod_e f_e
\]

is a “discrete measure” of the flow polytope \(P_G(x, 0)\).
Theorem. (CJM, 2009) The double Hurwitz number $H_r(x)$ is

$$H_r(x_1, \ldots, x_n) = \sum_{(n,r) \text{-Hurwitz } G} \frac{(-1)^{|E(G)|}}{|Aut G|} Q_G(x, 0),$$

where

$$Q_G(x, 0) = \sum_{(x,0) \text{-excessive } \mathbb{N}\text{-flows } f} \prod e f_e = \sum_{f \in P_G(x, 0) \cap \mathbb{Z}^E} \prod e f_e$$

is a "discrete measure" of the flow polytope $P_G(x, 0)$. 

3. Tropical Hurwitz numbers and polytope measures.
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After a lot of beautiful and quite intricate combinatorics (hyperplane arrangements, Gauss-Manin connection, cographical matroids, topological combinatorics), they get:

**Theorem.** (CJM, 2009)

- **(P.P.)** $\mathcal{H}_r(x_1, \ldots, x_n)$ is piecewise polynomial on $\text{Ch}(A_{n-1})$.
- **(W.C.)** Let $\tau_1$, $\tau_2$ be adjacent topes separated by the wall $\sum_{i \in E} x_i = 0$, $E \subseteq [n']$. For $x \in \tau_1$, $(\mathcal{H}^{\tau_1}_r - \mathcal{H}^{\tau_2}_r)(x)$ equals

$$
\sum_{s+t+u=r} \frac{(-1)^t}{\prod_i y_i} \mathcal{H}_s(x_E, y_T) \mathcal{H}_i^\bullet(-y_T, y_B) \mathcal{H}_u(-y_B, x_{E^c})
$$

where $\mathcal{H}^\bullet$ is the disconnected Hurwitz number.

Also: There is an “easy" tope $\tau$ where $\mathcal{H}^\tau_r$ is easy to compute.
4. Remarkable spaces.

Let $\Gamma$ be a lattice in a real vector space $V$. Let $A = \{a_1, \ldots, a_m\} \subseteq \Gamma$ be a vector configuration. The flats of $A$ are the subspaces of $V$ spanned by subsets of $A$.

Let $C[\Gamma]$ be the space of functions $f : \Gamma \to \mathbb{Z}$.

- Each $a \in \Gamma$ gives a difference operator
  \[ \nabla_a : C[\Gamma] \to C[\Gamma], \quad \nabla_a f(b) := f(b) - f(b - a) \]

- Each finite set $S \subseteq \Gamma$ gives a difference operator
  \[ \nabla_S : C[\Gamma] \to C[\Gamma], \quad \nabla_S = \prod_{a \in S} \nabla_a. \]

Think: “discrete derivatives”.

**Definition.** (De Concini-Procesi-Vergne, 2008)

The remarkable space of $A$ is
\[ \mathcal{F}(A) := \{ f \in C[\Gamma] \mid \nabla_{A-F} f \text{ is supported on } F \text{ for all flats } F \text{ of } A \}. \]

Origin: Atiyah-Singer’s index theorem for elliptic operators. ($A = \text{root system} \rightarrow \text{this is an equivariant K-theory group}$)
4. Remarkable spaces.

The remarkable space $\mathcal{F}(A)$ of $A$ is relevant to us because:

**Theorem.** (DPV, 2008) Any $f \in \mathcal{F}(A)$ satisfies

- (Piecewise quasipolynomiality) $f \in \mathcal{F}(A)$ is piecewise quasipolynomial on (ess.) the chamber complex $\text{Ch}(A)$.
- (Wall crossing formula) If $\tau_1$ and $\tau_2$ are adjacent topes of $\text{Ch}(A)$ separated by a wall $H$, then

$$f^{\tau_1} - f^{\tau_2} = (\mathcal{P}_{A-H}^{R_H} - \mathcal{P}_{A-H}^{-R_H}) * (\nabla_{A-H} f)^{\tau_1 \cap \tau_2}.$$ 

This is decoupled as a convolution of:
- the intrinsic wall-crossing functions $\mathcal{P}_{A-H}^{\pm R_H}$, which depend only on the configuration $A$ and the wall $H$.
- “discrete derivatives” of $f$.

**Ex.** The partition function $\Pi_A(b) = |P_A(b) \cap \mathbb{Z}^n|$ is in $\mathcal{F}(A)$. This recovers Paradan’s wall-crossing formula for $\Pi_A$.

**Ex.** The Dahmen-Micchelli space $DM(A) \subset \mathcal{F}(A)$. 
5. Remarkable spaces and double Hurwitz numbers.

Our main result:

**Proposition.** (A.) Wall-crossing formulas for double Hurwitz numbers follows from wall-crossing formulas in remarkable spaces.

**Sketch.** Let $G = \{ e_i - e_j, \text{ij edge of } G \}$, and $\overline{G} = G \cup -G$.

- Recall $\mathcal{H}_r(x) = \sum_G \frac{(-1)^{|E(G)|}}{|\text{Aut } G|} Q_G(x, 0)$.
- Check that $Q_G \in \mathcal{F}(\overline{G})$.
- Wall-crossing in $\mathcal{F}(\overline{G})$ now says:

$$Q^{\tau_1}_G - Q^{\tau_2}_G = (\mathcal{P}_G^{R_H} - \mathcal{P}_G^{-R_H}) * (\nabla_{G-H} Q_G)^{\tau_1 \cap \tau_2}.$$

- Intrinsic wall-crossing functions $\mathcal{P}$ for $\mathcal{F}(\overline{G})$ are just other $Q$s!
- Also, the "extrinsic" $\nabla_{G-H} Q_G = Q_H = Q_{H_a} * Q_{H_b}$!
- Then

$$Q^{\tau_1}_G - Q^{\tau_2}_G = Q^{\tau_a}_{H_a} * (Q_{(G-H)^+} - Q_{(G-H)^-}) * Q^{\tau_b}_{H_b}.$$
5. Remarkable spaces and double Hurwitz numbers.

**Proposition.** (A.) Wall-crossing formula for double Hurwitz numbers follows from wall-crossing formulas in remarkable spaces.

**Sketch.** (continued) \( \mathcal{H}_r(x) = \sum G \frac{(-1)^{|E(G)|}}{|\text{Aut } G|} Q_G(x, 0) \).

- Wall-crossing for \( Q_G \):
  \[ Q_{T_1} - Q_{T_2} = Q_{T_a} \ast (Q_{(G-H)^+} - Q_{(G-H)^-}) \ast Q_{T_b}. \]
- Now “just put them all together” to obtain something like
  \[ \mathcal{H}_{T_1}^{T_1} - \mathcal{H}_{T_2}^{T_2} = \sum_{s+t+u=r} (-1)^t \binom{r}{s, t, u} \mathcal{H}_s \ast \mathcal{H}_t^\bullet \ast \mathcal{H}_u. \]

**Some details to check.** Cutting and regluing the graphs, trivalence, plugging into the right topes, lower-dimensional wall-crossing, different chamber complexes \( \text{Ch}(\overline{G}) \), automorphisms.
5. Remarkable spaces and double Hurwitz numbers.

Other consequences:

**Conjecture.** (Goulden-Jackson-Vakil) The “Hurwitz polynomials” $\mathcal{H}_r(x_1, \ldots, x_n)$:

- are always either even or odd, and
- only have terms between degrees $r - 1$ and $2r + 1 - n$.

1. Since $\mathcal{H}_r(x) = \sum_G \frac{(-1)^{|E(G)|}}{|\text{Aut } G|} Q_G(x, 0)$ and $Q_G$ is the discrete measure of a flow polytope, this follows by weighted Ehrhart reciprocity. (A., 2009; CJM, 2009).

2. Is this related to spline smoothness? (P. Johnson recently announced a very different proof of 2.)
Again, the point.

Double Hurwitz numbers are piecewise polynomial functions. De Concini-Procesi-Vergne’s machinery of remarkable spaces allows us to understand this piecewise polynomial structure.

6. Questions.

• What do spline techniques tell us about Hurwitz numbers? Can they prove the second part of GJV’s conjecture?

• The functions $Q_G$ appear equally naturally:
  - as “tropical contributions” to double Hurwitz numbers,
  - as intrinsic wall-crossing functions in remarkable spaces, and
  - in the work of Atiyah and Singer.

  Why, really?

• What about higher Hurwitz numbers?
  (It would be enough to deal with triple Hurwitz numbers. This involves more subtle representation theory.)
many thanks!