Power Ideals of Hyperplane Arrangements

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Main objects: three families of algebraic objects associated to a hyperplane arrangement $\mathcal{A}$. (Today we focus on one of them.)

Main question: To what extent are their structure and their "size" (dimension, Hilbert series,...) determined by the combinatorics of $\mathcal{A}$?

$\mathcal{A} = \{H_1, \ldots, H_n\}$ - arrangement of hyperplanes in V

combin.	$M_{\mathcal{A}}$: matroid	$\rightarrow$	$T_{M_{\mathcal{A}}}(x, y)$: Tutte polynomial
	$\downarrow$?		$\downarrow$?
algebra	$A_{\mathcal{A}}$: alg. object	$\rightarrow$	$f(A_{\mathcal{A}})$: "measure" of $A_{\mathcal{A}}$

Algebra: The power ideals of $\mathcal{A}$.

$\mathcal{A} = \{H_1, \ldots, H_n\}$ - arrangement of hyperplanes in V
$H_i = \{x \in V \mid l_i(x) = 0\}$.
For $h \in V$, let

$$\rho_{\mathcal{A}}(h) = \text{ number of hyperplanes in } \mathcal{A} \text{ not containing } h.$$

The power ideals $I_{\mathcal{A},k}$ in $\mathbb{C}[V]$ are

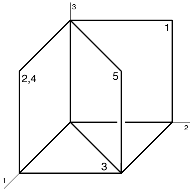
$$I_{\mathcal{A},k} := \left\langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in V, h \neq 0 \right\rangle.$$

The power inverse system $C_{\mathcal{A},k}$ is

$$\begin{aligned} C_{\mathcal{A},k} &:= \left\{ f(x) \in \mathbb{C}[V] \mid D_h^{\rho_{\mathcal{A}}(h)+k+1} f(x) = 0 \text{ for all } h \in V, h \neq 0 \right\} \\ &= \text{polynomials } f \text{ such that } \deg f|_h \leq \rho_{\mathcal{A}}(h) + k \text{ for all } h \end{aligned}$$

Example.

$\mathcal{G}$: $y_1 = 0, y_2 = 0, y_3 = 0, y_2 = 0, y_1 - y_2 = 0$



$$I_{\mathcal{G},0} = \left\langle h^{(\text{hyps. not containing } h)+1} \mid h \in V \right\rangle$$

$$= \left\langle x_1^3, x_2^4, x_3^2, (x_1+x_2)^4, \right.$$

$$\left. (x_1+ax_2)^5, (x_1+bx_3)^4, (x_2+cx_3)^5, (x_1+dx_2+ex_3)^6 \right\rangle$$

where a, b, c, d, e range over $\mathbb{C}$.

$$C_{\mathcal{G},0} = \{ f(\mathbf{y}) \mid g(\partial/\partial\mathbf{y}) f(\mathbf{y}) = 0 \text{ for all } g \in I_{\mathcal{G},0} \}$$

$$= \text{span}(1;\ y_1, y_2, y_3;\ y_1^2, y_2^2, y_1y_2, y_1y_3, y_2y_3;$$
$$y_2^3, y_1^2y_2, y_1^2y_3, y_1y_2^2, y_2^2y_3, y_1y_2y_3;$$
$$y_1y_2^3 - y_1^2y_2^2, y_2^3y_3, y_1^2y_2y_3, y_1y_2^2y_3;\ y_1y_2^3y_3 - y_1^2y_2^2y_3.)$$

and

$$\text{Hilb}\,(C_{\mathcal{G},0}; q) = 1 + 3q + 5q^2 + 6q^3 + 4q^4 + q^5.$$

$I_{\mathcal{A},k} := \langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in V, h \neq 0 \rangle$.
$C_{\mathcal{A},k} :=$ polynomials f such that $\deg f|_h \leq \rho_{\mathcal{A}} + k$ for all h

$I_{\mathcal{A},k}$ is homogeneous, so $I_{\mathcal{A},k}$ and $C_{\mathcal{A},k}$ are graded:
For $B = I_{\mathcal{A},k}, C_{\mathcal{A},k}$,

$$B_i = \{\text{homog. deg. } i \text{ elts.}\} \quad \to \quad B = \bigoplus_{i \geq 0} B_i;$$

Goal: Compute their Hilbert series $\text{Hilb}\,(B; q) = \sum \dim B_i q^i$.

Easy fact: It suffices to compute either one, because

$$\text{Hilb}\,(C_{\mathcal{A},k}; q) = \text{Hilb}\,(\mathbb{C}[V]/I_{\mathcal{A},k}; q) = \frac{1}{(1-q)^d} - \text{Hilb}\,(I_{\mathcal{A},k}; q)$$

Q: Why do we care about these spaces?
A: Because they appear in several different contexts.

Box splines (Dahmen-Michelli, Holtz-Ron, De Concini-Procesi)
Geometry (Postnikov-Shapiro-Shapiro)
Zonotopal Cox rings (Sturmfels-Xu, A)

A prototypical combinatorial result:

$\mathbb{Z}[x_1, \ldots, x_n] \,/\langle (x_{i_1} + \cdots + x_{i_k})^{k(n-k)+1} \,:\, 1 \leq i_1 < \cdots < i_k \leq n \rangle$
(which has a geometric meaning) has dimension $(n+1)^{n-1}$.
(Postnikov, Shapiro)

Combinatorics: The matroid of $\mathcal{A}$.

$\mathcal{A} = \{H_1, \ldots, H_n\}$ - arrangement of hyperplanes in V
$H_i = \{x \in V \mid l_i(x) = 0\}$.

The matroid $M_{\mathcal{A}}$ of $\mathcal{A}$ is the combinatorial data of
o how the H_is intersect, or
o how the l_is depend on each other.
It is given by the rank function

$$r : 2^{\mathcal{A}} \to \mathbb{Z}$$
$$r(\mathcal{B}) = \text{codim} \bigcap_{H_i \in \mathcal{B}} H_i = \text{rank}\{l_i \mid H_i \in \mathcal{B}\} \qquad \text{for } \mathcal{B} \subseteq \mathcal{A}$$

Combinatorics: The Tutte polynomial of $\mathcal{A}$.

The Tutte polynomial of $\mathcal{A}$ is

$$T_{\mathcal{A}}(x,y) = \sum_{\mathcal{B}\subseteq\mathcal{A}} (x-1)^{r(\mathcal{A})-r(\mathcal{B})}(y-1)^{|B|-r(\mathcal{B})}$$

It knows **a lot** about $\mathcal{A}$: (Zaslavsky, MacPherson, Crapo-Rota)

- $\mathbb{F}=\mathbb{R}$: number of regions of $\mathbb{R}^n\backslash\mathcal{A}$ is $|T(2,0)|$.
- $\mathbb{F}=\mathbb{C}$: cohomology ring of $\mathbb{C}^n\backslash\mathcal{A}$ has Hilb. ser. $q^nT(1+\frac{1}{q},0)$.
- $\mathbb{F}=\mathbb{F}_q$: number of points of $\mathbb{F}_q^n\backslash\mathcal{A}$ is $|T(1-q,0)|$.

The reason: It is a **universal deletion-contraction invariant**:
deletion: $\mathcal{A}\backslash H$; contraction: $\mathcal{A}/H = \{H'\cap H, H'\in\mathcal{A}-H\}$

Theorem. (Tutte) If $f(\mathcal{A})$ can be (nicely) expressed in terms of $f(\mathcal{A}\backslash H)$ and $f(\mathcal{A}/H)$, then $f(\mathcal{A})$ is an evaluation of $T_{\mathcal{A}}(x,y)$.

Some previous results.

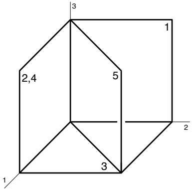
$C_{\mathcal{A},k} := \{f \text{ with } \deg f|_h \leq \rho_{\mathcal{A}}(h) + k \text{ for all lines } h\}$

$C'_{\mathcal{A},k} := \{f \text{ with } \deg f|_h \leq \rho_{\mathcal{A}}(h) + k \text{ for the lines } h \text{ of } \mathcal{A}.\}$
(only lines which are intersections of hyps. of $\mathcal{A}$)

Example.

$$I_{\mathcal{G},0} = \langle x_1^3, x_2^4, x_3^2, (x_1+x_2)^4, (x_1+ax_2)^5, (x_1+bx_3)^4, (x_2+cx_3)^5, (x_1+dx_2+ex_3)^6 \rangle$$

$$I'_{\mathcal{G},0} = \langle x_1^3, x_2^4, x_3^2, (x_1+x_2)^4 \rangle$$



Theorems.

$\text{Hilb}\,(C'_{\mathcal{A},-1}; q) = q^{n-r} T_{\mathcal{A}}(1, \frac{1}{q})$ (Dahmen-Miccheli 85)

$\text{Hilb}\,(C'_{\mathcal{A},0}; q) = q^{n-r} T_{\mathcal{A}}(1+q, \frac{1}{q})$ (Postnikov-Shapiro-Shapiro '99)

$\text{Hilb}\,(C_{\mathcal{A},-1}; q) = q^{n-r} T_{\mathcal{A}}(1, \frac{1}{q})$ (A.-Postnikov '02)

$\text{Hilb}\,(C'_{\mathcal{A},-2}; q) = q^{n-r} T_{\mathcal{A}}(0, \frac{1}{q})$ (Holtz-Ron '07)

$C_{\mathcal{A},k} := \{f \text{ with } \deg f|_h \leq \rho_{\mathcal{A}}(h) + k \text{ for all lines } h\}$
$C'_{\mathcal{A},k} := \{f \text{ with } \deg f|_h \leq \rho_{\mathcal{A}}(h) + k \text{ for the lines } h \text{ of } \mathcal{A}.\}$

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Theorems. (A.-Postnikov, '08)

- For $k \in \{0, -1, -2\}$, $C_{\mathcal{A},k} = C'_{\mathcal{A},k}$ and $\text{Hilb}\,(C_{\mathcal{A},k}; q)$ is as above.
- For $k \geq 0$,

$$\sum_{k \geq 0} \text{Hilb}\,(C_{\mathcal{A},k}; q) z^k = \frac{q^{n-r}}{(1-z)(1-qz)^{d-r}} T_{\mathcal{A}}\left(1 + \frac{q}{1-qz}\right)$$

(In general $C_{\mathcal{A},k} \neq C'_{\mathcal{A},k}$ for $k \geq 1$.)

Sketch of proof.

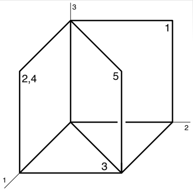
Goal. Compute $\mathrm{Hilb}(C_{\mathcal{A},k}; q)$ for $k \geq -2$.
Key. $\mathrm{Hilb}(C_{\mathcal{A},k}; q) = q\mathrm{Hilb}(C_{\mathcal{A}\setminus H,k}; q) + \mathrm{Hilb}(C_{\mathcal{A}/H,k}; q)$.

1. If $H \in \mathcal{A}$ is not a loop, then there is an exact sequence

$$0 \to C_{\mathcal{A}\setminus H,k}(-1) \to C_{\mathcal{A},k} \to C_{\mathcal{A}/H,k} \to 0$$

   of graded $\mathbb{C}$-vector spaces.
2. Let l_s be the linear form defining hyperplane H_s.
   - $C_{\mathcal{A},0} = \mathrm{span}\{\prod_{s\in S} l_s \mid S \subseteq [n]\}$
   - $C_{\mathcal{A},-1} = \mathrm{span}\{\prod_{s\in S} l_s \mid r(S) = r\}$
   - $C_{\mathcal{A},-2} = \mathrm{span}\{\prod_{s\in S} l_s \mid r(S - x) = r \text{ for all } x \notin S\}$
   - $(k \geq 1)$ $C_{\mathcal{A},k} = \mathrm{span}\{f \prod_{s\in S} l_s \mid \deg f \leq k, S \subseteq [n]\}$.

In 1. the difficulty is right exactness.
In 2. the difficulty is $\subseteq$.
These difficulties solve each other!! Proceed by joint induction.



Back to example.

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$$C_{\mathcal{G},0} = \{f(\mathbf{y}) \mid g(\partial/\partial\mathbf{y})f(\mathbf{y}) = 0 \text{ for all } g \in I_{\mathcal{G},0}\}$$

$$= \text{span}(1; \quad y_1, y_2, y_3; \quad y_1^2, y_2^2, y_1y_2, y_1y_3, y_2y_3;$$
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$$y_1y_2^3 - y_1^2y_2^2, y_2^3y_3, y_1^2y_2y_3, y_1y_2^2y_3; \quad y_1y_2^3y_3 - y_1^2y_2^2y_3.)$$

and

$$\text{Hilb}\,(C_{\mathcal{G},0}; q) = 1 + 3q + 5q^2 + 6q^3 + 4q^4 + q^5.$$

Application 1. **Spline theory.**

$A = \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^d$: unimodular set of vectors.
$(\det[a_{i_1}, \ldots, a_{i_d}] \in \{-1, 0, 1\}$ for all $i_1, \ldots, i_d)$.
$\mathcal{A}$: dual hyperplane arrangement

$Z(A)$: zonotope of $A := \sum_{a \in A} a$
B_A: box spline := convol. prod. of unif. measures on $a_1, \ldots, a_n$.
(piecewise polynomial, supported on $Z(A)$)

Holtz and Ron conjectured:

Theorem. (A.-Postnikov, '08)
Any real function on int $Z(A) \cap \mathbb{Z}^d$ extends uniquely to a polynomial function on $Z(A)$ of the form

$$p(\partial/\partial \mathbf{x}) B_A(\mathbf{x}) \qquad p \in C_{\mathcal{A},-2}.$$

In fact, $C_{\mathcal{A},-2}$ is the canonical least space with that property.

Application 2. **Zonotopal Cox rings.**

$\mathcal{A}$: hyperplane arrangement in $\mathbb{CP}^{d-1}$.

$B(\mathcal{A})$: In $\mathbb{CP}^{d-1}$ blowup the points of intersection of $\mathcal{A}$.
$Cox(B(\mathcal{A}))$: (multigraded) Cox ring of $B(\mathcal{A})$
Nagata: Not always fin. generated. (Hilbert's 14th problem)
$\mathcal{Z}(\mathcal{A}) \subset Cox(B(\mathcal{A}))$: zonotopal Cox ring of X (Sturmfels-Xu)

Theorem. (A.-Postnikov, '08)
A formula for the multigraded Hilbert series of $\mathcal{Z}(\mathcal{A})$ in terms of the multivariate Tutte polynomial of $\mathcal{A}$.

In progress: (A.) Analogous results for De Concini-Procesi's wonderful model $W(\mathcal{A})$ of $\mathcal{A}$. (In $\mathbb{CP}^{d-1}$ blowup all flats of $\mathcal{A}$.)

many thanks!

Combinatorics and geometry of power ideals
Federico Ardila and Alex Postnikov
Transactions of the AMS, to appear.

Available at:
`http://math.sfsu.edu/federico`
`http://front.math.ucdavis.edu`