Flag arrangements and tilings of simplices.

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The plan to follow: (or not to follow)

1. Arrangements of d flags in $\mathbb{C}^n$.
2. Rhombus tilings of equilateral triangles with holes.
3. Mixed subdivisions of $n\Delta_{d-1}$.
4. Applications to the flag Schubert calculus.
5. Tropical oriented matroids.

1. Arrangements of d flags in $\mathbb{C}^n$.

A complete flag $F_\bullet$ in $\mathbb{C}^n$ is

$$F_\bullet = \{\{0\} \subset \text{line} \subset \text{plane} \subset \cdots \subset \text{hyperplane} \subset \mathbb{C}^n\}.$$

Let $E^1_\bullet, \ldots, E^d_\bullet$ be d generically chosen complete flags in $\mathbb{C}^n$. Write

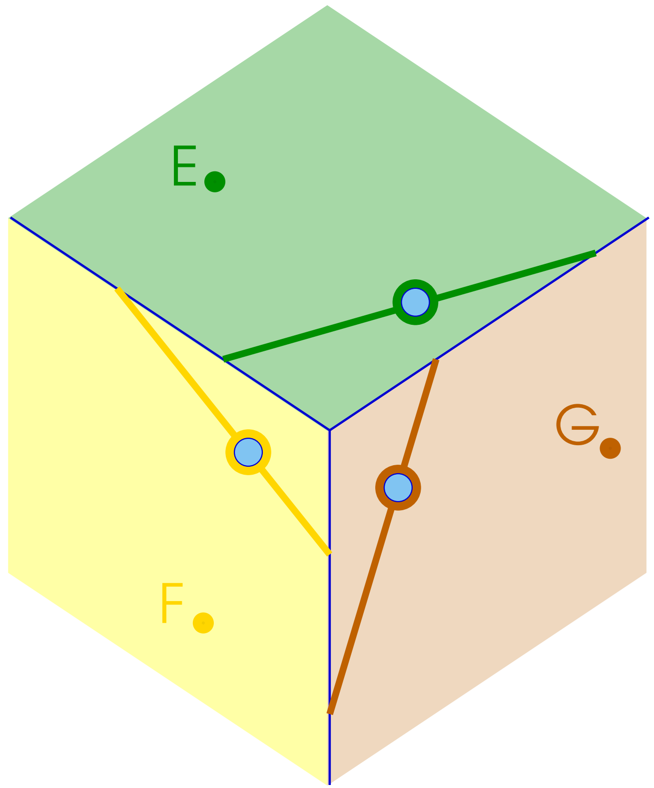
$$E^k_\bullet = \{\{0\} = E^k_0 \subset E^k_1 \subset \cdots \subset E^k_n = \mathbb{C}^n\},$$

where E^k_i is a vector space of dimension i.

Let $E_{\bullet}^{1}, \ldots, E_{\bullet}^{d}$ be d generically chosen complete flags in $\mathbb{C}^n$.

Example. $d = 3, n = 4$: flags $E_{\bullet}, F_{\bullet}, G_{\bullet}$ in $\mathbb{C}^4$ (projective picture)

Each flag is point $\subset$ line $\subset$ plane $\subset$ 3-space.



Goal. Study the set $\mathbf{E}_{n,d}$ of one-dimensional intersections determined by the flags; that is, all lines of the form

$$E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d},$$

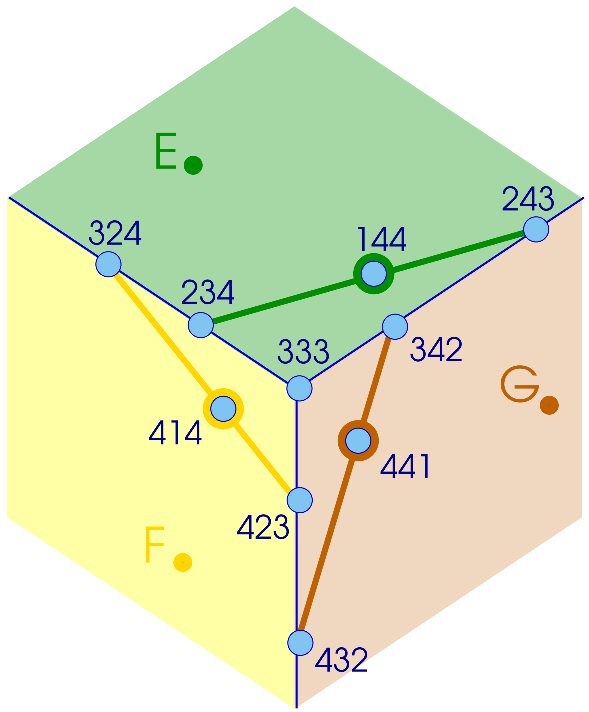
with $\sum(n - a_i) = n - 1$; that is, $\sum a_i = n(d-1) + 1$.

Example.

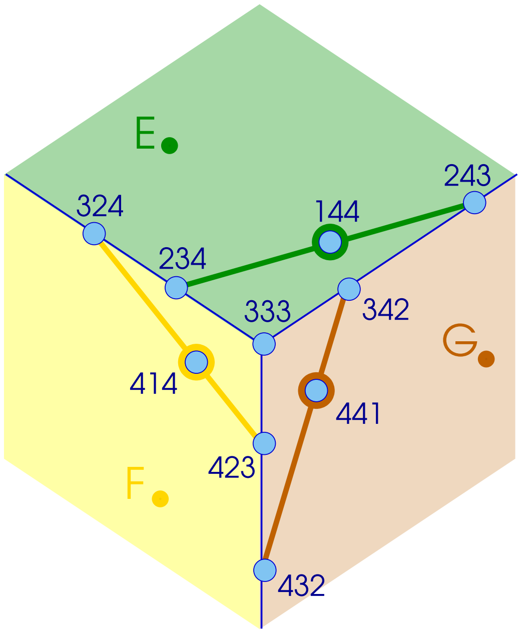
$\mathbf{E}_{4,3}$ consists of the ten lines:

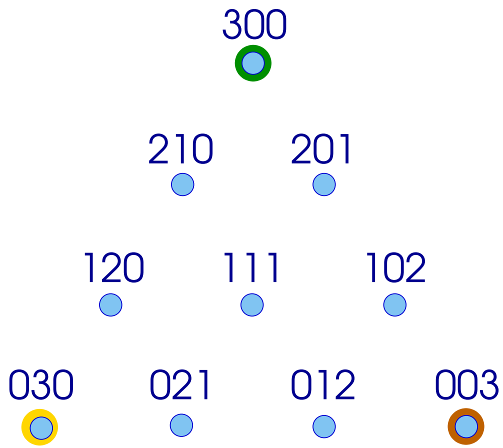
$$abc = E_a \cap F_b \cap G_c$$

for $a + b + c = 9$



Question. In $\mathbf{E}_{n,d}$, which sets are dependent/independent? What is the matroid?





First an encoding:

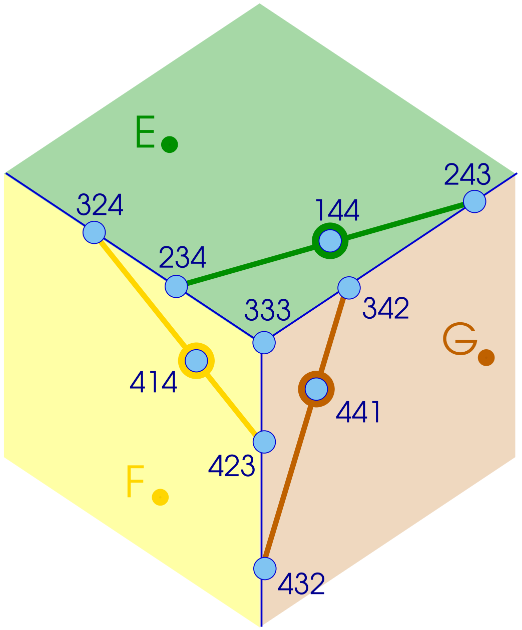
$$\text{lines in } \mathbf{E}_{n,d} \quad \leftrightarrow \quad \text{dots in “simplicial” array } T_{n,d}$$

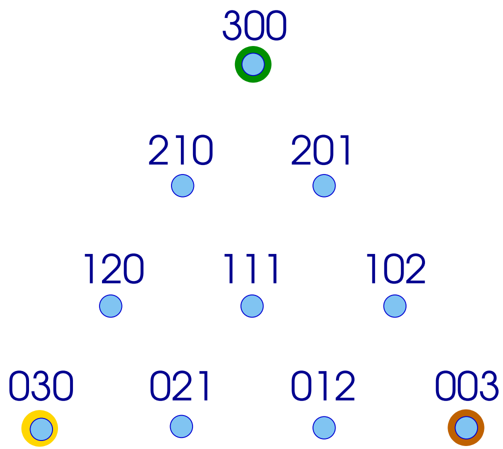
$$E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d} \quad \leftrightarrow \quad (n - a_1, \ldots, n - a_d)$$

Some easy dependence relations:

A k-dim $E^1_{b_1} \cap E^2_{b_2} \cap \cdots \cap E^d_{b_d}$ contains line $E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d}$ when $a_i \leq b_i$. Therefore, those lines have rank at most k.

Combinatorial dependence relation. Any $k+1$ dots in a simplex of size k are dependent.





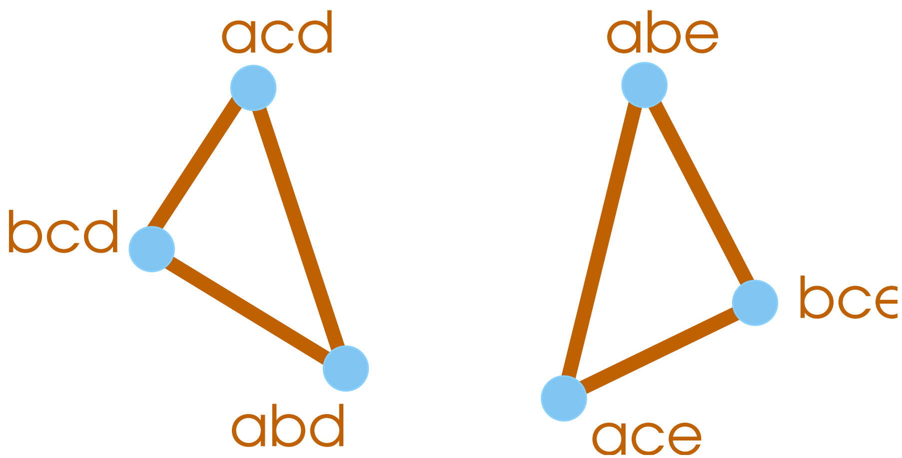
Question. Are these the only dependence relations?

Evidence that there may be other relations.

Five flags $A_\bullet, B_\bullet, C_\bullet, D_\bullet, E_\bullet$ in $\mathbb{C}^4$. We restrict our attention to:

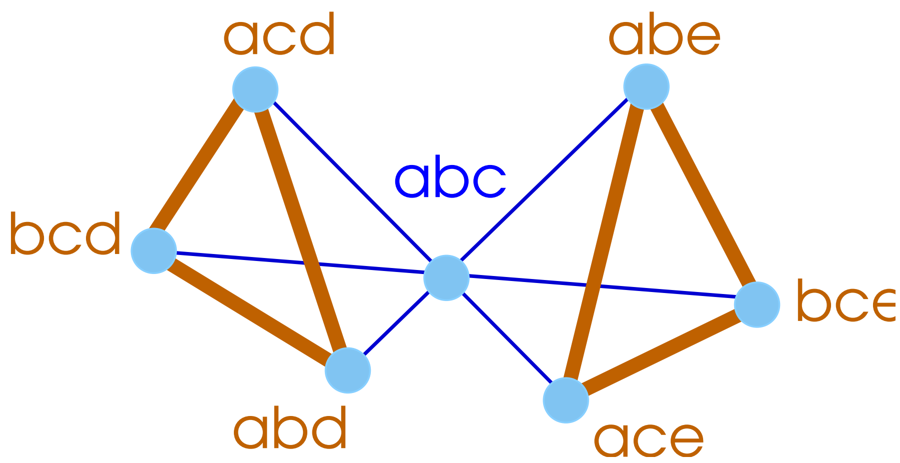
- the hyperplanes $a = A_3, b = B_3, \ldots$
- their points of intersection $abd = a \cap b \cap d, \ldots$

Combinatorial dependence relations: points abc, abd, abe are on line ab, etc. Are there others?



Consider the points on hyperplanes d and e.

Evidence that there may be other relations.

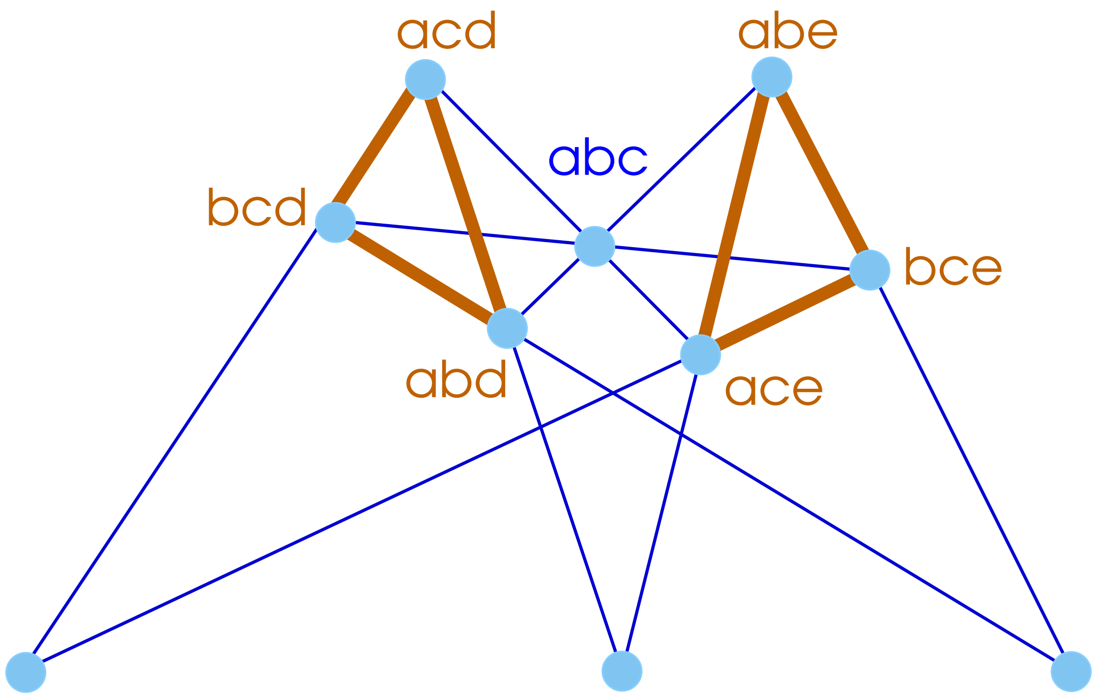


We are considering the triangles on hyperplanes d and e.

They are perspective with respect to point abc.

Evidence that there may be other relations.

By Desargues's theorem, we get three unexpected collinear points.

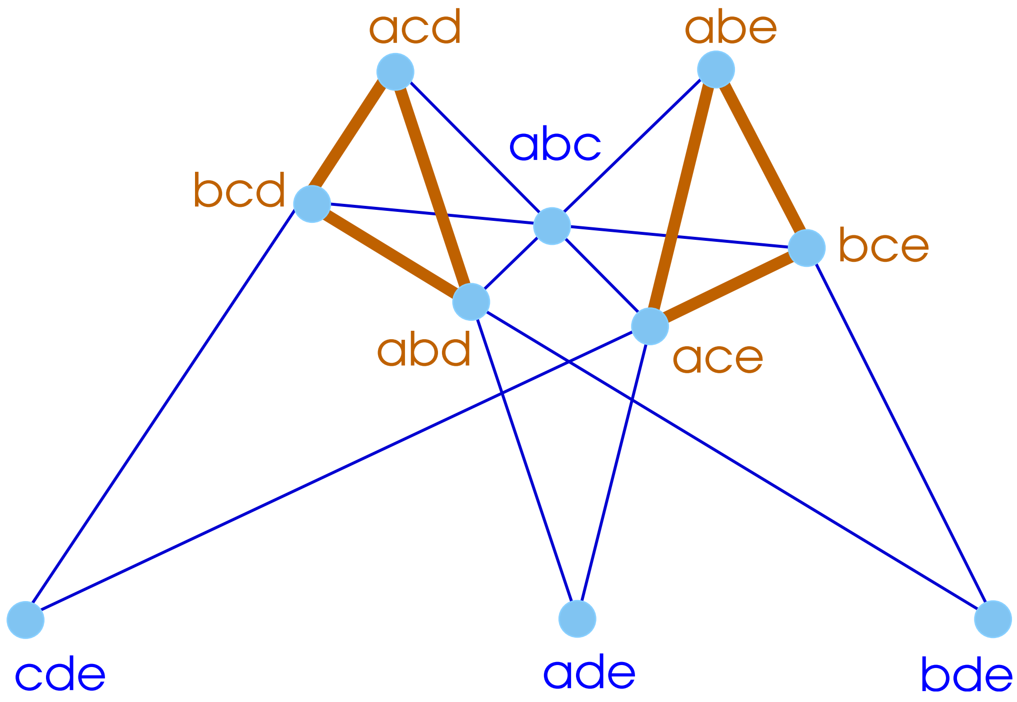


These three points are in our arrangement $\mathbf{E}_{5,4}$! The left one is *cde*.

Is this a new dependence relation?

Evidence that there may be other relations.

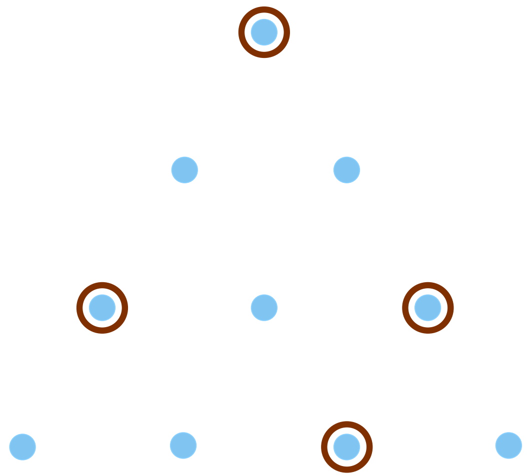
The three points are ade, bde, cde - collinearity is not unexpected.



- Desargues's theorem is really combinatorial, not geometric.
- For larger n, d, we might get nontrivial geometric configurations ($e.g.$, Pappus config.) which imply new dependence relations.

Having told you what to worry about, now I tell you not to worry about it. These **are** all the dependence relations.

Recall that $T_{n,d}$ is the $(d-1)$-dimensional simplicial array of dots of size n, which encodes the lines $\mathbf{E}_{n,d}$. Shown below is $T_{4,3}$.



Theorem. (Ardila, Billey, 2005.)

A set of dots in $T_{n,d}$ is independent if and only if no subarray $T_{k,d}$ of size k contains more than k dots.

The method of proof is constructive.

Goal:

How do we construct d “generic enough” flags in $\mathbb{C}^n$?

Reduce to:

How do we construct $(n-1)d$ “generic enough” hyperplanes in $\mathbb{C}^n$?

(Get a flag from $n-1$ hyps.: $A \supset (A \cap B) \supset (A \cap B \cap C) \supset \cdots$.)

Reduce to:

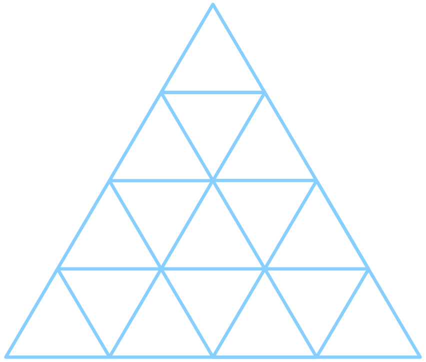
How do we construct a “generic enough” n-plane P in $\mathbb{C}^{(n-1)d}$?

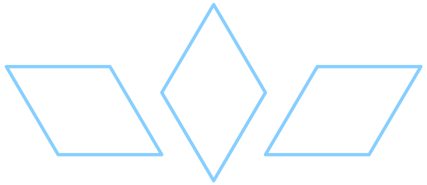
(Then intersect P with the nd coordinate hyperplanes in $\mathbb{C}^{(n-1)d}$.)

We do this using the theory of Dilworth truncations.

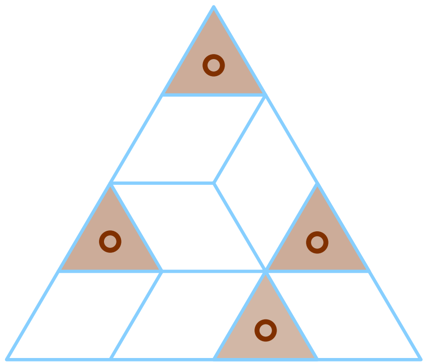
2. Rhombus tilings of triangles with holes.

To tile the equilateral triangle $T(n)$ of size n with unit rhombi,



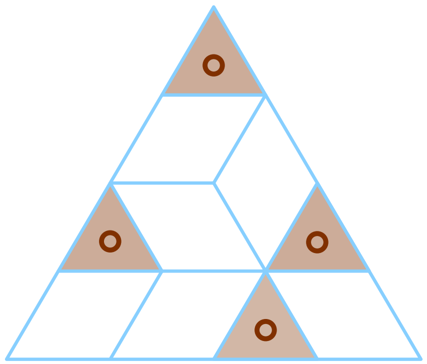


we first need to make $n = \binom{n+1}{2} - \binom{n}{2}$ holes.



Where can we put those holes?

Question. Given n holes in $T(n)$, is there a simple criterion to determine whether the resulting holey triangle can be tiled with unit rhombi?



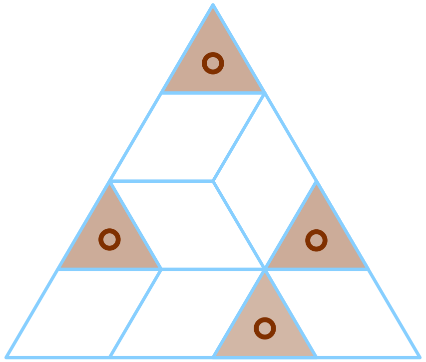
A rhombus tiling is equivalent to a complete matching, or marriage, of the $\binom{n}{2}$ downward triangles to some $\binom{n}{2}$ of the upward triangles.

The marriage theorem answers the question:

So-so answer. The holey $T(n)$ can be tiled if and only if any k downward triangles have at least k upward triangles to match to.

Question. Given n holes in $T(n)$, is there a simple criterion to determine whether the resulting holey triangle can be tiled with unit rhombi?

Answer 1 is valid for rhombus tilings of any region. However, the geometry of $T(n)$ allows for a nicer answer:



A necessary condition. If a holey triangle can be tiled with unit rhombi, then no $T(k)$ inside $T(n)$ contains more than k holes.

Proof. Count.

Question. Given n holes in $T(n)$, is there a simple criterion to determine whether the resulting holey triangle can be tiled with unit rhombi?

Better answer. (Ardila, Billey, 2005)

Consider a set of n holes in $T(n)$. The resulting holey triangle can be tiled with unit rhombi if and only if no $T(k)$ inside $T(n)$ contains more than k holes.

In other words:

The possible locations of the holes are precisely the bases of the matroid $\mathcal{T}_{n,3}$!

The method of proof is constructive. Given a "good" set of holes, we construct a tiling T with those holes. We start with a base tiling T_0, and arrive to T via local moves.

3. Mixed subdivisions of $n\Delta_{d-1}$.

Question.

(geometry of 3 flags) $\leftrightarrow$ (rhombus tilings of holey triangles)

(geometry of d flags) $\leftrightarrow$ (____________________________)

A fine mixed subdivision of the simplex $n\Delta_{d-1}$ is a subdivision using the following tiles:

$$(d-1)\text{-dimensional products of faces of } \Delta_{d-1}$$

Tiles: $(d-1)$-dimensional products of faces of Δ_{d-1}

Example. For $d = 3$, the tiles are:

- unit rhombus = (segment) $\times$ (segment)
- unit equilateral triangle

(fine mixed subdivisions of $n\Delta_2$) = (tilings of holey $T(n)$s)

Example. For $d = 4$, the tiles are:

- parallelepiped = (segm.) $\times$ (segm.) $\times$ (segm.)
- triangular prism = (triangle) $\times$ (segment)
- tetrahedron

We conjecture a higher-dimensional analog of our results on tilings.

Conjecture.

(geom. of 3 flags in $\mathbb{C}^n$) $\leftrightarrow$ (rhombus tilings of holey $T(n)$s)

(geom. of d flags in $\mathbb{C}^n$) $\leftrightarrow$ (fine mixed subdivs. of $n\Delta_{d-1}$)

More precisely:

Theorem. (Ardila, Billey, 2005)

In any fine mixed subdivision of $n\Delta_{d-1}$,

(a) there are exactly n tiles which are simplices, and

(b) no $k\Delta_{d-1}$ of size k in $n\Delta_{d-1}$ contains more than k simplices.

(c) (**Conjecture.**) If n unit simplices satisfy (a) and (b), they are the simplices in *some* fine mixed subdivision.

Theorem. (Ardila, Billey, 2005)

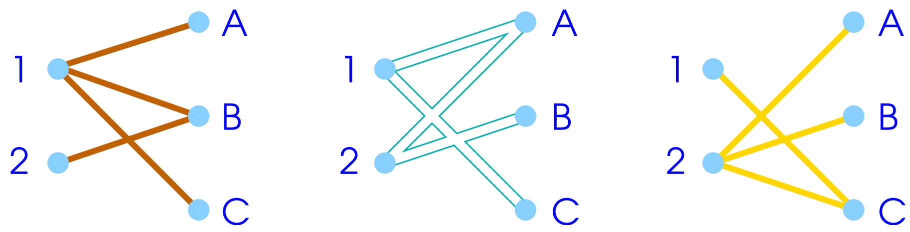
In any fine mixed subdivision of $n\Delta_{d-1}$,

(a) there are exactly n tiles which are simplices, and

(b) no $k\Delta_{d-1}$ of size k in $n\Delta_{d-1}$ contains more than k simplices.

To prove the theorem, we exhibit a bijection

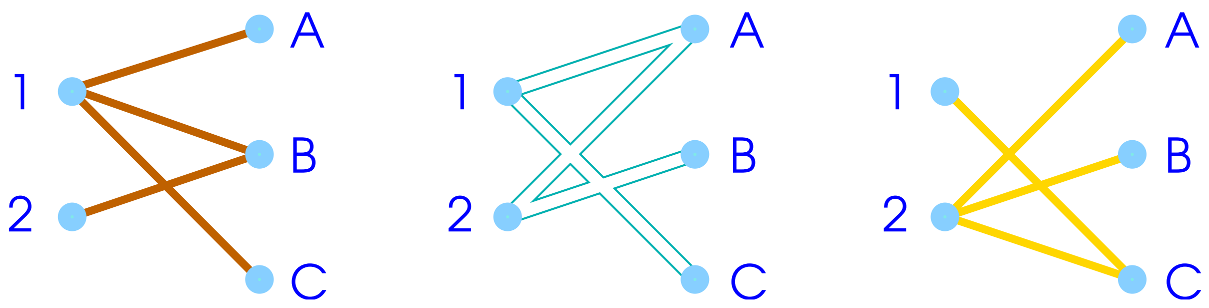
$$(\text{fine mixed subdivisions of } n\Delta_{d-1}) \leftrightarrow (\text{allowable sets of trees})$$



and translate (a) and (b) into combinatorial statements about trees.

Definition. A collection $t_1, \ldots, t_k$ of spanning trees of the complete bipartite graph $K_{n,d}$ is allowable if

1. For each t_i and each internal edge e of t_i, there exists an edge f and a tree t_j with $t_j = t_i - e \cup f$.
2. There do not exist two trees t_i and t_j, and a circuit C of $K_{n,d}$ which alternates between edges of t_i and edges of t_j.



Theorem. (Ardila, Billey, 2005)

The fine mixed subdivisions of $n\Delta_{d-1}$ are in one-to-one correspondence with the allowable sets of trees in $K_{n,d}$.

Conjecture.

(c) Any n unit simplices in $n\Delta_{d-1}$ which "are not too crowded" are the simplices in *some* fine mixed subdivision.

Even for $d = 4$, this is open and interesting:

Conjecture. Consider n unit tetrahedra in the tetrahedron of edge length n such that no k of them are contained in a tetrahedron of edge length k. The empty space left by these n tetrahedra can be exactly filled using triangular prisms and parallelepipeds.

To prove the conjecture, we need to construct fine mixed subdivisions of $n\Delta_{d-1}$ in a controlled way.

Tropical hyperplane arrangements (tropical polytopes) may be a good way to do it! (Mike Develin - Bernd Sturmfels, Paco Santos)

(More about this later.)

4. Applications to the flag Schubert calculus.

(Very) quick review of Schubert calculus of the flag manifold:

The relative position of two flags $E_\bullet$ and $F_\bullet$ in $\mathbb{C}^n$ is given by the $n \times n$ rank table whose (i, j) entry is $P[i, j] = \dim(E_i \cap F_j)$.

An example rank table:

$$P = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Each rank table comes from a permutation matrix:

$$P = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If $E_\bullet$ and $F_\bullet$ have rank table P, their relative position is $w = 53124$.

For fixed $E_\bullet$, divide all flags according to position with respect to $E_\bullet$:

The *Schubert cell* and *Schubert variety* be

$$\begin{aligned} X_w^\circ(E_\bullet) &= \{F_\bullet \mid E_\bullet \text{ and } F_\bullet \text{ have relative position } w\} \\ X_w(E_\bullet) &= \overline{X_w^\circ(E_\bullet)} \end{aligned}$$

Schubert problem. Given generic flags $E^1_\bullet, E^2_\bullet, E^3_\bullet$ in $\mathbb{C}^n$ and permutations u, v, w in S_n, how many flags $F_\bullet$ have relative positions u, v, w with respect to $E^1_\bullet, E^2_\bullet, E^3_\bullet$?

The answer, c_{uvw}, is independent of $E^1_\bullet, E^2_\bullet, E^3_\bullet$. The numbers c_{uvw} are very important. They are the multiplicative structure constants for the cohomology ring of the flag manifold.

Open problem. Given three permutations u, v, w, can we compute c_{uvw} combinatorially?

This question seems very difficult; the following may be easier:

Open problem. Can we describe the permutations u, v, w for which $c_{uvw} = 0$?

4.1. A vanishing criterion for c_{uvw}.

Assume we know the relative positions u, v, w of $F_\bullet$ with respect to $E^1_\bullet, E^2_\bullet, E^3_\bullet$. In other words, we know, for all a, b, c, j:

$$\dim(E^1_a \cap F_j), \qquad \dim(E^2_b \cap F_j), \qquad \dim(E^3_c \cap F_j).$$

Billey-Vakil: We can then compute, for all a, b, c, j,

$$\dim(E^1_a \cap E^2_b \cap E^3_c \cap F_j).$$

In particular, for each j, we know the set $L(u, v, w)_j$ of lines $E^1_a \cap E^2_b \cap E^3_c$ (where $a + b + c = 2n + 1$) which are in each F_j.

Observation. The matroid $\mathcal{T}_{n,3}$ tells us the rank of $L(u, v, w)_j$.

A very rough vanishing criterion. If for some j we have $\mathrm{rank}(L(u, v, w)_j) > j$ in the matroid $\mathcal{T}_{n,3}$, then $c_{uvw} = 0$.

(Already characterizes vanishing for $n \leq 5$, but only the beginning!)

4.2. Computing c_{uvw}.

Billey-Vakil: Using the numbers

$$\dim(E_a^1 \cap E_b^2 \cap E_c^3 \cap F_j),$$

we can write down an explicit set of equations cutting out the intersection

$$X = X_u(E_\bullet^1) \cap X_v(E_\bullet^2) \cap X_w(E_\bullet^3)$$

and just count the number of flags in X.

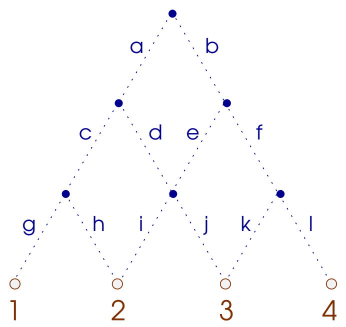
The equations are written in terms of the vectors:

$$v_{abc} = E_a^1 \cap E_b^2 \cap E_c^3$$

So it would be useful to have a nice choice of v_{abc}.

Ultimately, we want a nice representation of the matroid $\mathcal{T}_{n,3}$

We get this from $\mathcal{T}_{n,3}$ being a cotransversal matroid (via tilings!).



Assign weights to the edges. For each dot D, let $v_{D,i}$ be the sum of the weights of all paths from dot D to dot i on the bottom row.

For example, $v_{top} = (acg, ach + adi + bei, adj + bej + bfk, bfl)$.

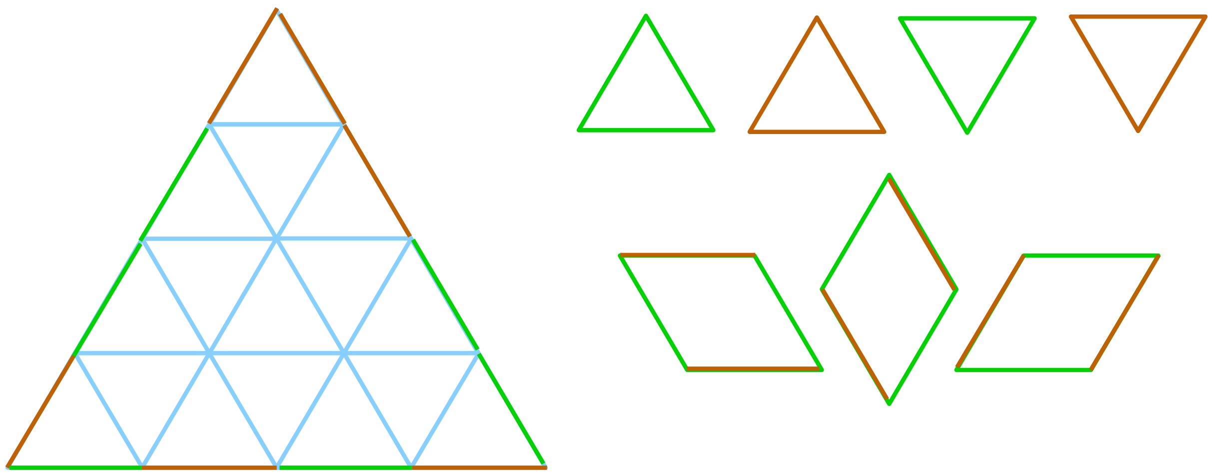
Theorem. (Ardila-Billey, 2005)

The vectors $v_D = (v_{D,1}, \ldots, v_{D,n})$ are a geometric representation of the matroid $\mathcal{T}_{n,3}$.

Result. (Billey-Vakil, 2004, Ardila-Billey, 2005) We get a method for computing c_{uvw} without reference to a fixed set of flags.

Reason to hope for more? (Knutson-Tao)

- In the corresponding problem for the Grassmannian, $c_{\lambda\mu\nu}$ is the number of puzzles; certain tilings of $T(n)$:



- **Saturation conjecture**: Explicit characterization of those λ, μ, ν for which $c_{\lambda\mu\nu} = 0$.

5. Tropical oriented matroids. (A research direction.)

Recall:

(f. m. subdivs. of $n\Delta_{d-1}$) $\updownarrow$ (C) (triangs. of $\Delta_{n-1} \times \Delta_{d-1}$)	(regular f. m. subdivs. of $n\Delta_{d-1}$) $\updownarrow$ (C) (regular triangs. of $\Delta_{n-1} \times \Delta_{d-1}$) $\updownarrow$(DS) (combin. types of generic arrs. of n tropical hyps. in $\mathbb{TP}^{d-1}$)

Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ appear in many different places.
(Babson-Billera, Bayer, Diaconis-Sturmfels, Haiman, Postnikov)

Now we have:

(allowable sets of trees)	(realizable allowable sets of trees)
$\updownarrow$(AB)	$\updownarrow$(AB)
(f. m. subdivs. of $n\Delta_{d-1}$)	(regular f. m. subdivs. of $n\Delta_{d-1}$)
$\updownarrow$ (C)	$\updownarrow$ (C)
(triangs. of $\Delta_{n-1} \times \Delta_{d-1}$)	(regular triangs. of $\Delta_{n-1} \times \Delta_{d-1}$)
	$\updownarrow$(DS)
	(combin. types of generic arrs. of n tropical hyps. in $\mathbb{TP}^{d-1}$)

Open question. Can the realizable allowable sets of trees be characterized combinatorially?

(tropical oriented matroids)	(realizable trop. or. matroids)
(allowable sets of trees) $\updownarrow$(AB) (f. m. subdivs. of $n\Delta_{d-1}$) $\updownarrow$ (C) (triangs. of $\Delta_{n-1} \times \Delta_{d-1}$) $\updownarrow$(ABD) (combin. types of gen. arrs. of d trop. pseudohyps. in $\mathbb{TP}^{n-1}$)	(realizable allowable sets of trees) $\updownarrow$(AB) (reg. f. m. subdivs. of $n\Delta_{d-1}$) $\updownarrow$ (C) (reg. triangs. of $\Delta_{n-1} \times \Delta_{d-1}$) $\updownarrow$(DS) (combin. types of generic arrs. of n tropical hyps. in $\mathbb{TP}^{d-1}$)

Project. (with Sara Billey, Mike Develin)

Develop a theory of these ubiquitous tropical oriented matroids.

Many open questions!

Thank you for your attention.

Preprint available at:

`math.sfsu.edu/federico`