Polinomios aritméticos de Tutte

Federico Ardila

San Francisco State University
Universidad de Los Andes, Bogotá, Colombia.

Congreso Colombiano de Matemáticas
Barranquilla, 16 de julio de 2013

Outline

1. Tutte polynomials
2. Hyperplane arrangements
3. Computing Tutte polynomials
4. Corte de comerciales.
5. Arithmetic Tutte polynomials
6. Toric arrangements
7. Computing arithmetic Tutte polynomials

Joint work with:

Federico Castillo (U. de Los Andes + U. of California at Davis)
Mike Henley (San Francisco State University)

1. THE TUTTE POLYNOMIAL.

Let $\mathcal{A} \subseteq \mathbb{K}^n$ be a collection of vectors.

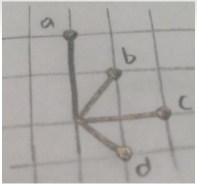
The **Tutte polynomial** of $\mathcal{A}$ is

$$T_{\mathcal{A}}(x, y) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (x-1)^{r(\mathcal{A})-r(\mathcal{B})}(y-1)^{|\mathcal{B}|-r(\mathcal{B})}.$$

where, for each $\mathcal{B} \subseteq \mathcal{A}$, the **rank of** $\mathcal{B}$ is

$$r(\mathcal{B}) = \dim \operatorname{span}(\mathcal{B})$$

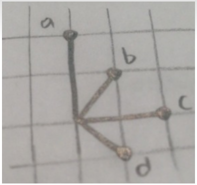
Example: $C_2 = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\} \subseteq \mathbb{R}^2$



subset	rank	contribution
$\emptyset$	0	$(x-1)^2(y-1)^0$
a, b, c, d	1	$(x-1)^1(y-1)^0$
ab, ac, ad, bc, bd, cd	2	$(x-1)^0(y-1)^0$
abc, abd, acd, bcd	2	$(x-1)^0(y-1)^1$
$abcd$	2	$(x-1)^0(y-1)^2$

$$\begin{aligned} T(x,y) &= (x-1)^2 + 4(x-1) + 6 + 4(y-1) + (y-1)^3 \\ &= x^2 + y^2 + 2x + 2y \end{aligned}$$

Ojo. Need Char $\mathbb{K} \neq 2$.



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WHY CARE ABOUT THE TUTTE POLYNOMIAL?

Many important invariants of $\mathcal{A}$ are evaluations of $T_{\mathcal{A}}(x, y)$.

For vector arrangements:

- $T(1, 1)$ = number of **bases.**
- $T(2, 1)$ = number of **independent sets.**
- $T(1, 2)$ = number of **spanning sets.**

These are nice, but maybe not terribly interesting.

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For graphs:

From a graph $G = (V, E)$ I get a vector arrangement $\mathcal{A}_G \in \mathbb{K}^V$:

$$\mathcal{A}_G = \{e_i - e_j \,:\, ij \text{ is an edge of } G\}$$

- $T(1, 1)$ = number of **spanning trees.**
- $T(2, 0)$ = number of **acyclic orientations** of edges.
- $T(0, 2)$ = number of **totally cyclic orientations** of edges.
- $(-1)^{v-c} q^c\, T(1 - q, 0)$ = **chromatic polynomial** = number of proper q-colorings of the vertices.
- $(-1)^{e-v+c}\, T(0, 1 - t)$ = **flow polynomial** = number of nowhere zero t-flows of the edges.

[Stanley, 1973, 1980] [Tutte, 1947] [Crapo, 1969]

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2. HYPERPLANE ARRANGEMENTS

For hyperplane arrangements:

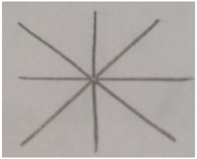
Vector $a \in \mathbb{K}^n \mapsto$ Hyperplane $H_a = \{x \in (\mathbb{K}^n)^* : a \cdot x = 0\}$.

Vector arr. $A \subseteq \mathbb{K}^n \mapsto$ Hyperplane arr. $H_{\mathcal{A}} = \{H_a : a \in \mathcal{A}\}$

Complement $V(\mathcal{A}) = \mathbb{K}^n \setminus \bigcup_{H \in \mathcal{A}} H$

Example.

$C_2 : 2x = 0,\ x + y = 0,\ 2y = 0,\ x - y = 0$



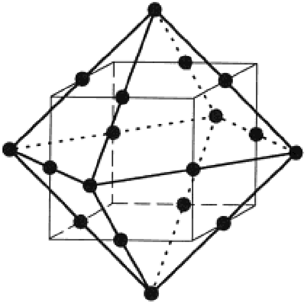
Example. C_3

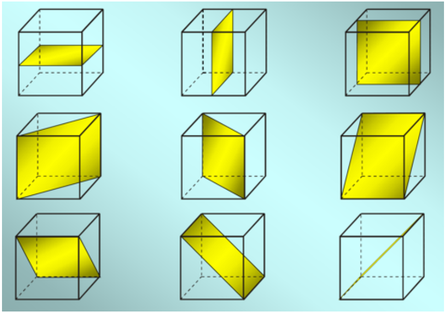
Root system:

- $\pm e_i \quad (1 \leq i \leq 3)$
- $\pm e_i \pm e_j \quad (1 \leq i < j \leq 3)$

Hyperplanes:

$$2x = 0, 2y = 0, 2z = 0$$
$$x + y = 0, y + z = 0, z + x = 0$$
$$x - y = 0, y - z = 0, z - x = 0$$





Many important invariants of $\mathcal{A}$ are evaluations of $T_{\mathcal{A}}(x, y)$.

For hyperplane arrangements:

- $(\mathbb{K} = \mathbb{R})$

$$(-1)^n T(2, 0) = \text{number of regions of } V(\mathcal{A})$$

[Zaslavsky, 1975]

- $(\mathbb{K} = \mathbb{C})$

$$T(1-q, 0) = \sum_i \dim H^i(V(A); \mathbb{Z})(-q)^i$$

[Orlik and Solomon, 1980, Goresky-MacPherson, 1988]

- $(\mathbb{K} = \mathbb{F}_q)$

$$|T(1-q, 0)| = |V(\mathcal{A})|$$

[Crapo and Rota, 1970]

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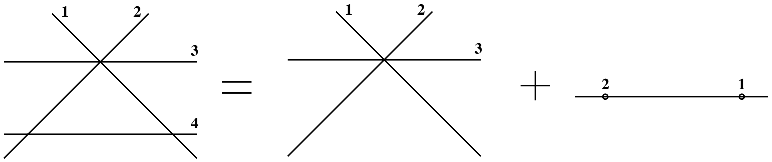
WHY IS THE TUTTE POLYNOMIAL IN SO MANY PLACES?

Given a hyperplane arrangement $\mathcal{A}$ and a hyperplane H:

Deletion: $\mathcal{A} \setminus H = \mathcal{A} \setminus H$
Contraction: $\mathcal{A}/H = \{H' \cap H \,:\, H' \in \mathcal{A}\}$

A **Tutte-Grothendieck** invariant is a function which behaves well under deletion and contraction:

$$f(\mathcal{A}) = f(\mathcal{A} \setminus H) + f(\mathcal{A}/H) \qquad (H \text{ nontrivial})$$



Theorem. (Brylawski, 1972) The Tutte polynomial is the universal T-G invariant. Every other one is an evaluation of $T_{\mathcal{A}}(x, y)$.

3. COMPUTING TUTTE POLYNOMIALS

Finite field method.

Let $\overline{\chi}(q,t) = (t-1)^r \, T\left(\frac{q+t-1}{t-1}, t\right)$.

Theorem. Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement. Let q be a large enough prime power, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then

$$q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}$$

where $h(p) =$ number of hyperplanes of $\mathcal{A}_q$ that p lies on.

Computing Tutte polynomials is #P-hard, so we cannot expect miracles from this method. Still, it is often very useful.

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An example.

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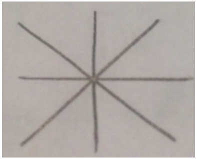


$$\begin{aligned}\overline{\chi}_{C_2}(q,t) &= \sum_{p \in \mathbb{F}_q^n} t^{h(p)} \\ &= t^4 + t^1[4(q-1)] + t^0[q^2 - 4q + 3]\end{aligned}$$

Corte de comerciales.

San Francisco State University – Colombia Combinatorics Initiative

Para más información sobre

- combinatoria enumerativa (el siguiente semestre),
- matroides,
- politopos,
- grupos de Coxeter,
- álgebra conmutativa combinatoria, y
- álgebras de Hopf en combinatoria,

pueden ver los (200+) videos y las notas de mis cursos de San Francisco State University y la Universidad de Los Andes:

http://math.sfsu.edu/federico/
http://youtube.com/user/federicoelmatematico

Decía que la aritmética...

4. THE ARITHMETIC TUTTE POLYNOMIAL.

Let $\mathcal{A} \subseteq \mathbb{Z}^n$ be a collection of vectors.

The **arithmetic Tutte polynomial** of $\mathcal{A}$ is

$$T_{\mathcal{A}}(x, y) = \sum_{\mathcal{B} \subseteq \mathcal{A}} m(B)(x-1)^{r(\mathcal{A})-r(\mathcal{B})}(y-1)^{|\mathcal{B}|-r(\mathcal{B})}.$$

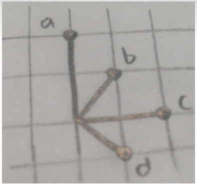
where, for each $\mathcal{B} \subseteq \mathcal{A}$, the **rank of** $\mathcal{B}$ is

$$r(\mathcal{B}) = \dim \operatorname{span}(\mathcal{B})$$

and the **multiplicity of** $\mathcal{B}$ is

$$m(\mathcal{B}) = \text{ index of } \mathbb{Z}\mathcal{B} \text{ inside } \mathbb{Z}^n \cap \operatorname{span}(\mathcal{B})$$

Example: $C_2 = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\} \subseteq \mathbb{Z}^2$



subset	rank	multiplicity	contribution
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$$\begin{aligned} M(x,y) &= 1(x-1)^2 + [2+2+1+1](x-1) + [4+2+2+ \\ &\quad +2+2+2] + [2+2+2+2](y-1) + 2(y-1)^3 \\ &= x^2 + 2y^2 + 4x + 4y + 3 \end{aligned}$$

5. HYPERTORIC ARRANGEMENTS

Let $\mathcal{T} = (\mathbb{K}^*)^n = (\mathbb{K} \setminus 0)^n$ be an n- torus.

For hypertoric arrangements:

Vector $a \in \mathbb{Z}^n \mapsto$ Hypertorus $T_a = \{x \in (\mathbb{K}^n)^* \,:\, x^a = 1\} \subset \mathcal{T}$.

Vector arr. $A \subseteq \mathbb{K}^n \mapsto$ Hyperplane arr. $\mathcal{T}_\mathcal{A} = \{H_a \,:\, a \in \mathcal{A}\}$

Complement $R(\mathcal{A}) = \mathcal{T} \setminus \bigcup_{T \in \mathcal{T}_\mathcal{A}} T$

Example.

C_2: $x^2 = 1,\ xy = 1,\ y^2 = 1,\ x/y = 1$

Several important invariants of $\mathcal{A}$ are evaluations of $M_{\mathcal{A}}(x, y)$.

For hypertoric arrangements:

- $(\mathbb{K} = \mathbb{R})$

  $(-1)^n M(1, 0)$ = number of regions of $R(\mathcal{A})$ in $\mathbb{S}_1^n$

  [Ehrenborg–Readdy–Sloane 2009, Moci, 2012]

- $(\mathbb{K} = \mathbb{C})$

  $$q^n M(2 + \tfrac{1}{q}, 0) = \sum_i \dim H^i(R(A); \mathbb{Z}) q^i$$

  [De Concini–Procesi 2005, Moci, 2012]

- $(\mathbb{K} = \mathbb{F}_{q+1}$, if $q \equiv 1 \pmod{N})$

  $$(-1)^n M(1 - q, 0) = |R(\mathcal{A})|$$

  [Bränden–Moci 2013, A.–Castillo–Henley 2013]

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Geometry:

1. [Stanley 1991] The **zonotope** of A is

$$Z(A) = \{\sum_{a \in A} \lambda_a \cdot a \,:\, 0 \leq \lambda \leq 1\}$$

- volume of $Z(A) = M(1, 1)$
- number of lattice points of $Z(A) = M(2, 1)$
- number of interior lattice points of $Z(A) = M(0, 1)$

Several important invariants of $\mathcal{A}$ are evaluations of $M_{\mathcal{A}}(x, y)$.

Box spline theory:

Numerical analysis $\bigcap$ Index theory $\bigcap$ Algebraic combinatorics

The Hilbert series of the **Dahmen – Micchelli space** and the **De Concini – Procesi – Vergne space** are:

- $\text{Hilb}(DM(A); q) = q^n M(\frac{1}{q}, 1)$. [Dahmen–Micchelli 1985]
- $\text{Hilb}(DPV(A); q) = q^n M(1 + \frac{1}{q}, 1)$ [A. 2012]

6. COMPUTING ARITHMETIC TUTTE POLYNOMIALS

Finite field method.

Let $\Psi(q,t) = (t-1)^r\, M\left(\frac{q+t-2}{t-1}, t\right).$

Theorem. [A. – Castillo – Henley 2012, Bränden – Moci 2012]
Let $\mathcal{A}$ be a toric arrangement. Let q be a large enough prime with $q \equiv 1 \pmod N$, and let $\mathcal{A}_q$ be the induced arrangement in $(\mathbb{F}_q^*)^n$. Then

$$q^{n-r}\Psi_{\mathcal{A}}(q,t) = \sum_{p \in (\mathbb{F}_q^*)^n} t^{h(p)}$$

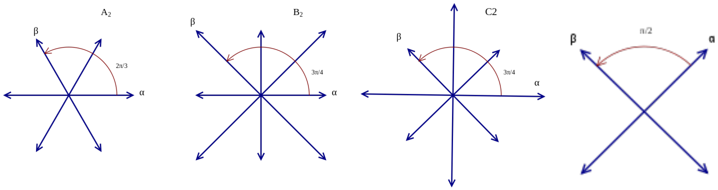
where $h(p)$ = number of hypertori of $\mathcal{A}_q$ that p lies on.

ROOT SYSTEMS

Root systems are arguably the most important vector configurations in mathematics. They are crucial in the classification of regular polytopes, simple Lie groups and Lie algebras, cluster algebras, etc.

“Classical root systems":

$$
\begin{aligned}
A_n^+ &= \{e_i - e_j \,:\, 1 \le i \le j \le n\} \\
B_n^+ &= \{e_i \pm e_j \,:\, 1 \le i \le j \le n\} \cup \{e_i \,:\, 1 \le i \le n\} \\
C_n^+ &= \{e_i \pm e_j \,:\, 1 \le i \le j \le n\} \cup \{2e_i \,:\, 1 \le i \le n\} \\
D_n^+ &= \{e_i \pm e_j \,:\, 1 \le i \le j \le n\}
\end{aligned}
$$



TUTTE POLYNOMIALS OF CLASSICAL ROOT SYSTEMS

We compute the (arithmetic) Tutte polynomials of A_n, B_n, C_n, D_n

Finite field method: Compute a(n) (arithmetic) Tutte polynomial by solving a counting problem over a finite field.

Example:

$$C_n : \quad x_i^2 = 1, \; x_i x_j = 1, \; x_i / x_j = 1$$

Count points $(p_1, \ldots, p_n) \in (\mathbb{F}_q^*)^n$ by the number of equations they satisfy. (Basic number theory, quadratic residues,...)

TUTTE POLYNOMIALS OF CLASSICAL ROOT SYSTEMS

The answers are best expressed in terms of the (arithmetic) **Tutte generating functions** $\Psi_A, \Psi_B, \Psi_C, \Psi_D$:

$$\Psi_A(x, y, z) = \sum_{n \geq 0} \Psi_{A_n}(x, y) \frac{z^n}{n!}$$

and the **two-variable Rogers-Ramanujan function**:

$$F(\alpha, \beta) = \sum_{n \geq 0} \frac{\alpha^n \beta^{\binom{n}{2}}}{n!}$$

Motivating example:

Theorem. [A. 2002] The Tutte generating function for the type A root systems is:

$$\Psi_A(x, y, z) = F(z, y)^x.$$

Similar (more complicated) formulas hold for types B, C, and D.

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TUTTE POLYNOMIALS OF CLASSICAL ROOT SYSTEMS

Theorem. [A.–Castillo–Henley 2013] The **arithmetic** Tutte generating functions for the classical root systems are:

$$\Psi_B = F(2Z,Y)^{\frac{X}{4}-1}F(Z,Y^2)F(YZ,Y^2)\left[F(2Z,Y)^{\frac{X}{4}}+F(-2Z,Y)^{\frac{X}{4}}\right]$$

$$\Psi_C = F(2Z,Y)^{\frac{X}{2}-1}F(YZ,Y^2)^2$$

$$\Psi_D = F(2Z,Y)^{\frac{X}{4}-1}F(Z,Y^2)^2\left[F(2Z,Y)^{\frac{X}{4}}+F(-2Z,Y)^{\frac{X}{4}}\right]$$

and

$$\Psi_A = \sum_{n\in\mathbb{N}}\varphi(n)\left(\left[F(Z,Y)F(\omega_n Z,Y)F(\omega_n^2 Z,Y)\cdots F(\omega_n^{n-1}Z,Y)\right]^{X/n}-1\right)$$

where $\varphi(n) = \#\{m \in \mathbb{N} : 1 \leq m \leq n, (m,n) = 1\}$ is Euler's totient function and ω_n is a primitive nth root of unity for each n.

Corollary. Formulas for zonotopes, DM and DPV-spaces, etc.

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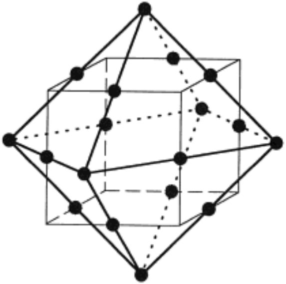
muchas gracias



El artículo está en:

http://arxiv.org/abs/1305.6621
http://math.sfsu.edu/federico