Severi degrees of toric surfaces are eventually polynomial

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November 24, 2010

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0. Outline.

1. Motivation.

– Enumerative geometry.

– Counting plane curves.

2. Our problem.

– Counting curves on toric surfaces.

– The main theorem.

3. The proof.

– Tropical curves.

– Floor diagrams.

– Template decomposition.

– Swap encoding.

– Discrete integrals over polytopes.

4. Comments, directions.

1. Motivation: Enumerative geometry

Enumerative Geometry:
Counting algebro-geometric objects with certain properties.

- "Traditional" technique: **Intersection theory**.

\- Form a moduli space of objects you wish to count
\- Compute in the cohomology ring of that moduli space

(k-planes $\rightarrow$ *Grassmannian* $\rightarrow$ $H^*(Gr_{n,k}) \rightarrow$ *Schubert calc.* $\rightarrow$ *Littlewood-Richardson rule* $\rightarrow$ *...)*

- "New" technique: **Tropical geometry**.

\- "Just" tropicalize! $\rightarrow$ Define tropical objects and count them instead.
\- Actually count them!

(plane curves $\rightarrow$ *tropical curves* $\rightarrow$ *floor diagrams* $\rightarrow$ *...)*

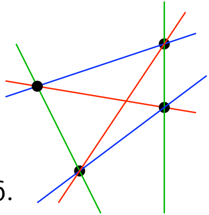
This talk: Enumeration of curves on toric surfaces via tropical geometry.

1. Motivation: Counting plane curves.

Question

How many (possibly reducible) nodal algebraic curves in $\mathbb{CP}^2$ have degree d, δ nodes, and pass through $\frac{(d+3)d}{2} - \delta$ generic points?

Number of such curves is the *Severi degree* $N^{d,\delta}$.



$N^{1,0} = \#\{\text{lines through 2 points}\} = 1.$

$N^{2,1} = \#\{\text{1-nodal conics through 4 points}\} = 3$

$N^{4,4} = \#\{\text{4-nodal quartics through 10 points}\} = 666.$

If $d \geq \delta + 2$, such a curve is irreducible by Bézout's Theorem, so

$N^{d,\delta} =$ *Gromov-Witten invariant* $N_{d,g}$, with $g = \frac{(d-1)(d-2)}{2} - \delta$.

1. Motivation: Counting plane curves.

Theorem (Fomin–Mikhalkin, 2009)

For a fixed δ, we have

$$N^{d,\delta} = N_\delta(d),$$

for a combinatorially defined polynomial $N_\delta(d) \in \mathbb{Q}[d]$, provided $d \geq 2\delta$.

Conjectured by Di Francesco– Itzykson (1994) and by Göttsche (1997).

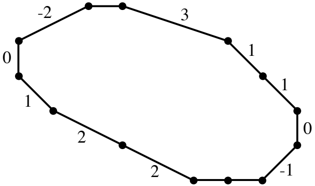
- J. Steiner (1848): $N^{d,1} = 3(d-1)^2$
- A. Cayley (1863): $N^{d,2} = \frac{3}{2}(d-1)(d-2)(3d^2-3d-11)$
- S. Roberts (1867): $\delta = 3$
- I. Vainsencher (1995): $\delta = 4, 5, 6$
- S. Kleiman–R. Piene (2001): $\delta = 7, 8$
- F. Block (2010): $\delta = 9, 10, 11, 12, 13, 14$

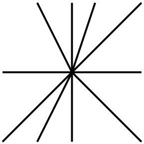
2. What we do: Counting curves on toric surfaces.

Goal: Study the analogous problem for toric surfaces.

Given a lattice polygon $\Delta = \Delta(\mathbf{c}, \mathbf{d})$, get: (**c**=slopes, **d**=lengths) :

- a (projective) toric surface $S = \mathrm{Tor}(\mathbf{c})$. (think: **c**=surface)
- a(n ample) line bundle $\mathcal{L} = \mathcal{L}(\mathbf{c}, \mathbf{d})$ on S (think: **d**="multidegree")
- a (complete) linear system $|\mathcal{L}|$





The Severi degree $N^{\Delta,\delta}$ is the answer to:

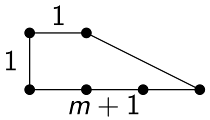
Question

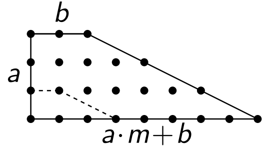
- *How many nodal algebraic curves in $|\mathcal{L}|$ have δ nodes, and pass through a generic set of $|\Delta \cap \mathbb{Z}^2| - 1$ points in S?*
- *How many curves in the torus $(\mathbb{C}^*)^2$ have δ nodes, Newton polygon Δ, and pass through $|\Delta \cap \mathbb{Z}^2| - 1$ generic points in $(\mathbb{C}^*)^2$?*

2. What we do: Counting curves on toric surfaces.

Goal: Study the Severi degrees $N^{\Delta,\delta}$ for polygons (toric surfaces) Δ.

Ex. Σ_m = Hirzebruch surface, (a, b) = bidegree





$N^{\Delta,\delta} =$ # curves in Σ_m of bi-degree (a, b) with δ nodes through an appropriate number of generic points

We find:

$$
\begin{aligned}
N^{\Delta,1} &= 1, \\
N^{\Delta,2} &= 3bm^2 + 6ab - 2bm - 4a - 4b + 4, \\
N^{\Delta,3} &= \frac{9}{2}b^4m^2 - 6b^3m^2 + 18abm^3 + 18a^2b^2 - 24abm^2 \\
\vdots \quad & \quad +2b^2m^2 - 12bm^3 - 24a^2b
\end{aligned}
$$

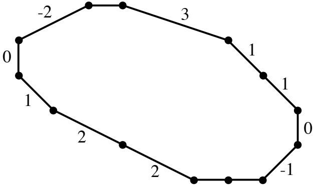
Note: Severi degrees are polynomial in a and b, **and also in** m!

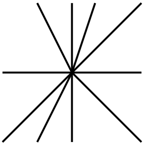
Our main result.

Assume $\Delta(\mathbf{c}, \mathbf{d})$ is *h-transverse*: (slopes of normal fan are integers).

$\mathbf{c} = ((3, 1, 0, -1); (-2, 0, 1, 2))$
(slopes)

$\mathbf{d} = ((1, 2, 1, 1); (1, 1, 1, 2); 1)$
(lengths)





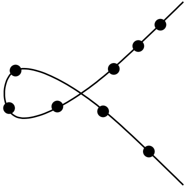
Theorem (Polynomiality of Severi degrees) (A. - Block, 2010)

Fix $m, n \geq 1$ and $\delta \geq 1$. There exists a universal and combinatorially defined polynomial $p_\delta(\mathbf{c}, \mathbf{d})$ such that the Severi degree $N^{\Delta,\delta}$ is given by

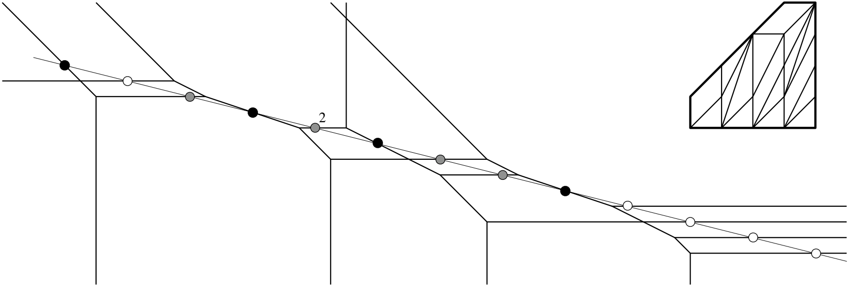
$$N^{\Delta,\delta} = p_\delta(\mathbf{c}, \mathbf{d}). \tag{1}$$

for any sufficiently large and spread out $\mathbf{c} \in \mathbb{Z}^{m+n}$ and $\mathbf{d} \in \mathbb{Z}^{m+n+1}_{\geq 0}$.

Mikhalkin's Correspondence Theorem (2005)
replaces enumeration of algebraic curves on $|\mathcal{L}|$:

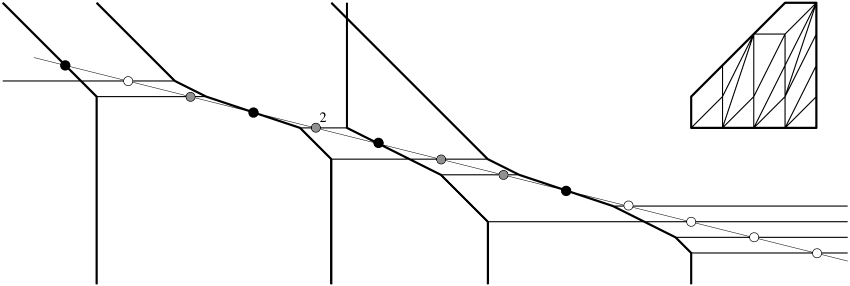


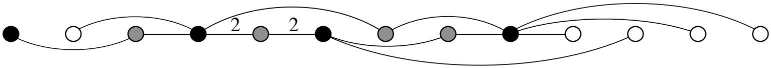
by weighted enumeration of tropical curves dual to subdivisions of Δ:



3.2. Proof: Floor diagrams

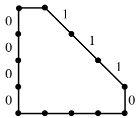
Mikhalkin-Brugalle-Fomin ('07, '09) replace enum. of "Δ–tropical curves" with enum. of combinatorial gadgets, (marked) Δ–floor diagrams.

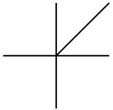




3.2. Proof: Floor diagrams

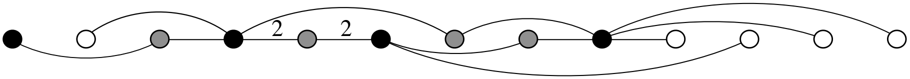
D_l, D_r: left and right slopes (with multiplicities)
t: top length
$D_l = \{0,0,0,0\}, D_r = \{1,1,1,0\}, \ t = 1.$





Marked Δ-floor diagram:

- permutations $\mathbf{l}, \mathbf{r}$ of D_l, D_r
- weak ordered partition $\mathbf{s}$ of t
- a bipartite graph on ({●, …, ●}, {◍, …, ◍, ○, …, ○}), linearly ordered and directed L-to-R, with $\mathbb{Z}^+$ edge weights such that:
  - ○s are sinks, ◍s have one incoming and one outgoing edge,
  - div(◍) = 0, div(○) = -1, div(●$_i$) $\leq (\mathbf{r} - \mathbf{l} + \mathbf{s})_i$.



$\mathbf{r} = (1,1,0,1), \mathbf{l} = (0,0,0,0), \mathbf{s} = (0,1,0,0), \mathbf{r} - \mathbf{l} + \mathbf{s} = (1,2,0,1)$

3.2. Proof: Floor diagrams.

Theorem (Brugallé–Mikhalkin 2007)

$$N^{\Delta,\delta} = \sum_{\mathcal{D}} weight(\mathcal{D}),$$

summing over all marked Δ-floor diagrams $\mathcal{D}$ with d **black** *vertices, and "cogenus" $\delta(\mathcal{D}) := \frac{d(d-3)}{2} + b_0(\mathcal{D}) - b_1(\mathcal{D}) = \delta$.*

weight$(\mathcal{D}) = \prod_{e \text{ edge}}$ weight(e)
$b_0(\mathcal{D})$ = number of connected components of $\mathcal{D}$.
$b_1(\mathcal{D})$ = sum of the genera of the connected components of $\mathcal{D}$.

3.2. Proof: Floor diagrams.

Theorem (Brugallé–Mikhalkin 2007)

$$N^{\Delta,\delta} = \sum_{\delta(\mathcal{D})=\delta} weight(\mathcal{D}),$$

*summing over all marked Δ-floor diagrams $\mathcal{D}$ with d **black** vertices, and "cogenus" $\delta(\mathcal{D}) := \frac{d(d-3)}{2} + b_0(\mathcal{D}) - b_1(\mathcal{D}) = \delta$.*

Ok, this is combinatorial, but unmanageable in many ways!
Some of the difficulties:

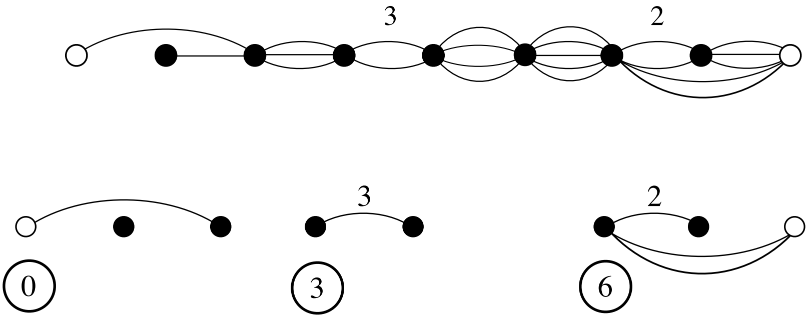
- **l**, **r** permutations of sets of a variable length
- **s** partition of a variable number
- floor diagrams are hard to control - number of vertices isn't even fixed.

3.3. Proof: Template decomposition

Decompose marked floor diagrams into templates:

– Drop gray vertices, merge white vertices into a top and a bottom one.
– Drop "short" edges and isolated black vertices.
– Keep track of starting points of templates.

This process is reversible.



3.3. Proof: Template decomposition

Brugalle - Mikhalkin: $N^{\Delta,\delta} = \sum \text{weight}(\mathcal{D})$

Δ-floor diagram $\mathcal{D}$ $\rightarrow$
– permutations $\mathbf{l}, \mathbf{r}$ of D_l, D_r
– templates $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$
– starting points $\mathbf{k} = (k_1, \ldots, k_m)$.

Theorem (Fomin–Mikhalkin 2009 (plane curves) / A.–Block 2010)

$$N^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} \sum_{\mathbf{\Gamma}\,:\,\delta(\mathbf{\Gamma})=\delta-\delta(\mathbf{l},\mathbf{r})} \sum_{\mathbf{k}\in A(\mathbf{\Gamma},\mathbf{a})} P_{\mathbf{\Gamma}}(\mathbf{a},\mathbf{k})\mathit{weight}(\mathbf{\Gamma})$$

– # of preimages: combinatorially defined function $P_{\mathbf{\Gamma}}(\mathbf{a}, \mathbf{k})$.
– cogenus gets distributed: $\delta(\Delta) = \delta(\mathbf{\Gamma}) + \delta(\mathbf{l}, \mathbf{r})$,
– $A(\mathbf{\Gamma}, \mathbf{a})$ = set of possible $\mathbf{k}$ (depends on $\mathbf{\Gamma}$ and $\mathbf{a} := \mathbf{r} - \mathbf{l} + \mathbf{s}$).

$\rightarrow$ The set of possible $\mathbf{\Gamma}$s is now finite.
$\rightarrow$ Still, the set of possible $(\mathbf{l}, \mathbf{r})$ is troublesome.

3.4. Proof: Swap encoding

$$N^{\Delta,\delta} = \sum_{(\mathbf{l},\mathbf{r})} \sum_{\mathbf{\Gamma} : \delta(\mathbf{\Gamma}) = \delta - \delta(\mathbf{l},\mathbf{r})} \sum_{\mathbf{k} \in A(\mathbf{\Gamma},\mathbf{a})} P_{\mathbf{\Gamma}}(\mathbf{a},\mathbf{k}) \mathrm{weight}(\mathbf{\Gamma})$$

Problem: $\mathbf{l}, \mathbf{r}$ are permuts. of D_l, D_r, where each c_i appears d_i times.
In principle, exponentially many of them. What limits them:

$$\delta(\mathbf{l},\mathbf{r}) = \textstyle\sum_{(i,j) \in Rev(\mathbf{r})} (r_j - r_i) + \sum_{(i,j) \in Rev(-\mathbf{l})} (l_i - l_j) < \delta.$$

For simplicity of notation, let's assume $D_l = \{0, ..., 0\}$.
Say $\mathbf{c} = (5, 3, 2, 1), \mathbf{d} = (4, 6, 4, 3)$, so $D_r = \{5^4, 3^6, 2^4, 1^3\}$.
A permutation: $\mathbf{r} = 55335353233212121$

Key observations:
– If all $d_i > \delta$, there are no reversals of c_i and c_j $(i \geq j + 2)$.
– If all $c_i - c_{i+1} > \delta$, there are no reversals at all.

3.4. Proof: Swap encoding

Let's assume $d_i > \delta$. Then just record the reversals of c_i and c_{i+1}.

Say $\mathbf{c} = (5,3,2,1), \mathbf{d} = (4,6,4,3)$, $\mathbf{r} = 55335353233212121$

$\mathbf{r}$	$=$	5	5	3	3	5	3	5	3	2	3	3	2	1	2	1	2	1
1	:	-1	-1	1	1	-1	1	-1	1		1	1						
2	:			-1	-1		-1		-1	1	-1	-1	1		1		1	
3	:									-1			-1	1	-1	1	-1	1

$$\pi_1 = (1,1,-1,1,-1), \quad \pi_2 = (1,-1,-1), \quad \pi_3 = (1,-1,1,-1)$$

– $\delta(\mathbf{r}) = \sum_{Rev(\mathbf{r})}(r_j - r_i) = \sum_i \mathrm{inv}(\pi_i)(c_i - c_{i+1})$ only depends on $\mathbf{c}$ and π.
– There are finitely many $\pi = (\pi_1, \ldots, \pi_k)$ with $\delta(\mathbf{r}) \leq \delta$.

$$N^{\Delta,\delta} = \sum_{\pi} \sum_{\Gamma:\ \delta(\Gamma)=\delta-\delta(\mathbf{l},\mathbf{r})} \left(\sum_{\mathbf{k}\in A(\Gamma,\mathbf{a})\cap\mathbb{Z}^m} P_\Gamma(\mathbf{a},\mathbf{k}) \right) \mathrm{weight}(\Gamma)$$

Now the first two sums are finite! Lastly, we need to deal with $(\sum P_\Gamma)$.

3.5. Proof: Discrete integrals over polytopes

Goal:

$$f^{\pi,\mathbf{\Gamma}}(\mathbf{a}) = \sum_{\mathbf{k} \in A(\mathbf{\Gamma},\mathbf{a}) \cap \mathbb{Z}^m} P_{\mathbf{\Gamma}}(\mathbf{a}, \mathbf{k}) \text{ is polynomial in } \mathbf{c}, \mathbf{d},$$

where $\mathbf{a} = \mathbf{r} - \mathbf{l} + \mathbf{s}$ and $\mathbf{r}, \mathbf{l}$ are encoded by π.
Easy: $\mathbf{a}$ is piecewise linear in $\mathbf{c}$ and $\mathbf{d}$.

Recall: (marked floor diagram $\mathcal{D}$) $\mapsto$ (templates $\mathbf{\Gamma}$, starting points $\mathbf{k}$)

– $P_{\mathbf{\Gamma}}(\mathbf{a}, \mathbf{k})$ is the number of preimages of $(\mathbf{\Gamma}, \mathbf{k})$.
– $A(\mathbf{\Gamma}, \mathbf{a})$ is the set of possible $\mathbf{k}$ for a given $\mathbf{\Gamma}$

It turns out that

– $P_{\mathbf{\Gamma}}(\mathbf{a}, \mathbf{k})$ is a product of binomial coeffs, polynomial in $\mathbf{a}$.
– $A(\mathbf{\Gamma}, \mathbf{a})$ is generally unmanageable. But if $\mathbf{d}$ is large enough, it is a polytope with fixed facet directions and variable facet parameters.

Lemma

$$f^{\pi,\mathbf{\Gamma}}(\mathbf{a}) = \sum_{\mathbf{k}\in A(\mathbf{\Gamma},\mathbf{a})\cap\mathbb{Z}^m} P_{\mathbf{\Gamma}}(\mathbf{a},\mathbf{k})$$ *is polynomial in* **c** *and* **d**

– $P_{\mathbf{\Gamma}}(\mathbf{a},\mathbf{k})$: polynomial in **a**.
– $A(\mathbf{\Gamma},\mathbf{a})$: polytope with fixed facet directions and variable parameters.

Then $f^{\pi,\mathbf{\Gamma}}(\mathbf{a})$ is the discrete integral of a polynomial over a polytope, so it is piecewise quasipolynomial in **a** (and hence in **c** and **d**).

piecewise?
An intricate analysis shows we stay in one region of quasipolynomiality.

quasi?
$P_{\mathbf{\Gamma}}(\mathbf{a},\mathbf{k})$ is a Delzant polytope, so we actually get a polynomial. □

3. The main result, again.

Theorem (Polynomiality of Severi degrees) (A.-Block, 2010.)

Fix $m, n \geq 1$ and $\delta \geq 1$ and an h-transverse toric surface $Tor(\mathbf{c})$. There exists a combinatorially defined polynomial $p_{\delta,\mathbf{c}}(\mathbf{d})$ such that the Severi degree $N^{\Delta,\delta}$ is given by

$$N^{\Delta,\delta} = p_{\delta,\mathbf{c}}(\mathbf{d}).$$

for any sufficiently large and spread out $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{m+n+1}$.

With a bit more work,

Theorem (Universal Polynomiality) (A.-Block, 2010.)

Fix $m, n \geq 1$ and $\delta \geq 1$. There is a universal and combinatorially defined polynomial $p_\delta(\mathbf{c}, \mathbf{d})$ such that the Severi degree $N^{\Delta,\delta}$ of any h-transverse toric surface is given by

$$N^{\Delta,\delta} = p_\delta(\mathbf{c}, \mathbf{d}).$$

for any sufficiently large and spread out $\mathbf{c} \in \mathbb{Z}^{m+n}$ and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{m+n+1}$.

4. Comments.

• Only for *h*-transverse polygons? Probably not.
We inherit this restriction from Brugalle-Mikhalkin (2007):
(count curves) $\rightarrow$ (count tropical curves) $\rightarrow$ (count floor diagrams)
Problem: Modify floor diagrams to make this work in general.

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• Only for large $\mathbf{c}$ and $\mathbf{d}$? Yes.
But the thresholds can probably be improved.

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(count curves) $\rightarrow$ (count tropical curves) $\rightarrow$ (count floor diagrams)
Problem: Modify floor diagrams to make this work in general.

• Only for large $\mathbf{c}$ and $\mathbf{d}$? Yes.
But the thresholds can probably be improved.

• Only for toric surfaces? No.
Göttsche's Conjecture (1998) / Tzeng's Theorem (2010):
For any smooth surface S and $(5\delta - 1)$-very ample line bundle $\mathcal{L}$,

$$N^\delta = T_\delta(c_1(\mathcal{L})^2, c_1(\mathcal{L}) \cdot c_1(K_S), c_1(K_S)^2, c_2(T_S)),$$

for a universal polynomial T_δ. (K_S, T_S : canonical, tangent bundle of S.)
$\rightarrow$ Our surfaces are almost never smooth.
$\rightarrow$ The Chern class $c_2(T_S)$ is not polynomial in $\mathbf{c}$ and $\mathbf{d}$. The others are.

There is surely much more to this story...

Thank you.

The slides are available upon request.

The paper will be posted in the next few days to the arXiv and to:

http://math.sfsu.edu/federico