The Hopf monoid of generalized permutahedra

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The plan.

1. Species.

2. Hopf monoids.

- Product, coproduct.
- Three important examples: posets, graphs, matroids.
- Antipode, characters, and invariants.

3. Generalized permutahedra.

- The Hopf monoid: product, coproduct.
- Restriction to posets, graphs, and matroids.
- The antipode and reciprocity theorems.

1. Species.

(Joyal) A species P consists of:

- For each finite set I, a vector space $\mathsf{P}[I]$.
- For each bijection $\sigma : I \to J$, a linear map

$$\mathsf{P}[\sigma] : \mathsf{P}[I] \to \mathsf{P}[J]$$

  such that

$$\mathsf{P}[\sigma \circ \tau] = \mathsf{P}[\sigma] \circ \mathsf{P}[\tau] \quad \text{and} \quad \mathsf{P}[\mathrm{id}] = \mathrm{id}.$$

Think: ways of putting a certain combinatorial structure on a set I.

$\mathsf{P}(I) = \text{span}\{\text{combinatorial structures on } I\}$

Examples.

$\mathsf{Q}[I] := \text{span}\{\text{posets on } I\}$.
$\mathsf{G}[I] := \text{span}\{\text{graphs on vertex set } I\}$.
$\mathsf{M}[I] := \text{span}\{\text{matroids on ground set } I\}$.

2. Hopf monoids.

(Aguiar-Mahajan, *Monoidal functors, species, and Hopf algebras.*)

A (connected) Hopf monoid $(\mathrm{P}, \mu, \Delta)$ consists of:

- A species P.
- For each $I = S \sqcup T$, product and coproduct maps

$$\mathrm{P}[S] \otimes \mathrm{P}[T] \xrightarrow{\mu_{S,T}} \mathrm{P}[I] \quad \text{and} \quad \mathrm{P}[I] \xrightarrow{\Delta_{S,T}} \mathrm{P}[S] \otimes \mathrm{P}[T].$$

- Two inverse maps

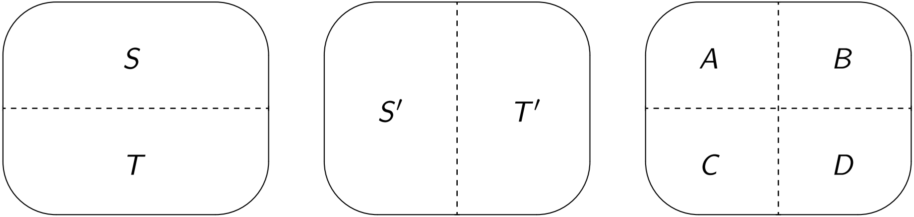
$$\Bbbk \longrightarrow \mathrm{P}[\emptyset] \quad \text{and} \quad \mathrm{P}[\emptyset] \longrightarrow \Bbbk.$$

Think: rules for "merging" and "breaking" our structures.

These maps should satisfy various axioms, including the following:

2. Hopf monoids. Compatibility axiom.

Fix decompositions $S \sqcup T = I = S' \sqcup T'$, and let A, B, C, D be:



Then this diagram must commute:

$$\begin{array}{ccc}
P[A] \otimes P[B] \otimes P[C] \otimes P[D] & \xrightarrow{\mathrm{id} \otimes \mathrm{switch} \otimes \mathrm{id}} & P[A] \otimes P[C] \otimes P[B] \otimes P[D] \\
\uparrow {\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & \downarrow {\scriptstyle \mu_{A,C} \otimes \mu_{B,D}} \\
P[S] \otimes P[T] \xrightarrow[\mu_{S,T}]{} & P[I] & \xrightarrow[\Delta_{S',T'}]{} P[S'] \otimes P[T']
\end{array}$$

2. Hopf monoids. The Hopf monoid of posets.

$Q[I] :=$ vector space with basis the set of posets on I.

The species of posets Q is a Hopf monoid, under:

Product: direct sum.

$$\begin{aligned} \mu_{S,T} : Q[S] \otimes Q[T] &\longrightarrow Q[I] \\ q_S \otimes q_T &\longmapsto q_S \oplus q_T \end{aligned}$$

Coproduct: splitting.

$$\Delta_{S,T} : Q[I] \longrightarrow Q[S] \otimes Q[T]$$

$$q \longmapsto \begin{cases} q|_S \otimes q|_T & \text{if } S \text{ is an order ideal of } q \\ 0 & \text{otherwise} \end{cases}$$

2. Hopf monoids. The Hopf monoid of matroids.

$\mathrm{M}[I] :=$ vector space with basis the set of matroids on I.

The species of matroids M is a Hopf monoid with

$$\mathrm{M}[S] \otimes \mathrm{M}[T] \xrightarrow{\mu_{S,T}} \mathrm{M}[I] \qquad\qquad \mathrm{M}[I] \xrightarrow{\Delta_{S,T}} \mathrm{M}[S] \otimes \mathrm{M}[T]$$

$$m_1 \otimes m_2 \longmapsto m_1 \oplus m_2 \qquad\qquad m \longmapsto m|_S \otimes m/_S$$

where

$$\begin{aligned} m_1 \oplus m_2 &= \text{direct sum of } m_1 \text{ and } m_2, \\ m|_S &= \text{restriction of } m \text{ to } S, \\ m/_S &= \text{contraction of } S \text{ from } m. \end{aligned}$$

Recall: A matroid on I is a collection $\mathcal{B}$ of r-subsets ("bases") such that:

If $A, B \in \mathcal{B}$ and $a \in A - B$, there exists $b \in B - A$ with $A - a \cup b \in \mathcal{B}$.

Prototypical example:

I = collection of vectors in V, $\mathcal{B}$ = subsets of I which are bases of V

2. Hopf monoids. The Hopf monoid of graphs.

$\mathrm{G}[I] :=$ vector space with basis
the set of graphs (with half edges) with vertex set I.

The species of graphs G is a Hopf monoid with

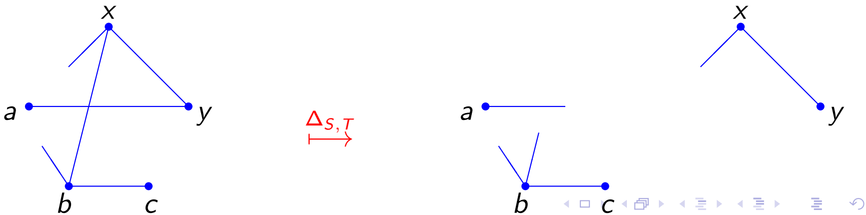
$$\mathrm{G}[S] \otimes \mathrm{G}[T] \xrightarrow{\mu_{S,T}} \mathrm{G}[I] \qquad\qquad \mathrm{G}[I] \xrightarrow{\Delta_{S,T}} \mathrm{G}[S] \otimes \mathrm{G}[T]$$

$$g_1 \otimes g_2 \longmapsto g_1 \sqcup g_2 \qquad\qquad g \longmapsto g|_S \otimes g/_S$$

where

$$\begin{aligned} g_1 \sqcup g_2 &= \text{disjoint union of } g_1 \text{ and } g_2, \\ g|_S &= \text{keep everything incident to } S, \\ g/_S &= \text{remove everything incident to } S. \end{aligned}$$

$I = \{a, b, c, x, y\}$ $\qquad S = \{a, b, c\}$ $\qquad T = \{x, y\}$



2. Hopf monoids. Antipode.

The antipode of a connected Hopf monoid P consists of the maps

$$\mathrm{s}_I : \mathrm{P}[I] \to \mathrm{P}[I]$$

given by

$$\mathrm{s}_I = \sum_{\substack{S_1,\dots,S_k \\ k\geq 1}} (-1)^k \, \mu_{S_1,\dots,S_k} \circ \Delta_{S_1,\dots,S_k}.$$

The sum is over all *ordered* decompositions

$$I = S_1 \sqcup \cdots \sqcup S_k$$

into *nonempty disjoint* subsets. For each such decomposition,

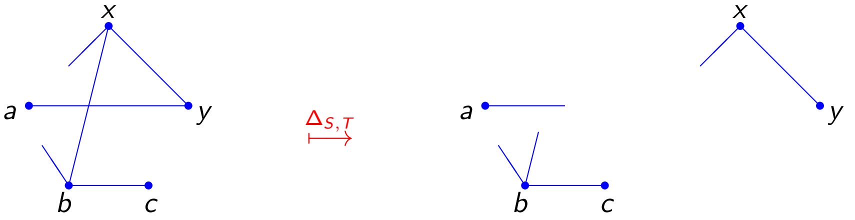
$$\mathrm{P}[I] \xrightarrow{\Delta_{S_1,\dots,S_k}} \mathrm{P}[S_1] \otimes \cdots \otimes \mathrm{P}[S_k] \xrightarrow{\mu_{S_1,\dots,S_k}} \mathrm{P}[I]$$

General problem. Find a simple formula for the antipode of a Hopf monoid.

(Very often there is **much** cancellation in the definition above.)

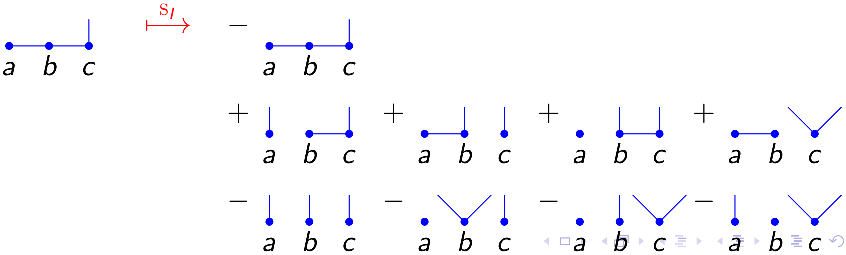
2. Hopf monoids. Antipode for graphs.

$I = \{a, b, c, x, y\}$ $\qquad$ $S = \{a, b, c\}$ $\qquad$ $T = \{x, y\}$



$g \qquad g|_S \qquad g/_S$

Ex. For $n = 3$, we expect 13 terms (ordered Bell number), but



2. Hopf monoids. Characters.

Let P be a Hopf monoid. A character ζ consists of maps

$$\zeta_I : \mathrm{P}[I] \to \Bbbk$$

such that for each $I = S \sqcup T$,

$$\begin{array}{ccc} \mathrm{P}[S] \otimes \mathrm{P}[T] & \xrightarrow{\mu_{S,T}} & \mathrm{P}[I] \\ \downarrow{\scriptstyle \zeta_S \otimes \zeta_T} & & \downarrow{\scriptstyle \zeta_I} \\ \Bbbk \otimes \Bbbk & \xrightarrow[\cong]{} & \Bbbk \end{array}$$

and

$$\mathrm{P}[\emptyset] \xrightarrow{\zeta_\emptyset = \mathrm{id}} \Bbbk.$$

2. Hopf monoids. Polynomial invariants.

Let P be a Hopf monoid and ζ a character.
Define, for each $x \in \mathrm{P}[I]$ and $n \in \mathbb{N}$,

$$\chi_I(x)(n) \;:=\; \sum_{S_1 \sqcup \cdots \sqcup S_n = I} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\dots,S_n}(x).$$

Sum over all ordered decompositions of I into n disjoint subsets.

Proposition.

1. $\chi_I(x)$ is a polynomial function of n.
2. $\chi_I(x)(1) = \zeta_I(x)$.
3. $\chi_I(x)(-1) = \zeta_I\Big(\mathrm{s}(x)\Big)$. ($\to$ reciprocity theorems)

The function $\chi_I(x)$ is a polynomial invariant of the structure x (canonically associated to P and ζ).

2. Hopf monoids. Invariants of graphs and matroids.

- Let $\zeta_I : \mathrm{G}[I] \to \Bbbk$ be
$$\zeta_I(g) := \begin{cases} 1 & \text{if } g \text{ consists of half-edges only,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(g) =$ chromatic polynomial of g.
(For $n \in \mathbb{N}$, counts proper colorings of g with $[n]$.)

- Let $\zeta_I : \mathrm{M}[I] \to \Bbbk$ be
$$\zeta_I(m) := \begin{cases} 1 & \text{if } m \text{ has a unique basis,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(m) =$ polynomial defined by Billera-Jia-Reiner (2006).
(For $n \in \mathbb{N}$, counts m-generic functions $f : I \to [n]$.)

- Let $\zeta_I : \mathrm{Q}[I] \to \Bbbk$ be
$$\zeta_I(q) := \begin{cases} 1 & \text{if } q \text{ is an antichain} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(q) =$ (strict) order polynomial of Stanley (1970 or 1971).
(For $n \in \mathbb{N}$, counts order-preserving labelings of q with $[n]$.)

3. Generalized permutahedra.

Euclidean space $\mathbb{R}^I := \{\text{functions } x : I \to \mathbb{R}\}$.

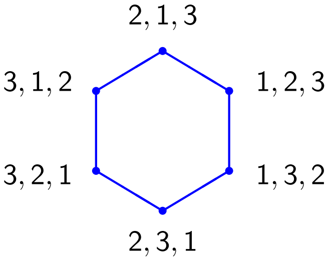
The standard permutahedron

$$\pi_I := \text{Convex Hull}\{\text{bijective functions } x : I \to [n]\} \subseteq \mathbb{R}^I$$

(where $n = |I|$).

$I = \{a, b, c\}$ $\qquad$ $\{x_a + x_b + x_c = 6\}$

$\pi_I =$

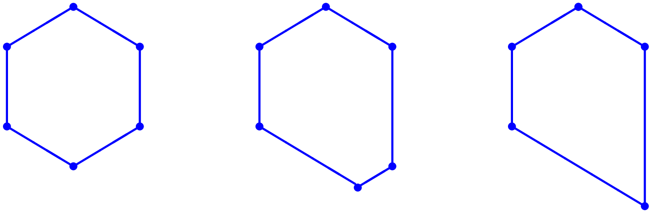


3. Generalized permutahedra.

- Postnikov (2005).
- Postnikov, Reiner, Williams (2007).
- A., Benedetti, Doker (2008).

Move vertices of π_I keeping new edges parallel to old ones.

Generalized permutahedra



3. Generalized permutahedra. Restriction and contraction.

Given a polytope $P \subseteq \mathbb{R}^I$ and $v \in \mathbb{R}^I$, let

$$P_v \ := \ \text{face of } P \text{ where } \langle v, - \rangle \text{ is maximum.}$$

Given $I = S \sqcup T$, let $v_{S,T} \in \mathbb{R}^I$ be any vector such that

$$\begin{cases} v_i = v_j & \text{if } i, j \in S \text{ or } i, j \in T, \\ v_i > v_j & \text{if } i \in S \text{ and } j \in T. \end{cases}$$

Proposition. (AA, 08) Let P be a generalized permutahedron.

1. $P_{v_{S,T}} =: P_{S,T}$ depends only on S and T.
2. There are generalized permutahedra $P_1 \subseteq \mathbb{R}^S$ and $P_2 \subseteq \mathbb{R}^T$ such that $P_{S,T} = P_1 \times P_2$. (Note $\mathbb{R}^I = \mathbb{R}^S \times \mathbb{R}^T$.)

Define

$$P|_S \ := \ P_1 \subseteq \mathbb{R}^S, \qquad P/_S \ := \ P_2 \subseteq \mathbb{R}^T.$$

3. Generalized permutahedra. The Hopf monoid.

GP$[I]$:= vector space with basis the set of generalized permutahedra in $\mathbb{R}^I$.

> **Theorem.** (AA, 08) The species of generalized permutahedra GP is a Hopf monoid with:

$$\mathrm{GP}[S] \otimes \mathrm{GP}[T] \xrightarrow{\mu_{S,T}} \mathrm{GP}[I] \qquad \mathrm{GP}[I] \xrightarrow{\Delta_{S,T}} \mathrm{GP}[S] \otimes \mathrm{GP}[T]$$

$$P \otimes Q \longmapsto P \times Q \qquad P \longmapsto P|_S \otimes P/_S$$

> **Theorem.** (AA, 08)
> GP is a Hopf monoid.

Let $\zeta_I : \mathrm{GP}[I] \to \Bbbk$ be

$$\zeta_I(P) := \begin{cases} 1 & \text{if } P \text{ is a point,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_I(P) =$ polynomial defined by Billera-Jia-Reiner (2006).

3. Generalized permutahedra. Graphs and matroids.

There are $\mathrm{G}[I] \to \mathrm{GP}[I]$ and $\mathrm{M}[I] \to \mathrm{GP}[I]$ and $\mathrm{Q}[I] \to \mathrm{GP}[I]$.

G graph $\to$ graphic zonotope $Z(G)$

$$Z(G) = \sum_{ij \in G} (e_i - e_j) = \{\sum_{ij \in G} a_{ij}(e_i - e_j) : 0 \leq a_{ij} \leq 1\}.$$

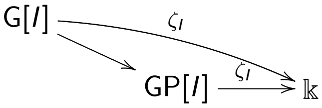
M matroid $\to$ matroid polytope P_M

$$P_M = \mathrm{conv}\{e_{i_1} + \cdots + e_{i_k} \mid \{i_1, \ldots, i_k\} \in \mathcal{B}\}.$$

Q poset $\to$ poset polyhedron P_Q

$$P_Q : \sum x_i = 0; \sum_{i \in A} x_i < 0 \text{ for order ideals } A \subset Q.$$

Proposition. (AA, 08) These are morphisms of Hopf monoids. In addition, the diagram



commutes, and similarly for $\mathrm{M}[I]$ and $\mathrm{Q}[I]$.

3. Generalized permutahedra. The antipode.

Theorem. Let P be a generalized permutahedron.

$$\mathrm{s}_I(P) \;=\; (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim Q}\, Q.$$

The sum is over all faces Q of P.

Proof. Antipode formula gives a big alternating sum of faces of P:

$$\mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(P) \;=\; P_{S_1,\ldots,S_k},$$

with lots of repeated terms. The coefficient of each face is the reduced Euler characteristic of a sphere. □

Note: This is the best possible formula. No cancellation at all! (One advantage of working over species...)

3. Generalized permutahedra. Reciprocity theorems .

Corollary.

$$\chi_I(P)(-1) = (-1)^{|I|}\,\#\{\text{vertices of } P\}.$$

Some instances:

Graphs: $\chi_I(g)$ = chromatic polynomial

$\chi_I(g)(-1) = (-1)^{|I|}\,\#\{\text{acyclic orientations of } g\}$ (Stanley).

Matroids: $\chi_M(g)$ = BJR matroid polynomial

$\chi_I(m)(-1) = (-1)^{|I|}\,\#\{\text{bases of } m\}$ (Billera-Jia-Reiner).

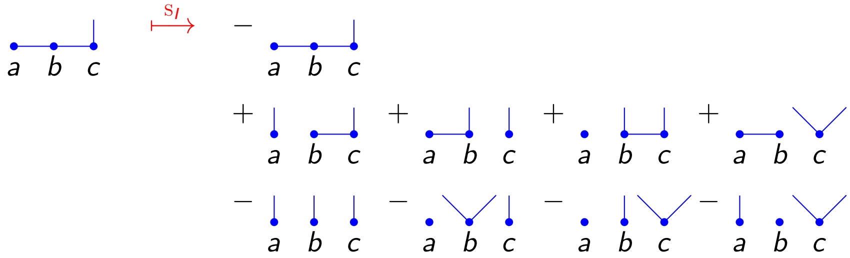
Posets: $\chi_Q(q)$ = strict order polynomial ; $\Omega(q)$ = order poly.

$\chi_Q(q)(-1) = (-1)^{|I|}$

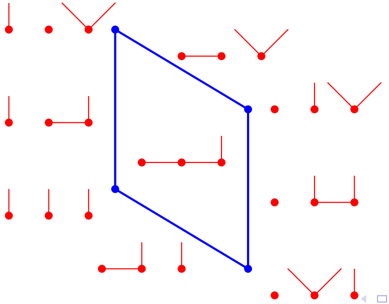
$\chi_Q(q)(-n) = (-1)^{|I|}\Omega_Q(q)(n)$ (Stanley).

Others: polymatroids, hypergraphs,...

3. Generalized permutahedra. Graphs and the antipode.



A polytopal explanation:



The graphic zonotope
$[a, b] + [b, c] + \{c\}$

Thank you.