TODXS CUENTAN
Difference, Humanity, and Belonging in the Mathematics Classroom

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0. CULTURE SETTING, TOGETHER
The week before class starts...

The first day of class...
Quietly, with your neighbor (2 min):

Write 3 — 5 words that come to mind when you hear this.
My students’ words:

community . joy . polyrhythm . family
crescendo . playful . encouraging
unexpected . churchlike . inviting . dancing
conversation . courage . motivation
cheerful . Spanish . learning . rhythm
celebration . style . culture . festive
Carlos Embales y los Roncos Chiquitos - Quítalo del Rincón
Carlos Embales y los Roncos Chiquitos

Quíralo del Rincón

Ricardo Díaz

Que contesta el que lo sepa y que aprenda el que no sabe
Que venga otro a la pizarra que se sepa la lección
Y al que no quiera aprender
Le enseñamos, ¡muy contentos!

20+3?  23  30+6?  36
30+3?  33  30+6?  36

Quíralo del rincón y pásalo a la ventana
Y ya verás como mañana él aprende la lección
Al momento, ¡muy contento!

20+3?  23  30+6?  36
30+3?  33  30+6?  36

Esta lección la supo Juana, sin llegar a la ventana
Cuando estudio lo hago en serio, yo estudio con jarana
Si uno no quiere creer, preguntárselo a mi hermana
Sumo, multiplico, y resto, y solo en una semana
La rumba la baila Juana, sentada en la palangana
Esta rumba no termina, por lo menos hasta mañana
If you know the answer, answer
If you don’t know it, learn.
And if anyone doesn’t want to learn,
We’ll teach them, very happily!

20+3? 23  30+6? 36
30+3? 33  30+6? 36

Move them away from the corner,
and put them in front of the window.
You’ll see how they’ll learn the lesson
Right away, very happily!

20+3? 23  30+6? 36
30+3? 33  30+6? 36

If you don’t believe an answer, you can ask my sister.
I add, multiply, and subtract, all in one week!
When I study I do it seriously, I study with my guitar
This party does not end, at least until tomorrow.
This is a math class:
community . joy . polyrhythm . family
crescendo . playful . encouraging
unexpected . churchlike . inviting . dancing
conversation . courage . motivation
cheerful . Spanish . learning . rhythm
celebration . style . culture . festive

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Talk to your neighbor: (2 mins)

○ Choose 1 — 2 of these words that apply in your math class.
○ What are concrete practices you use to help that happen?
My students’ choices and mine:

- community
- joy
- polyrhythm
- family
- crescendo
- playful
- encouraging
- unexpected
- churchlike
- inviting
- dancing
- conversation
- courage
- motivation
- cheerful
- Spanish
- learning
- rhythm
- celebration
- style
- culture
- festive
Concrete practices  (Student suggestions for me)

• Offer many group assignments where we get to work with different people each time.

• Have the assessment and the grading reflect the emphasis on growth and teamwork.
Concrete practices  (Student suggestions for each other)

- Be very mindful of how we communicate with each other. Stay encouraging.
- Do not take the joy of discovery away from others.
- Stay honest and vulnerable. If I don’t think I understand, ask for help.
- Be excited to help our classmates learn.
- Each instrument plays a different rhythm; together they create a beautiful piece of art. Similarly, every brain works differently; let’s use our math community to make learning more fun, and more creative.
- In that guaguancó we can only hear the musicians, but we’re pretty sure the community is dancing in front of them. Try to accomplish that in our class.
- Make space, take space.
MAKE SPACE, TAKE SPACE
Rochelle Gutiérrez

Embracing Nepantla

Nepantla: The space in between

Access

Dominant

Mathematics

Identity

Critical

Play the game

Mathematics

Achievement

Change the game

Power

Embracing Nepantla
Today:

COMBINATORICS

PEDAGOGY
MATH 420/720: Combinatorics

• Only combinatorics class offered regularly at SFSU. (Yearly) Not a graduation requirement.

• Students:
  • (undergrads who just took proofs) → (combinatorics M.A. students)
  • 1/3 to M.A./Ph.D., 1/3 to teaching, 1/3 to industry
  • very diverse student population

• SFSU:
  • large urban public university, part of CSU
  • first Ethnic Studies department in the US
Pedagogical practices

• Cross-generational, non-hierarchical mathematical space. Open doors.

• Students (and guests) DJ: music that makes them feel comfortable, happy, at home.

• Every day every student does active work, with a different partner each day.

• Each pair discusses and summarizes the main point of the class, records it in a diary.

• Ideas are welcome. Open problems each week, usually coming out of class discussions.

• Include many kinds of work in assessment: HW, groupwork, essays, diary, project.

• Don’t shy away from politics.
Some challenges
(reflected in student feedback)

• A (small, but nontrivial) minority of students felt uncomfortable with so much groupwork with preset groups, and/or that coordinating schedules was hard.

• I incorporated student feedback in course design. A few students experienced this as lack of clarity in course objectives and expectations.

• Time! “He tries really hard to engage with everyone and that paradoxically means that he doesn't have a lot of time for an individual student sometimes.”
1. BELONGING AND COMMUNITY

The danger of a single story
1. BELONGING AND COMMUNITY
The power of many stories
As a classroom community, our capacity to generate excitement is deeply affected by our interest in one another, in hearing one another's voices, in recognizing one another's presence. Any radical pedagogy must insist that everyone's presence is acknowledged. That insistence cannot be simply stated. It has to be demonstrated through pedagogical practices. There must be an ongoing recognition that everyone influences the classroom dynamic, that everyone contributes. Often before this process can begin there has to be some deconstruction of the traditional notion that only the professor is responsible for classroom dynamics.
MATHEMATICAL OBJECTS HAVE FEELINGS, TOO
The towers of Brahma / Hanoi
The towers of Brahma / Hanoi

- **Goal**: Move all the pegs to the right tower
The towers of Brahma / Hanoi

- **Goal**: Move all the pegs to the right tower

- **Rule**: Never put a bigger disk over a smaller disk.
The towers of Brahma / Hanoi

• **Goal**: Move all the pegs to the right tower

• **Rule**: Never put a bigger disk over a smaller disk.

\[ a_n = \# \text{ of steps needed to solve the puzzle with } n \text{ disks} \]
The towers of *Brahma / Hanoi*

• Invented by Édouard Lucas (Paris, 1883)

• “in a Hindu temple, the puzzle was supposedly used to increase the mental discipline of young priests.”

• When Lucas went to market the puzzle, Hanoi had just been seized by the French in 1882. Lucas chose Hanoi because it was in the newspaper headlines. (Hinz et.al. 2013)
The towers of **Brahma / Hanoi**

- When Lucas went to market the puzzle, Hanoi had just been seized by the French in 1882. He chose it for marketing reasons.
The towers of Lucas
Meanwhile…

???

Meanwhile…
Booooring...
Start over.

Point of view: Bottom Disk
When are these people gonna get off me, so I can move?
When are these people gonna get off me, so I can move?

\[ a_n = a_{n-1} + \cdots \]
\[ a_n = a_{n-1} + 1 + \cdots \]

Cool.
Now I can finally move.
Ok, I’ll let them back on. How long will that take?

\[ a_n = a_{n-1} + 1 + \cdots \]
\[ a_n = a_{n-1} + 1 + a_{n-1} \]

Ok, I’ll let them back on.
How long will that take?

\( a_{n-1} \) steps.
\[ a_n = 2a_{n-1} + 1, \quad n \geq 2 \]
\[ a_1 = 1 \]

1, 3, 7, 15, 31, 63, 127, \ldots

\[ a_n = 2^n - 1 \]
Ok, we can solve Lucas’s puzzle in $2^n - 1$ steps.

We are done!

Or are we?
Ok, we can solve Lucas’s puzzle in $2^n - 1$ steps.

We are done!

Whose point of view are we missing?
Start over.  

Point of view: Top Disk

How does the small disk experience this game?
Start over. 

Point of view: Top Disk

How does the small disk experience this game?

Talk to your neighbor. (2 mins.)
How does the small disk experience this game?

I just moved.
I just moved.

I know what the next move must be.
I just moved.

I know what the next move must be.
I just moved.

I know what the next move must be. Only I can move next.
I just moved.

I know what the next move must be.

Only I can move next.
Every other move is mine.
I must move exactly $2^{n-1}$ times!
Wait a second...
I control everything!!!
I need to figure out how to move,
and the other disks follow me.
Ok, we can solve Lucas’s puzzle in $2^n - 1$ moves. The top disk makes half the moves, others follow.

But this is still not the full story.

Whose point of view are we missing?
Moving 8 disks takes 255 moves.

How much do you think the other disks move?

Talk to your neighbor, and make a guess.

(1 min.)
Moving 8 disks takes 255 moves.

Can you prove this?

Take this home. :)

128
64
32
16
8
4
2
1
Ok, we can solve Lucas’s puzzle in $2^n-1$ moves. The $i$-th disk makes $2^{i-1}$ moves.

But this is still not the full story.

Whose point of view are we missing?
This was almost all news to me, and there is much more to be said here!

(Open-ended HW / Project.)
This was almost all news to me, and there is much more to be said here!

Amira Alkeswani, Anastasiya Timchenko, Bryan Swartout did this and much more.
The map of possibilities for n=2

Amira Alkeswani, Anastasiya Timchenko, Bryan Swartout did this and much more.
The map of possibilities for $n=2,3,4$

Amira Alkeswani, Anastasiya Timchenko, Bryan Swartout
• Taking an already solved problem and solving it in other ways leads to really different results – not just different answers, but different questions.

• History and math books are written by victors. There are benefits of considering many perspectives.

• We discussed the importance of the point of view in seeing things and in solving math problems. Never ever reduce the importance of your point of view. As well, consider other people's ways of looking at things.
2. DIFFERENCE
As women, we have been taught either to ignore our differences, or to view them as causes for separation and suspicion rather than as forces for change. Without community there is no liberation. But community must not mean a shedding of our differences, nor the pathetic pretense that these differences do not exist.

Advocating the mere tolerance of difference is a total denial of the creative function of difference in our lives. Difference must be not merely tolerated, but seen as a fund of necessary polarities between which our creativity can spark like a dialectic.
Domino Tilings

**Study**: The tilings of a $2 \times n$ rectangle with dominoes.
Exercise 1. List all the domino tilings of a $2 \times n$ rectangle.
13 domino tilings of a 2 x 6 rectangle.

Do we have them all?
Exercise 2. Count the domino tilings of a $2 \times n$ rectangle.
\[ A_n = \text{# of tilings of a } 2 \times n \text{ rectangle.} \]

Points of view help again!

The region

The tiles
$A_n = \# \text{ of tilings of a } 2 \times n \text{ rectangle}.$
A_n = \# of tilings of a 2 \times n rectangle.

Points of view help!
$A_n = \# \text{ of tilings of a } 2 \times n \text{ rectangle.}$

Points of view help!

I am covered by

or

$A_{n-1}$

tilings

or
\( A_n = \# \text{ of tilings of a } 2 \times n \text{ rectangle.} \)

I am covered by \( \text{ or } \) .

Points of view help!
$A_n = \# $ of tilings of a $2 \times n$ rectangle.

Points of view help!

I am covered by $\begin{array}{c} \text{or} \\ \begin{array}{c} \text{or} \end{array} \end{array}$.
$A_n =$ # of tilings of a $2 \times n$ rectangle.

Points of view help!

$I$ am covered by $\square$ or $\square$.

$A_n = A_{n-1} + A_{n-2}$

Fibonacci numbers
\[ A_n = \# \text{ of tilings of a } 2 \times n \text{ rectangle.} \]  

Points of view help!

\[ A_n = A_{n-1} + A_{n-2} \]  

Pingala numbers (200 BC)
Design Exercise. Organize the tilings of a 2 x n rectangle, in a way that is beautiful, useful, instructive,….

with: May-Li Khoe
Step 1. Individually, organize the domino tilings of a 2 x n rectangle in a way that is beautiful, useful, instructive, ...
Step 1. Individually, organize the domino tilings of a 2 x n rectangle in a way that is beautiful, useful, instructive, …

**MLK:** Good design serves a purpose.

**Some examples:**
- Ease of use
- Better understanding
- Aesthetics
Step 2. Get in pairs.

A: I notice ___ and I like ___ about your design.

B: I was trying to ___. Questions / challenges I faced are ___.

A+B: We could improve our designs by ___.

Some math questions that come up are ___.

Step 3. Present your partner’s design. What questions does it raise?
Step 4. (HW) What did you learn about these domino tilings? Try to answer some questions raised by your designs.

How many tilings have exactly $2k$ horizontal tiles?
How many tilings have exactly $2k$ horizontal tiles? $\binom{n-k}{k}$
How many tilings have exactly $2k$ horizontal tiles? \[ F_{n+1} = \sum_k \binom{n-k}{k} \]
Step 4. (HW) What did you learn about these domino tilings? Try to answer some questions raised by your designs.

How many tilings are symmetric?
How many tilings are symmetric?

\[ F_{\sim \frac{n}{2}} \sim \sqrt{F_n} \]
Step 4. (HW) What did you learn about these domino tilings? Try to answer some questions raised by your designs.

How do we generate the tilings recursively and beautifully?
Step 5. (Open HW/Project) Go deeper into q’s raised by your design.

How do we generate the tilings recursively and beautifully?
How do we generate the tilings recursively and beautifully?

Nicolás Willey, Tsz Shan Wong
How do we generate the tilings recursively and beautifully?

Nicolás Willey, Tsz Shan Wong
How do we generate the tilings recursively and beautifully?

Nicolás Willey, Tsz Shan Wong
DESIGN AND COMBINATORICS
Student Feedback

• I've learned that math is beautiful. Design is not only a study of how to make things look pretty. But also how to organize things [...] to make it carry a meaning.

• Different organizational structures lead to unique insights concerning math problems.

• Work with others can bring you more new idea to improve what you already have. Combining idea with others will make things more interesting.
3. HUMANITY
Our full selves are welcome in the classroom
3. HUMANITY
Our full selves are essential in the classroom
Rochelle Gutiérrez

Some Rehumanizing Practices:

• **Positioning**: Question hierarchies. Authority shifts from text/teacher to students as meaning makers.

• **Windows / Mirrors**: Students see themselves in the curriculum, reconnect with their own histories.

• **Creating**: Students invent new forms of math, not just reproduce what came before.

• **Broadening**: What counts as mathematics?

• **Ownership**: Math as something one does for oneself, not for others. “Express oneself.”
Tiling the Aztec diamond

**Goal**: Study the tilings of a region with dominoes. (My REU!)
Tiling the Aztec diamond

Goal: Study the tilings of a region with dominoes. (My REU!)

A domino tiling of the Aztec diamond $AD_3$
The 64 tilings of the Aztec diamond $\text{AD}_3$
Three stories about the Aztec diamond

1. My first public talk.
Three stories about the Aztec diamond

2. Mi primera charla pública.
Three stories about the **Mayan** diamond

2. Mi primera charla pública.
El Castillo de Chichén Itzá
El Castillo de Chichén Itzá

Ricardo Ceja, Vy Doyle, Duyeong Kim
Three stories about the Mayan diamond

3. En California.
Three stories about the Mayan diamond

3. En California.
Three stories about the Mayan diamond

3. En California.
Two months later: Final Project

Tiling the UFW Symbol

Jodi McWhirter
George Santellano
Joel Gallegos

May 23, 2017
United Farm Workers

INTRODUCTION

The United Farm Workers Union

- Founded in 1962, the UFW Union is the nation's first and largest farm workers' union. The UFW was founded by Cesar Chavez, Dolores Huerta, and Gilbert Padilla.
- The UFW was founded to empower migrant farm workers, improve their working conditions, and provide adequate wages.
- For more information, visit ufw.org.

Information from [ufw].

George Santellano: presentation at Latinx in Math
Tiling the United Farm Workers symbol

UFW

UFW₃

UFW₄
Tiling the United Farm Workers symbol

**Theorem.** (Gallegos, McWhirter, Santellano 2017) The number of domino tilings of the United Farm Worker eagle $UFW_{2k+1}$ is $S_k^2$, where $S_k$ is the $k$-th large Schröder number.

$S_3 = 22$
Theorem. (Gallegos, McWhirter, Santellano 2017) The number of domino tilings of the United Farm Worker eagle UFW_{2k+1} is S_k^2, where S_k is the k-th large Schröder number.

Proof (?):

Overeager FA: Oh, cool! There’s a general method to prove that, for many regions, the number of domino tilings is N^2 or 2N^2. Try it!

JG, JMW, GS: We could do that, or...
Theorem. (Gallegos, McWhirter, Santellano 2017) The number of domino tilings of the United Farm Worker eagle $UFW_{2k+1}$ is $S_k^2$, where $S_k$ is the $k$-th large Schröder number.

Proof:
DIFFERENCE, HUMANITY, BELONGING
A FRAMEWORK: TODXS CUENTAN

Axiom 1. Mathematical talent is distributed equally among different groups, irrespective of geographic, demographic, and economic boundaries.

Axiom 2. Everyone can have joyful, meaningful, and empowering mathematical experiences.

Axiom 3. Mathematics is a powerful, malleable tool that can be shaped and used differently by various communities to serve their needs.

Axiom 4. Every student deserves to be treated with dignity and respect.

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FA: This talk.
TODXS CUENTAN

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FA: CAT(0) Robots, Geometry, and Society, Dec 2019
TODXS CUENTAN

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DIFFERENCE, HUMANITY, BELONGING

I am an optimist, but we still have very far to go.
Rochelle Gutiérrez

Mathematics education cannot truly improve until it adequately addresses the very students who the system has most failed. We need a central focus on students who are Latinx, Black, and Indigenous… developing practices and measures that feel humane to those specific communities as a means to guide the field.

Rehumanizing Mathematics
Those of us who are poor, who are lesbians, who are Black, who are older – know that survival is not an academic skill. It is learning how to take our differences and make them strengths. For the master’s tools will never dismantle the master’s house. They may allow us temporarily to beat him at his own game, but they will never enable us to bring about genuine change. And this fact is only threatening to those … who still define the master's house as their only source of support.

The Master’s Tools Will Never Dismantle The Master’s House
For black folks teaching – educating – was fundamentally political because it was rooted in antiracist struggle. Almost all our teachers at Booker T. Washington were black women. Teachers worked with and for us to ensure that we would fulfill our intellectual destiny and by so doing uplift the race. My teachers were on a mission. Attending school then was sheer joy. I loved being a student. I loved learning.

School changed utterly with racial integration. Bussed to white schools, we soon learned that obedience, and not a zealous will to learn, was what was expected of us. That shift from beloved, all-black schools to white schools where black students were always seen as interlopers, as not really belonging, taught me the difference between education as the practice of freedom and education that merely strives to reinforce domination.
At Stanford University, the primary lesson was reinforced: we were to learn obedience to authority. The vast majority of our professors lacked basic communication skills, they were not self-actualized and they often used the classroom to enact rituals of control that were about domination and the unjust exercise of power. In these settings I learned a lot about the kind of teacher I did not want to become.

In graduate school I wanted to become a critical thinker. Yet that longing was often seen as a threat to authority. Individual white male students who were seen as "exceptional," were often allowed to chart their intellectual journeys, but the rest of us (and particularly those from marginal groups) were always expected to conform.
An assignment:

A HW in my class:

“Let’s continue playing some music before class and in between things, to bring some more light into that classroom. On your designated day, please choose a song to share that makes you feel comfortable / joyful / at home. If you’d like to, you can tell us a bit about the song or why it’s meaningful to you. “
An assignment:

“Choose a song to share that makes you feel comfortable / joyful / at home.”
An assignment:

“Choose a song to share that makes you feel comfortable / joyful / at home.”

It was surprising to me how personal their choices were. Students want to be seen, as humans, in our classrooms.
This can be a math class:

difference . humanity . belonging
liberation
This can be a math class:

difference . humanity . belonging . liberation

Talk to your neighbor: (2 mins)

What are concrete practices you can use to help that happen?
¡¡¡ muchas gracias !!!