
matroides
orientadas
tropicales

Tropical Oriented Matroids

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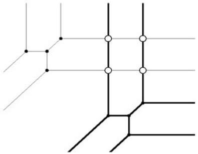

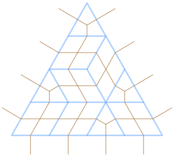

Ten Questions in Tropical Geometry
(ADFMMMPSY - MSRI, 2006)

A list of ten key open problems in (the algebraic and combinatorial side of) tropical geometry.

Question 8. What is a tropical oriented matroid?
Construct a combinatorial model which captures the fundamental properties of a tropical hyperplane arrangement.

The plan

1. Tropical hyperplane arrangements
   - What is tropical geometry?
   - Tropical hyperplane arrangements
2. Oriented matroids
   - Definition
   - Uses
3. Tropical oriented matroids
   - Covectors
   - Toolkit
   - Geometric models
   - Open problems

Tropical Geometry

What is tropical geometry?

It depends on who you ask. One point of view:

$$\begin{array}{rcl} \text{algebraic variety} & \mapsto & \text{tropical variety} \\ V & \mapsto & \text{Trop}(V). \end{array}$$

Idea: Obtain information about V from Trop(V).

o Trop(V) is simpler, but contains much information about V.

o Trop(V) is a polyhedral fan, where we can do combinatorics.

Example: Gromov-Witten invariants of $\mathbb{CP}^2$ can be computed by tropicalizing, *i.e.*, combinatorially. (Mikhalkin)

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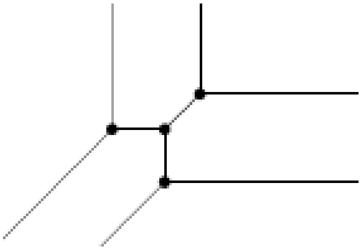
$$x \oplus y = \max(x, y) \qquad x \odot y = x + y$$

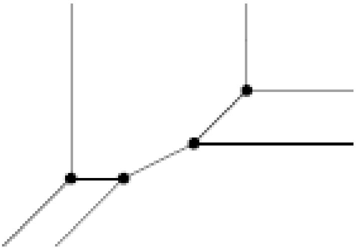
Example 1. Tropical conics in $\mathbb{TP}^2$:

$AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = 0 \mapsto$

$\max(a + 2x, b + 2y, \ldots, e + x + z, f + y + z)$ achieved twice.

Two tropical conics:



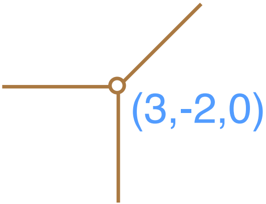


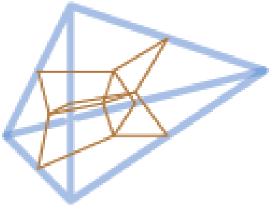
(Tropical projective plane $\mathbb{TP}^2$: $(a, b, c) \sim (a - c, b - c, 0)$)

Example 2. Tropical hyperplanes in $\mathbb{TP}^{n-1}$.

$A_1X_1 + \ldots + A_nX_n = 0 \mapsto$ $\max(x_1 + a_1, \ldots, x_n + a_n)$ ach. twice

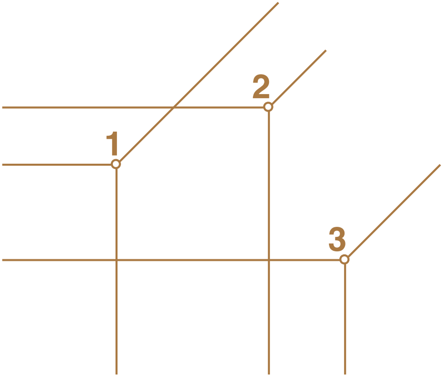
$\mathbb{TP}^2$: $\max(x - 3, y + 2, z)$ twice $\mathbb{TP}^3$: $\max(x_1, x_2, x_3, x_4)$ twice

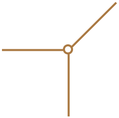




This is the polar fan of the simplex centered at $-(a_1, \ldots, a_n)$.
It divides $\mathbb{TP}^n$ into $n + 1$ regions.

Goal: To study **tropical hyperplane arrangements.**





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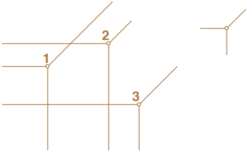

Why? Some reasons:

- Tropical polytopes "=" arrangements. (Develin, Sturmfels)
- Tropical linear spaces are **very** interesting and not well understood. (A., Klivans; A., Reiner, Williams; Speyer)
  They live inside arrangements. Connections:
  - Lafforgue's surgery on Grassmannians, matroid subdivs.
  - De Concini-Procesi's wonderful compactifications.
- Subdivs. of $\Delta_{n-1} \times \Delta_{d-1}$ and the Schubert calculus of the flag manifold. (A., Billey)
- Convexity in Bruhat-Tits buildings. (Joswig, Sturmfels,Yu)
- The rich theory of hyperplane arrangements.

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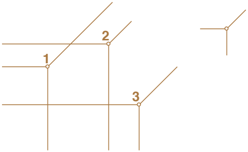

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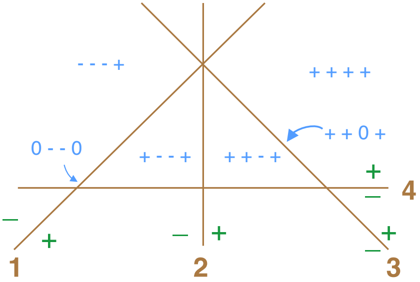
Oriented Matroids

$\mathcal{A}$ - hyperplane arrangement in $\mathbb{R}^n$.
$M_{\mathcal{A}}$ - **oriented matroid** - captures its combinatorial structure.



A **covector** for each face of $\mathcal{A}$:
¿On what side of each hyperplane am I?

What is an oriented matroid?

A collection of **covectors** in $\{+,-,0\}^n$ such that:

- (Zero) 0 is a covector
- (Symmetry) If v is a covector, so is $-v$.
- (Surrounding) If u, v are covectors, so is $u \circ v$.
- (Elimination) If u, v are covectors and $j \in S(u, v)$, there is a covector w with$w_j = 0$ and $w_i = (u \circ v)_i$ for $i \notin S(u, v)$.

Here:

$$(u \circ v)_i := \begin{cases} u_i \text{ si } u_i \in \{+,-\} \\ v_i \text{ si } u_i = 0. \end{cases}$$

and

$$S(u, v) := \{i : u_i = -v_i \neq 0\}$$

Fine, but what is an oriented matroid?

A combinatorial model for real hyperplane arrangements.
Each axiom abstracts a geometric property of arrangements.

It is a **great** model:

- Almost no matroid comes from hyperplane arrangements, but they all come from a pseudo-hyperplane arrangement.
- Almost any combinatorial theorem about hyperplane arrangements is true for matroids.
- It is applicable to vector configs, graphs, polytopes,. . .
- A powerful toolkit has been develoepd:
    - equivalent points of view (independence, cycles, . . .)
    - constructions (duality, sum, intersection, . . .)

Uses. Topology of complex hyperplane arrangements, combinatorial differential manifolds, random walks, . .

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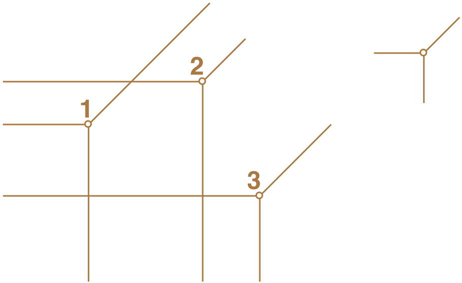
Tropical oriented matroids

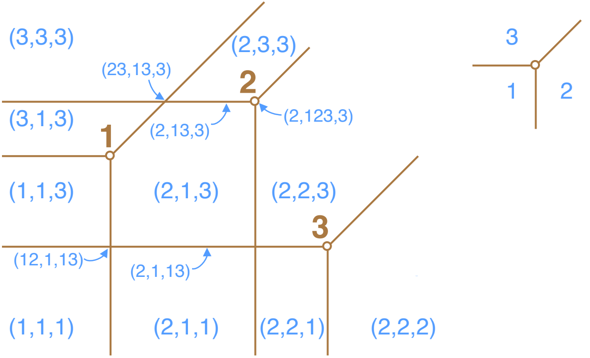
The project.

- Build a combinatorial model for tropical hyperplane arrangements.
- Develop the theory of tropical oriented matroids.



Tropical oriented matroids

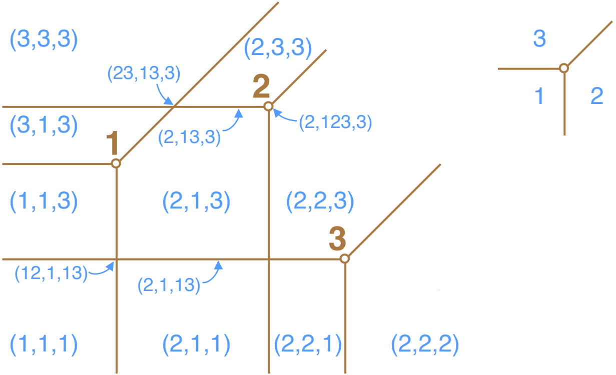
A hyperplane splits $\mathbb{TP}^{d-1}$ into d sectors. Which one am I in?
n hyperplanes in $\mathbb{TP}^{d-1} \mapsto$ covectors $(A_1, \ldots, A_n)$, $A_i \subseteq [d]$.



Properties of covectors?

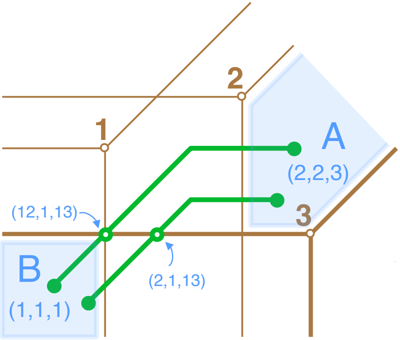
Property 1. (Boundary)
$(i, \ldots, i)$ is a covector for all $1 \leq i \leq d$.

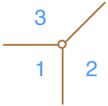
I go to infinity in the *i*th direction.



Property 2. (Elimination)
If A and B are covectors and i is a coordinate, there is a covector C such that $C_i = A_i \cup B_i$, and $C_j \in \{A_j, B_j, A_j \cup B_j\}$ for all j.

C is the intersection of H_i with the line segment from A to B.



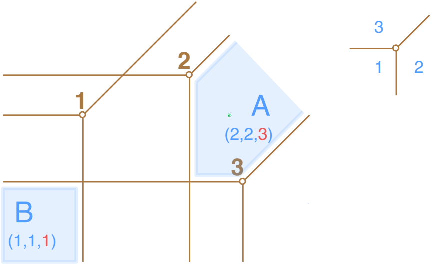


Property 3. (Comparability)

Two covectors cannot form a cycle:

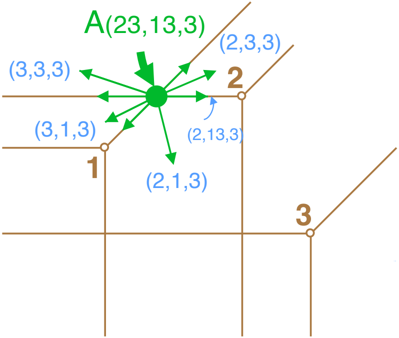
$$A = (347, 25, 2, 3, 23, 16)$$
$$B = (36, 456, 6, 46, 1, 367)$$

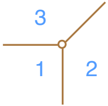
If I walk from A to B, I move less in direction 3 than in 1.



Property 4. (Surrounding)
For any covector $A = (3457, 256, 124, 357, 25, 16)$ and any ordered partition $3,5 \ < \ 1,4,7 \ < \ 2,6$, the "minimal covector" $A_< = (35, 5, 14, 35, 5, 1)$ is a covector.

Walk from A in the direction specified by $< \ \rightarrow$ get to face $A_<$.





Definition/Theorem. (A., Develin, 2007)

Definition. A tropical oriented matroid is a set of covectors of the form $(A_1, \ldots, A_n)$, with $A_i \subseteq [d]$, such that:

- (Boundary) $(i, \ldots, i)$ is a covector for all $1 \leq i \leq d$.
- (Elimination) If A and B are covectors and $1 \leq i \leq d$, there is a covector C with $C_i = A_i \cup B_i$ and $C_j \in \{A_j, B_j, A_j \cup B_j\}$ for $j \neq i$.
- (Comparability) No two covectors form a cycle.
- (Surrounding) For any covector A and any ordered partition $<$ of $[d]$, the "minimal covector" $A_<$ is a covector.

Theorem. The covectors of a tropical hyperplane arrangement form a tropical oriented matroid.

Tropical oriented matroids (TOMs) are a good combinatorial model for tropical hyperplane arrangements:

- Sufficiently weak to include tropical arrangements.
- Sufficiently strong to prove interesting theorems.

Toolkit.

1. Constructions:

- Theorem. The deletion $M\backslash i$ is a TOM.
  Erase coordinate i from each covector. $(1 \leq i \leq n)$
- Theorem. The contraction M/i is a TOM.
  Consider only the covectors not containing j. $(1 \leq j \leq d)$
- Conjecture. The dual of a TOM is a TOM.
  Transpose: $(124, 1, 134, 23, 124) \mapsto (1235, 145, 34, 135)$

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2. Convexity. (joint work with Anna Brown, 07)

A **convex geometry** is a combinatorial model that captures the common features of many notions of convexity, in the same way that matroids model independence. (Edelman et. al.)

- Theorem. (Björner, Edelman, Ziegler) An oriented matroid M determines a convex geometry on the elements of M.
- Theorem. (A., Brown) A tropical oriented matroid M determines a convex geometry on the elements of M.
- Ongoing project. Investigating these “tropical convex geometries".

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3. Equivalent points of view.

- Theorem.
  The regions (maximal covectors) of M determine it.
- Theorem.
  The vertices (minimal covectors) of M determine it.
- Question. Axioms for regions? for vertices?
- Conjecture. Two geometric models:
  - Subdivisions of the polytope $\Delta_{n-1} \times \Delta_{d-1}$.
  - "Pseudohyperplane" arrangements.

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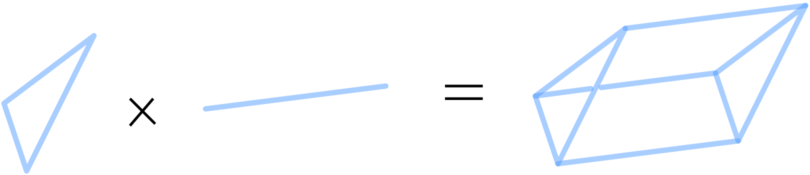
Subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$

The product of simplices $\Delta_{n-1} \times \Delta_{d-1}$ is the polytope in $\mathbb{R}^{n+d}$ whose nd vertices are, for $1 \leq i \leq n, 1 \leq j \leq d$,

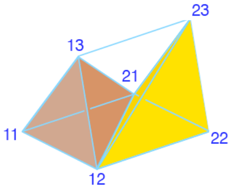
$$e_i + f_j = (0, \ldots, 0, 1, 0, \ldots, 0; 0, \ldots, 0, 1, 0, \ldots, 0)$$



Subdivision of P: A tiling $P = P_1 \cup \ldots \cup P_k$ where the P_is are subpolytopes, and $P_i \cap P_j$ is empty or is a face of P_i and P_j.

Triangulation: A subdivision into simplices.

Subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$.



- Very nice structure. (Gelfand - Kapranov - Zelevinsky)
- Toric varieties of transportation polytopes. (Sturmfels)
- Disconnected toric Hilbert schemes. (Santos)
- $d = 3$: Schubert calc.: criterion for Littlewood - Richardson numbers $c_{uvw} = 0$ in the flag manifold (A., Billey)
- (A., Beck, Hosten, Pfeifle, Seashore; Babson, Billera; Bayer; Develin-Sturmfels; Haiman; Postnikov; etc.)

Conjecture. (A.-Develin, 2007)
Tropical oriented matroids of (n, d) = Subdivs. of $\Delta_{n-1} \times \Delta_{d-1}$

Theorems:

- tropical oriented matroids $\subseteq$ subdivisions
- tropical oriented matroids $=$ subdivisions ($d = 3$).
- subdivisions satisfy the boundary, comparability, and surrounding axioms.

Only one axiom is missing!

- Conjecture. Subdivisions satisfy the elimination axiom.

Difficulty: To “navigate" a subdivision in a controlled manner.

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Sketch of the bijection.

Subdivisions ↔ Tropical oriented matroids

(mixed) faces ↔ covectors

11, 12, 13, 21 ↔ (123, 1)

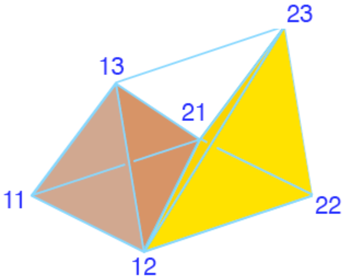
12, 13, 21, 23 ↔ (23, 13)

12, 21, 22, 23 ↔ (2, 123)

13, 21, 23 ↔ (3, 13)

12, 21 ↔ (2, 1)

22, 23 ↔ (∅, 23) We ignore it.



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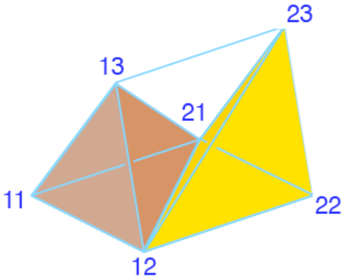
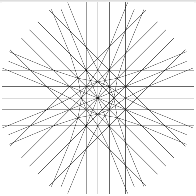


Tropical pseudohyperplane arrangements

Topological Representation Theorem. (Folkman, Lawrence, 1978)
Any oriented matroid can be represented by an arrangement of pseudohyperplanes.



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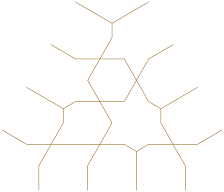

Topological Representation Conjecture. (A., Develin)
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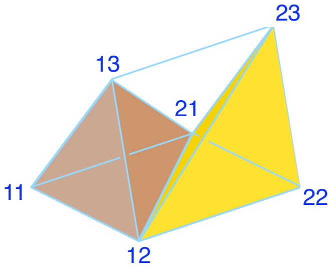
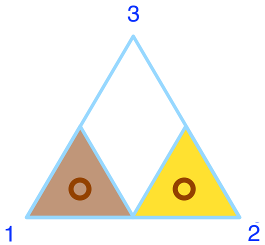
Sketch of a proof. Step 1.

The Cayley trick gives us a bijection:

$$\text{subdivs. of } \Delta_{n-1} \times \Delta_{d-1} \leftrightarrow \text{mixed subdivs. of } n\Delta_{d-1}$$





Topological Representation Conjecture. (A., Develin)
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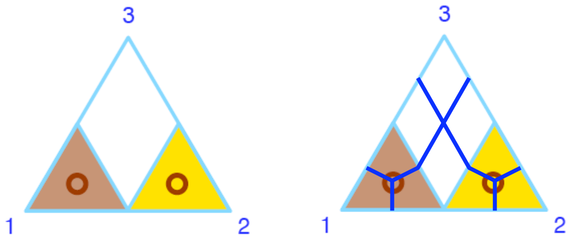
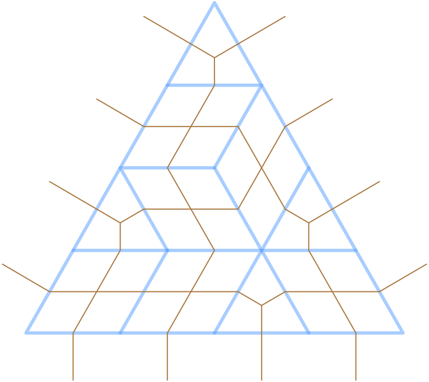
Sketch of a proof. Step 2.

Draw the mixed Voronoi subdivision of each cell.

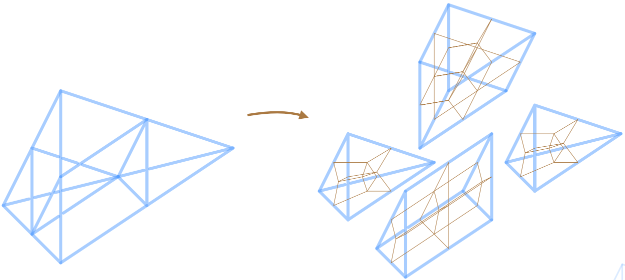


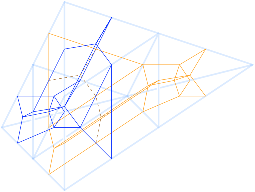
The lower-dimensional faces determine the arrangement of tropical pseudohyperplanes.

An example:
A mixed subdivison of $n\Delta_2$ is a tiling with triangles and unit rhombi. The corresponding arrangement:

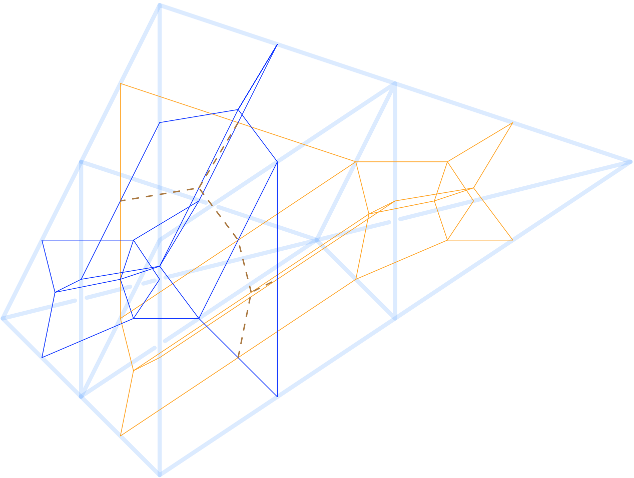


A mixed subdivision of $n\Delta_3$ and the corresponding arrangement:





A tropical pseudohyperplane arrangement in $\mathbb{TP}^3$:



Open problems.

Combinatorics

- Prove TOM duality.
- Define morphisms (strong maps) between TOMs.
- Define tropical (unoriented) matroids.

Geometry

- Bijection between TOMs and subdivs of $\Delta_{n-1} \times \Delta_{d-1}$.
- Make precise connection with Schubert calculus of $\mathcal{F}\ell_n$.

Topology

- Prove the topological representation conjecture.
- Study the topology of a TOM (face poset, etc.)
- Use TOMs and their morphisms to give a combinatorial model of the space of tropical hyperplane arrangements.

many thanks!

The article is available at:

http://math.sfsu.edu/federico

http://front.math.ucdavis.edu/0706.2920