The algebraic and combinatorial structure of generalized permutahedra

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...or: how I came to appreciate Hopf monoids (and algebras)

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...or: how I came to appreciate Hopf monoids (and algebras)

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This is work from 2008-2017, closely related to papers by:

Carolina Benedetti, Nantel Bergeron, Lou Billera, Eric Bucher, Harm Derksen, Alex Fink, Rafael González D'León, Vladimir Grujić, Joshua Hallam, Brandon Humpert, Ning Jia, Carly Klivans, John Machacek, Swapneel Mahajan, Jeremy Martin, Vic Reiner, Bruce Sagan, Tanja Stojadinović, Jacob White...

0. Inverting formal power series: Multiplication

A new take on an old question:

How do we invert a formal power series?

Let $A(x) = \sum a_n \frac{x^n}{n!}$ and $B(x) = \sum b_n \frac{x^n}{n!}$ be multiplicative inverses. Assume $a_0 = b_0 = 1$.

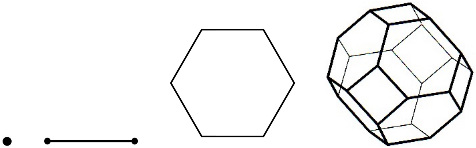
Then $B(x) = 1/A(x)$ is given by

$$
\begin{aligned}
b_1 &= -a_1 \\
b_2 &= -a_2 + 2a_1^2 \\
b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\
b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 24a_1^4 \\
&\vdots
\end{aligned}
$$

How to make sense of these numbers?

0. Inverting formal power series: Multiplication.

Permutahedra:



π_1: point, π_2: segment, π_3: hexagon, π_4: truncated octahedron...

For exponential generating functions, $B(x) = 1/A(x)$ is given by

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\ b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 24a_1^4 \end{aligned}$$

Faces of π_4:

- 1 truncated octahedron π_4
- 8 hexagons $\pi_3 \times \pi_1$ and 6 squares $\pi_2 \times \pi_2$
- 36 segments $\pi_2 \times \pi_1 \times \pi_1$
- 24 points $\pi_1 \times \pi_1 \times \pi_1 \times \pi_1$

0. Inverting formal power series: Composition.

A new take on an old question:

How do we invert a formal power series?

$A(x) = \sum a_{n-1}x^n$, $B(x) = \sum b_{n-1}x^n$: **compositional** inverses.
Assume $a_0 = b_0 = 1$.

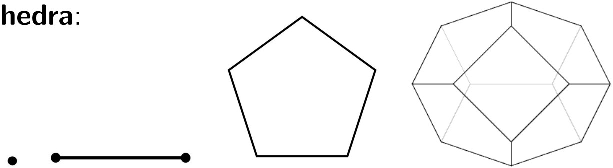
Then $B(x) = A(x)^{\langle -1 \rangle}$ is given by Lagrange inversion:

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\ b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4 \\ &\vdots \end{aligned}$$

How to make sense of these numbers?

0. Inverting formal power series: Composition.

Associahedra:



$\mathfrak{a}_1$: point, $\mathfrak{a}_2$: segment, $\mathfrak{a}_3$: pentagon, $\mathfrak{a}_4$: 3-associahedron...

For ordinary generating functions, $B(x) = A(x)^{\langle -1 \rangle}$ is given by

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\ b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4 \end{aligned}$$

Faces of $\mathfrak{a}_4$:
- 1 3-associahedron $\mathfrak{a}_4$
- 6 pentagons $\mathfrak{a}_3 \times \mathfrak{a}_1$ and 3 squares $\mathfrak{a}_2 \times \mathfrak{a}_2$
- 21 segments $\mathfrak{a}_2 \times \mathfrak{a}_1 \times \mathfrak{a}_1$
- 14 points $\mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1$

Hopf algebras? Hopf monoids? A remark.

Hopf monoids are a bit more abstract than Hopf algebras, but better suited for many combinatorial purposes. We have a functor

$$\text{Hopf monoids} \longrightarrow \text{Hopf algebras}$$

so there are Hopf algebra analogs of all of our results.

1. Species and Hopf monoids.

(Joyal) A species P consists of:

- For each finite set I, a vector space $\mathrm{P}[I]$.
- For each bijection $\sigma : I \to J$, a linear map $\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J]$, consistent with composition. (Relabeling.)

Think: ways of putting a certain combinatorial structure on set I.

$\mathrm{P}[I] = \mathrm{span}\{$combinatorial structures of type P on $I\}$

Examples.

$\mathrm{G}[I] := \mathrm{span}\{$graphs on vertex set $I\}$.
$\mathrm{Q}[I] := \mathrm{span}\{$posets on $I\}$.
$\mathrm{M}[I] := \mathrm{span}\{$matroids on ground set $I\}$.

Hopf monoids.

(Joni-Rota, *Coalgebras and bialgebras in combinatorics.*)
(Aguiar-Mahajan, *Monoidal functors, species, and Hopf algebras.*)

A Hopf monoid $(\mathrm{P}, \mu, \Delta)$ consists of:

- A species P.
- For each $I = S \sqcup T$, product and coproduct maps

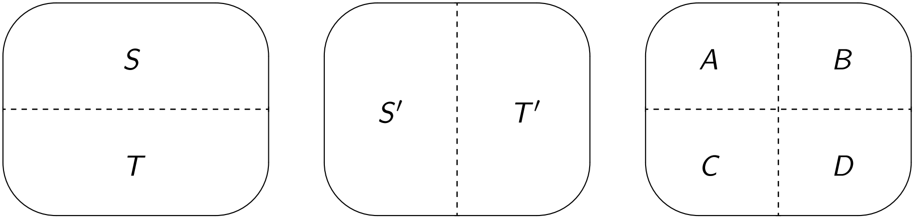
$$\mathrm{P}[S] \otimes \mathrm{P}[T] \xrightarrow{\mu_{S,T}} \mathrm{P}[I] \quad \text{and} \quad \mathrm{P}[I] \xrightarrow{\Delta_{S,T}} \mathrm{P}[S] \otimes \mathrm{P}[T].$$

Think: We have rules for "merging" and "breaking" our structures.

These maps should satisfy various axioms, including the following:

Hopf monoids. Compatibility axiom.

Fix two decompositions $S \sqcup T = I = S' \sqcup T'$, and let A, B, C, D be:



Then this diagram must commute:

$$
\begin{array}{ccccc}
\mathrm{P}[A] \otimes \mathrm{P}[B] \otimes \mathrm{P}[C] \otimes \mathrm{P}[D] & & \xrightarrow{\mathrm{id} \otimes \mathrm{switch} \otimes \mathrm{id}} & & \mathrm{P}[A] \otimes \mathrm{P}[C] \otimes \mathrm{P}[B] \otimes \mathrm{P}[D] \\
\uparrow {\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & & & \downarrow {\scriptstyle \mu_{A,C} \otimes \mu_{B,D}} \\
\mathrm{P}[S] \otimes \mathrm{P}[T] & \xrightarrow[\mu_{S,T}]{} & \mathrm{P}[I] & \xrightarrow[\Delta_{S',T'}]{} & \mathrm{P}[S'] \otimes \mathrm{P}[T']
\end{array}
$$

Think: “merge, then break” = ”break, then merge”

Examples of Hopf monoids: Graphs.

$\mathrm{G}[I] :=$ span{graphs (with half edges) on vertex set I}.

The species of graphs G is a Hopf monoid with

$$\mathrm{G}[S] \otimes \mathrm{G}[T] \xrightarrow{\mu_{S,T}} \mathrm{G}[I] \qquad \mathrm{G}[I] \xrightarrow{\Delta_{S,T}} \mathrm{G}[S] \otimes \mathrm{G}[T]$$

$$g_1 \otimes g_2 \longmapsto g_1 \sqcup g_2 \qquad g \longmapsto g|_S \otimes g/_S$$

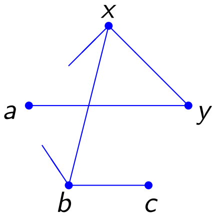
where

$g_1 \sqcup g_2$ = disjoint union of g_1 and g_2,

$g|_S$ = keep everything incident to S,

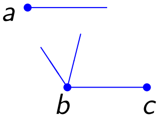
$g/_S$ = remove everything incident to S.

$I = \{a, b, c, x, y\}$

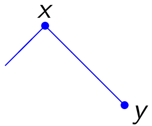


$\overset{\Delta_{S,T}}{\longmapsto}$

$S = \{a, b, c\}$



$T = \{x, y\}$



Examples of Hopf monoids: Posets.

$Q[I] := \text{span}\{\text{posets on } I\}$.

The species of posets Q is a Hopf monoid, under:

Product: disjoint union.

$$\mu_{S,T} : Q[S] \otimes Q[T] \longrightarrow Q[I]$$
$$q_1 \otimes q_2 \longmapsto q_1 \sqcup q_2$$

Coproduct: splitting.

$$\Delta_{S,T} : Q[I] \longrightarrow Q[S] \otimes Q[T]$$
$$q \longmapsto \begin{cases} q|_S \otimes q|_T & \text{if } S \text{ is an order ideal of } q \\ 0 & \text{otherwise} \end{cases}$$

Examples of Hopf monoids: Matroids.

$\mathrm{M}[I] := \mathrm{span}\{\text{matroids on } I\}$.

The species of matroids M is a Hopf monoid with

$$\mathrm{M}[S] \otimes \mathrm{M}[T] \xrightarrow{\mu_{S,T}} \mathrm{M}[I] \qquad\qquad \mathrm{M}[I] \xrightarrow{\Delta_{S,T}} \mathrm{M}[S] \otimes \mathrm{M}[T]$$

$$m_1 \otimes m_2 \longmapsto m_1 \oplus m_2 \qquad\qquad m \longmapsto m|_S \otimes m/_S$$

where

$$\begin{aligned} m_1 \oplus m_2 &= \text{direct sum of } m_1 \text{ and } m_2, \\ m|_S &= \text{restriction of } m \text{ to } S, \\ m/_S &= \text{contraction of } S \text{ from } m. \end{aligned}$$

Recall: A matroid on I is a collection $\mathcal{B}$ of r-subsets ("bases") such that:

If $A, B \in \mathcal{B}$ and $a \in A - B$, there is $b \in B - A$ with $A - a \cup b \in \mathcal{B}$.

Prototypical example:

I = collection of vectors in V, $\mathcal{B}$ = subsets which are bases of V

Other Hopf monoids.

There are many other Hopf monoids of interest in combinatorics.
A few of them:

- graphs G
- posets P
- matroids M
- set partitions Π (symmetric functions)
- paths A (Faá di Bruno)
- simplicial complexes SC
- hypergraphs HG
- building sets BS

3. The antipode of a Hopf monoid.

The antipode of a connected Hopf monoid P consists of the maps

$$\mathrm{s}_I : \mathrm{P}[I] \to \mathrm{P}[I]$$

$$\mathrm{s}_I = \sum_{\substack{I=S_1\sqcup\cdots\sqcup S_k \\ k\geq 1}} (-1)^k \, \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}.$$

summing over all ordered set partitions $I = S_1 \sqcup \cdots \sqcup S_k$. $(S_i \neq \emptyset)$

Think:	groups $\mapsto$ inverses
	Hopf monoids $\mapsto$ antipodes

$(\mathrm{s}^2 = \mathrm{id})$

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$(\mathrm{s}^2 = \mathrm{id})$

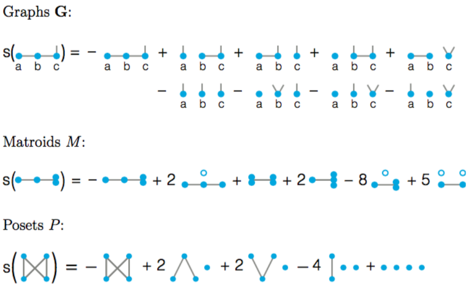
General problem. Find the simplest possible formula for the antipode of a Hopf monoid.

(Usually there is **much** cancellation in the definition above.)

Examples: The antipode of a graph, matroid, poset.

Ex. Takeuchi: $\mathrm{s}_I = \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}$.

For $n = 3, 4$ this sum has 13, 73 terms. However,



How do we explain (and predict) the simplification?

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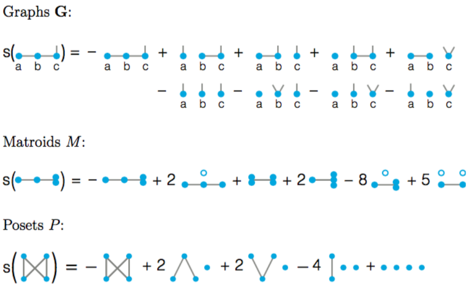

How do we explain (and predict) the simplification?

- Inclusion-exclusion
- Sign-reversing involution
- Möbius functions

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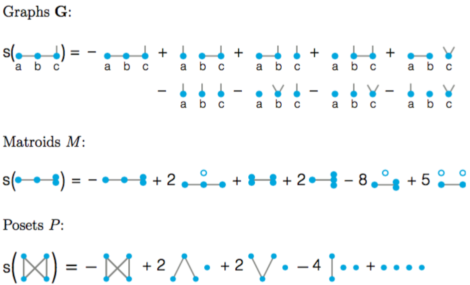
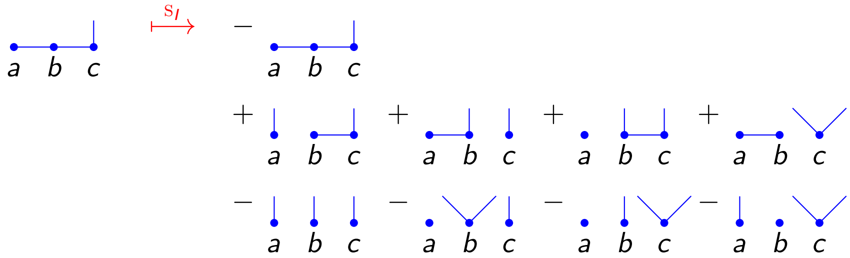


How do we explain (and predict) the simplification?

- Inclusion-exclusion
- Sign-reversing involution
- Möbius functions
- Euler characteristics

Example: The antipode of a graph.

Ex. Takeuchi: $\mathrm{s}_I = \sum_{I=S_1 \sqcup \cdots \sqcup S_k} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}$.



Theorem. (Aguiar–A., Humpert–Martin, Benedetti–Sagan)

$$\mathrm{s}(g) = \sum_{f \text{ flat}} (-1)^{|I|-r(f)} a(g/f) f$$

where $a(h)$ = number of acyclic orientations of h.

Other examples of antipodes.

There are many other Hopf monoids of interest.
Only some of their antipodes were known.

- graphs G: Humpert–Martin 10, Benedetti–Sagan 15
- posets P: Schmitt, 94
- matroids M: ?
- set partitions / symm fns. Π: Aguiar–Mahajan 10
- paths A: ?
- simplicial complexes SC: Benedetti–Hallam–Michalak 16
- hypergraphs HG: ?
- building sets BS: ?

Goal 1. a unified approach to compute these and other antipodes.
(We do this).

4. Generalized permutahedra.

Euclidean space $\mathbb{R}^I \;:=\; \{\text{functions } x : I \to \mathbb{R}\}$.

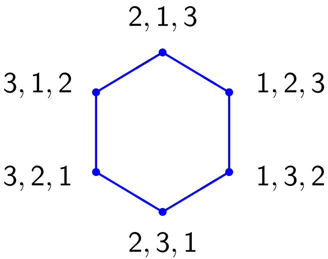
The standard permutahedron is

$$\pi_I \;:=\; \text{Convex Hull}\{\text{bijective functions } x : I \to [n]\} \subseteq \mathbb{R}^I$$

(where $n = |I|$).

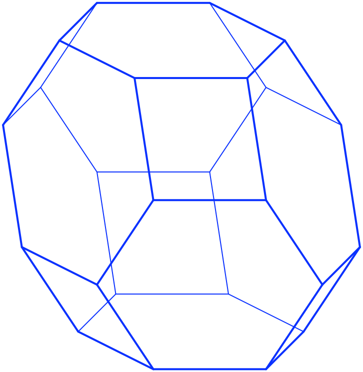
$I \;=\; \{a, b, c\}$ $\qquad\qquad$ $\{x_a + x_b + x_c = 6\}$

$\pi_I \;=$



The standard permutahedron.

The standard permutahedron π_I for $|I| = 4$:



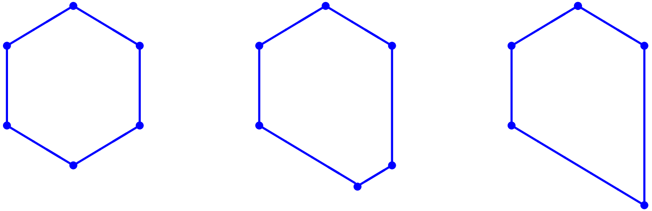
Generalized permutahedra.

Edmonds (70), Postnikov (05), Postnikov–Reiner–Williams (07),...

Equivalent formulations:

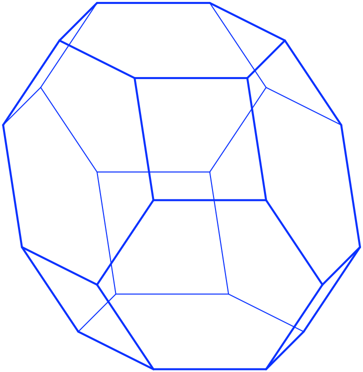
- Move facets of standard permutahedron without passing vertices.
- Move vertices while preserving edge directions.
- Change polytope while coarsening the normal fan.

Generalized permutahedra:

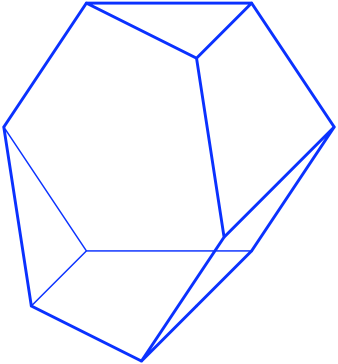


Generalized permutahedra in 3-D.

The permutahedron π_4.



A generalized permutahedron.



Generalized permutahedra. Product.

Key Lemma. If P and Q are generalized permutahedra in $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively, and $I = S \sqcup T$, then

$$P \times Q$$

is a generalized permutahedron in $\mathbb{R}^I$.

Define the **multiplication** of two generalized permutahedra to be

$$P \otimes Q := P \times Q$$

Generalized permutahedra. Restriction and contraction.

Given a polytope $P \subseteq \mathbb{R}^I$ and $v \in \mathbb{R}^I$, let

$$P_v \ := \ \text{face of } P \text{ where } \langle v, - \rangle \text{ is maximum.}$$

Given $I = S \sqcup T$, let $e_{S,T} \in \mathbb{R}^I$ have coordinates:

$$\begin{cases} e_s = 1 & \text{for } s \in S \\ e_t = 0 & \text{for } t \in T \end{cases}$$

Key Lemma. If P is a generalized permutahedron and $I = S \sqcup T$,

$$P_{e_{S,T}} = P_1 \times P_2.$$

for generalized permutahedra $P_1 \subseteq \mathbb{R}^S$ and $P_2 \subseteq \mathbb{R}^T$.

Define the **restriction** and **contraction**

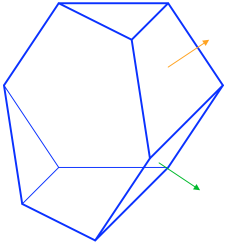
$$P|_S \ := \ P_1 \subseteq \mathbb{R}^S, \qquad P/_S \ := \ P_2 \subseteq \mathbb{R}^T.$$

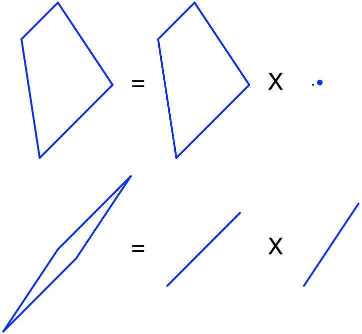
Generalized permutahedra: Restriction and contraction.

Given $I = S \sqcup T$, the vector $e_{S,T} \in \mathbb{R}^I$ has $e_s = 1, e_t = 0$. Then

$$P_{e_{S,T}} = P|_S \times P/_S$$

Two examples:





5. The Hopf monoid of generalized permutahedra.

GP$[I]$:= span {generalized permutahedra in $\mathbb{R}^I$}.

Define a product and coproduct:

$$\mathrm{GP}[S] \otimes \mathrm{GP}[T] \xrightarrow{\mu_{S,T}} \mathrm{GP}[I] \qquad \mathrm{GP}[I] \xrightarrow{\Delta_{S,T}} \mathrm{GP}[S] \otimes \mathrm{GP}[T]$$

$$P \otimes Q \longmapsto P \times Q \qquad P \longmapsto P|_S \otimes P/_S$$

Theorem. (Aguiar–A.)
GP is a Hopf monoid.

(Derksen–Fink 10 proved a very similar result.)

Generalized permutahedra: Posets, graphs, matroids.

There are generalized permutahedra associated to:

G graph $\to$ graphic zonotope $Z(G)$

$Z(G) = \sum_{ij \in G}(e_i - e_j)$

M matroid $\to$ matroid polytope P_M

$P_M = \text{conv}\{e_{i_1} + \cdots + e_{i_k} \mid \{i_1, \ldots, i_k\}$ is a basis of $M\}$.

P poset $\to$ poset cone C_P

$C_P : \text{cone}\{e_i - e_j \ : \ i < j \text{ in } P\}$.

Proposition. (Aguiar–A.) These give inclusions of Hopf monoids:

$$\mathrm{G} \hookrightarrow \mathrm{GP}, \qquad \mathrm{M} \hookrightarrow \mathrm{GP}, \qquad \mathrm{P} \hookrightarrow \mathrm{GP}$$

.

The antipode of GP.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{s}(P) \;=\; (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim Q}\, Q.$$

The sum is over all faces Q of P.

The antipode of GP.

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$$\mathrm{s}(P) \;=\; (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim Q}\, Q.$$

The sum is over all faces Q of P.

Proof. Takeuchi:

$$\mathrm{s}(P) \;=\; \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k\, P_{S_1,\ldots,S_k}$$

where $P_{S_1,\ldots,S_k}$ is the maximum face of P in direction $S_1|\cdots|S_k$.
Coeff. of a face Q: huge sum of 1s and -1s. How to simplify it?
It is the reduced Euler characteristic of a sphere! $\square$

This is the best possible formula. No cancellation or grouping!
(One advantage of working over species!)

The antipodes of GP, Q, G, M.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{s}(P) \;=\; (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim Q}\, Q.$$

The sum is over all faces Q of P.

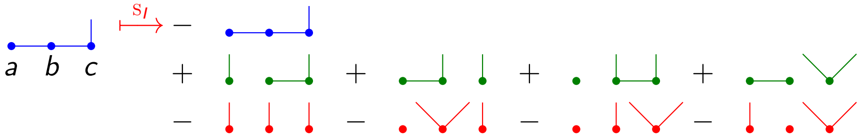
As a consequence, we get

Corollary. Best possible formulas for the antipodes of:

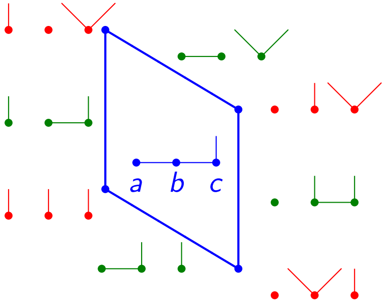
- posets (Schmitt, 94)
- graphs (new – also Humpert–Martin, 12)
- matroids (new)

Example: The antipode of graphs.

The (post-cancellation) antipode of a graph:



Geometric explanation: The faces of the graphic zonotope!



Many antipode formulas.

objects	polytopes	Hopf structure	antipode
set partitions	permutahedra	Joni-Rota	Joni-Rota
paths	associahedra	Joni-Rota, new	Haiman-Schmitt, new
graphs	graphic zonotopes	Schmitt	new, Humpert-Martin
matroids	matroid polytopes	Schmitt	new
posets	braid cones	Schmitt	new
submodular fns	gen. permutahedra	Derksen-Fink	new
hypergraphs	hg-polytopes	new	new
simplicial cxes	new: sc-polytopes	Benedetti et al	Benedetti et al
building sets	nestohedra	new, Grujić et al	new
weaves (graphs)	graph associahedra	new	new

Lots of interesting algebra and combinatorics.

6. Characters of Hopf monoids.

Let P be a Hopf monoid. A character ζ consists of maps

$$\zeta_I : \mathrm{P}[I] \to \mathbb{k}$$

which are multiplicative: for each $I = S \sqcup T$, $s \in P[S], t \in P[T]$:

$$\zeta_S(s)\zeta_T(t) = \zeta_I(s \cdot t).$$

Think: character = multiplicative function on our objects

Group of characters, inversion.

The group of characters of a Hopf monoid P:

Operation: For $p \in \mathrm{P}[I]$

$$\zeta_1 * \zeta_2(p) = \sum_{I = S \sqcup T} \zeta_1(p|_S)\zeta_2(p/_S)$$

Identity:

$$u(p) = \begin{cases} 1 & \text{if } p = 1 \in \mathrm{P}[\emptyset] \\ 0 & \text{otherwise.} \end{cases}$$

Inverses:

$$\zeta^{-1} = \zeta \circ \mathrm{S}$$

- This group is hard to describe in general.
- To understand it, we must understand the antipode.

Group of characters, reciprocity.

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operation: convolution (via coproduct) inverse: antipode

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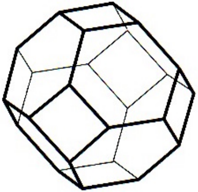
1. The Hopf monoid of permutahedra.

vertices: permutations
faces: ordered set partitions
(Schoute, 1911)



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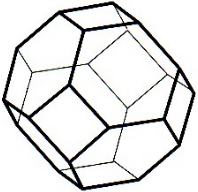
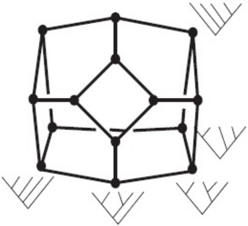


2. The Hopf monoid of associahedra.

vertices: binary parenthesizations of a word
faces: arbitrary parenthesizations
(Tamari 51, Stasheff-Milnor 63,
Haiman-Lee 89, Loday 02,...)



Group of characters, reciprocity for permutahedra.

1. Let Π = submonoid of GP generated by permutahedra.

Proposition. The group of characters of Π is isomorphic to the group of exponential generating functions $1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots$ under multiplication.

Sketch of Proof.

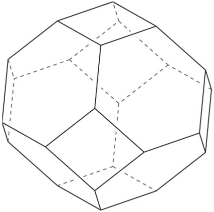
char. of $\Pi \leftrightarrow$ seq. $1, a_1, a_2, \ldots \leftrightarrow$ egf $A(x) = 1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots$

Group operation:

characters: $a * b(p) = \sum_{I = S \sqcup T} a(p|_S) b(p/_S)$

sequences: $c_n = \sum \binom{n}{k} a_k b_{n-k}$

power series: $C(x) = A(x)B(x)$



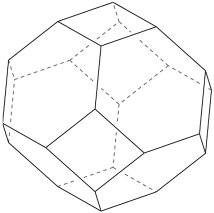
Group of characters, reciprocity for permutahedra.

Proposition. The group of characters of Π is the multiplicative group of exponential generating functions $1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots$.

inversion of egfs $\longleftrightarrow$ antipode of permutahedra

For exponential generating functions, $B(x) = 1/A(x)$ is given by

$$b_1 = -a_1$$
$$b_2 = -a_2 + 2a_1^2$$
$$b_3 = -a_3 + 6a_2 a_1 - 6a_1^3$$
$$b_4 = -a_4 + 8a_3 a_1 + 6a_2^2 - 36a_2 a_1^2 + 24a_1^4$$
$$\vdots$$



These numbers come from the antipode of the permutahedron:

$$\mathrm{S}(\pi_4) = -\pi_4 + 8\pi_3\pi_1 + 6\pi_2^2 - 36\pi_2\pi_1^2 + 24\pi_1^4$$

(1 perm., 8 hexagons and 6 squares, 36 segments, 24 points.)

Group of characters, reciprocity for associahedra.

2. Let A = submonoid of GP generated by Loday's associahedra.

> **Proposition.** The group of characters of A is isomorphic to the group of ordinary generating functions $x + a_1x^2 + a_2x^3 + \cdots$ under composition.

Sketch of Proof.

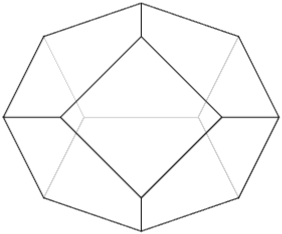
char. of $\Pi \leftrightarrow$ seq $1, a_1, a_2, \ldots \leftrightarrow$ ogf $A(x) = x + a_1x^2 + a_2x^3 + \cdots$

Group operation:

characters: $a * b(p) = \sum_{I=S\sqcup T} a(p|_S)b(p/_S)$

sequences: $c_n = \sum a_{k-1} b_{i_1} b_{i_2} \cdots b_{i_k}$

power series: $C(x) = A(B(x))$



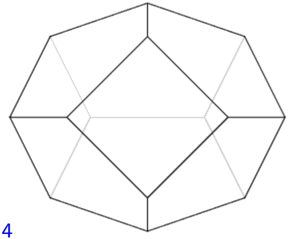
Group of characters, reciprocity for associahedra.

Proposition. The group of characters of Π is the compositional group of generating functions $x + a_1x^2 + a_2x^3 + \cdots$.

compositional inversion of gfs $\longleftrightarrow$ antipode of associahedra

For ordinary generating functions, $B(x) = A(x)^{\langle -1 \rangle}$ is given by

$$b_1 = -a_1$$
$$b_2 = -a_2 + 2a_1^2$$
$$b_3 = -a_3 + 5a_2a_1 - 5a_1^3$$
$$b_4 = -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4$$
$$\vdots$$



These numbers come from the antipode of the associahedron:

$$\mathrm{S}(\mathfrak{a}_4) = -\mathfrak{a}_4 + 6\mathfrak{a}_3\mathfrak{a}_1 + 3\mathfrak{a}_2^2 - 21\mathfrak{a}_2\mathfrak{a}_1^2 + 14\mathfrak{a}_1^4$$

(1 assoc., 6 pentagons and 3 squares, 21 segments, 14 points.)

Group of characters, reciprocity for associahedra.

This reformulation of the Lagrange inversion formula for

$$B(x) = A(x)^{\langle -1 \rangle}$$

may be seen as an answer to Loday's question:

> *"There exists a short operadic proof of the [Lagrange inversion] formula which explicitly involves the parenthesizings, but it would be interesting to find one which involves the topological structure of the associahedron."* *(Loday, 2005)*

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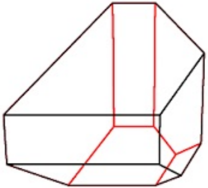
Project. (A.–Benedetti–González.)
Compute the group of characters and reciprocity rules for other interesting submonoids of GP.



Group of characters, reciprocity for associahedra.

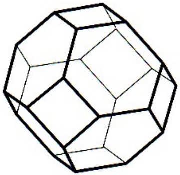
Two (of many) interesting enumerative consequences:

1. The Hopf monoid of permutahedra.

Recall: $\zeta(g)$ is 1 if g has no edges and 0 otherwise.

Proposition. (Aguiar–A.)
$(-1)^n \zeta^{-1}(K_n) = D_n$ (# of derangements)
(Conjectured by Humpert + Martin 12)

Key idea:
The polytope of K_n is the permutahedron π_n, so we can compute in the group of characters of Π.



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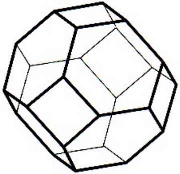
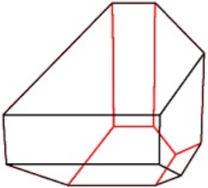


2. The Hopf monoid of associahedra.

Proposition. (Aguiar–A.)
The number of face parallelism classes of Loday's associahedron $\mathfrak{a}_n$ is Catalan # C_n.
Counted by dimension: Narayana numbers.



This follows from the antipode of $\mathfrak{a}_n$.

7. Polynomial invariants from characters.

Each character gives a polynomial invariant.

Let P be a Hopf monoid and ζ a character.
Define, for each $p \in \mathrm{P}[I]$ and $n \in \mathbb{N}$,

$$\chi(p)(n) := \sum_{S_1 \sqcup \cdots \sqcup S_n = I} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(p),$$

summing over all weak ordered set partitions $I = S_1 \sqcup \cdots \sqcup S_n$ (where S_i could be empty).

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summing over all weak ordered set partitions $I = S_1 \sqcup \cdots \sqcup S_n$ (where S_i could be empty).

Proposition.

1. $\chi(p)(n)$ is a polynomial function of n.
2. $\chi(p)(-n) = \chi\Big(\mathrm{S}(p)\Big)(n)$. (antipode → reciprocity thms)

$\chi(p)$: a polynomial invariant of the structure p.

Examples: Invariants of posets, graphs, matroids.

- Graphs:

$$\zeta(g) := \begin{cases} 1 & \text{if } g \text{ has no edges,} \\ 0 & \text{otherwise.} \end{cases}$$

$\chi(g)$ = chromatic polynomial of g. (Birkhoff, 1912).
For $n \in \mathbb{N}$, it counts proper colorings of g with $[n]$.

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$$\zeta(q) := \begin{cases} 1 & \text{if } q \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

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Goal 2. A unified approach to these and other results.

Generalized permutahedra: Posets, graphs, matroids.

We have Hopf submonoids:

$$\mathrm{P} \hookrightarrow \mathrm{GP}, \qquad \mathrm{G} \hookrightarrow \mathrm{GP}, \qquad \mathrm{M} \hookrightarrow \mathrm{GP}.$$

Theorem. (Aguiar–A., Billera–Jia–Reiner) The characters:

$$\zeta(q) \text{ (posets)}, \quad \zeta(g) \text{ (graphs)}, \quad \zeta(m) \text{ (matroids)}$$

are "shadows" of the same character on GP:

$$\zeta_I(P) := \begin{cases} 1 & \text{if } P \text{ is a point,} \\ 0 & \text{otherwise.} \end{cases}$$

The (strict order)/(chromatic)/(BJR matroid) polynomials:

$$\chi(q)(\mathrm{n}) \text{ (posets)}, \quad \chi(g)(\mathrm{n}) \text{ (graphs)}, \quad \chi(m)(\mathrm{n}) \text{ (matroids)}$$

are "shadows" of the same polynomial on GP:

$$\chi(P)(n) := \# \text{ of } P\text{-generic functions } f : I \to [n]$$

Reciprocity.

For any character ζ and associated polynomial $\chi(P)$ on GP,

$$\chi(P)(-1) = \zeta\Big(\mathrm{s}(P)\Big) \qquad \chi(P)(-n) = \chi\Big(\mathrm{s}(P)\Big)(n).$$

Corollary. For $\zeta_I(P) = 1$ (if P is a point) or 0 (otherwise),

$$\chi(P)(-1) = (-1)^{|I|}\,\#\{\text{vertices of } P\}.$$

This gives a unified explanation of:

Graphs: $\chi(g)$ = chromatic polynomial
$\chi(g)(-1) = (-1)^{|I|}\,\#\{\text{acyclic orientations of } g\}$ (Stanley).

Posets: $\chi(q)$ = strict order polynomial ; $\Omega(q)$ = order poly.
$\chi(q)(-n) = (-1)^{|I|}\Omega(q)(n)$ (Stanley).

Matroids: $\chi(m)$ = BJR matroid polynomial
$\chi(m)(-1) = (-1)^{|I|}\,\#\{\text{bases of } m\}$ (Billera-Jia-Reiner).

8. A current direction: the polytope algebra.

- In the polytope algebra:

The antipode of P is

$$\mathrm{s}(P) \;=\; (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim Q}\, Q = -P^o$$

where P^o = rel. interior of P. An involution! (Ehrhart,McMullen)

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- Brion's theorem: $P = \displaystyle\sum_{v \text{ vertex}} \mathrm{cone}_v(P)$

Theorem. (Aguiar – A.) There is a Brion morphism $B : \mathrm{GP} \to \mathrm{P}$

$$P \mapsto \sum_{v \text{ vertex}} \mathrm{poset}_v(P).$$

This restricts to B: W (weaves) $\longrightarrow$ Connes-Kreimer (rooted trees).
Several interesting algebraic and combinatorial consequences.

Current and future directions.

For GP and its submonoids P:

- Describe the Lie algebra $\mathcal{P}(P^*)$ of primitive elements of P*, which determines P via Cartier–Milnor–Moore.

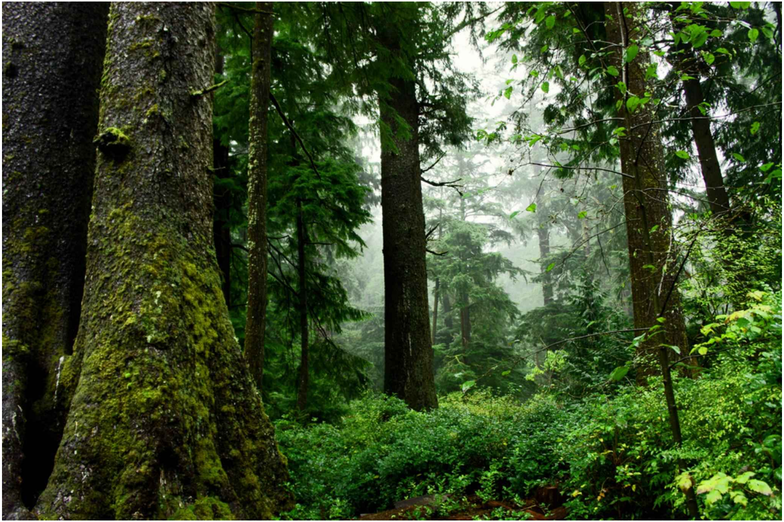
- Describe the Brion map $B^* : \mathcal{P}(P^*) \to \mathcal{P}(GP^*)$ of Lie algebras.

- Extend this to deformations GP_W of Coxeter permutahedra π_W, apply them to Coxeter associahedra, Coxeter matroids,... This requires extending Hopf monoids to type W. (Aguiar–Mahajan)

- Extend to deformations of any simple polytope. This requires extending Hopf monoids to any arrangement. (Aguiar–Mahajan)

- Study the connection with polyhedral valuations on generalized permutahedra. (Derksen–Fink)

¡Muchas gracias!



The preprints are coming soon!