Consider a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \).

A Gelfand-Tsetlin pattern for \( \lambda \) (or simply a GT(\( \lambda \)) pattern) is an array of integers satisfying the inequalities on the left panel of Figure 1 below.

A Feigin-Fourier-Littelmann-Vinberg pattern for \( \lambda \) (or simply a FFLV(\( \lambda \)) pattern) is an array of non-negative integers such that, for any Dyck path from \( \lambda_1 \) to \( \lambda_n \), the sum of the numbers along the path is at most \( \lambda_1 - \lambda_n \).

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \]

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \]

Figure 1: Left: GT patterns. Right: FFLV patterns.

\[ \text{Theorem 1. (ABS '10, Feigin-Fourier-Littelmann '10) (Conjecture: Vinberg '05) } \]

The number of GT(\( \lambda \)) patterns equals the number of FFLV(\( \lambda \)) patterns.

\[ \text{Motivation} \]

This work is rooted in the representation theory of the special linear Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) = \{ n \times n \text{ matrices of trace 0} \} \). \[ [A, B] = AB - BA \]

The irreducible representations \( V(\lambda) \) of \( \mathfrak{sl}_n(\mathbb{C}) \) are in bijection with partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) (modulo addition of \( (1, 1, \ldots, 1) \)).

- (1950) Gelfand and Tsetlin constructed a basis for \( V(\lambda) \) indexed by the GT patterns for \( \lambda \). Therefore

\[ \dim V(\lambda) = \text{number of GT(\( \lambda \))-patterns}. \]

- (2005) Vinberg proposed a conjectural construction of a basis for \( V(\lambda) \) indexed by the FFLV patterns for \( \lambda \).

- (2010) Feigin, Fourier, and Littelmann proved that Vinberg’s conjectural basis is independent and spanning. (via two subtle algebraic arguments.) Thus

\[ \dim V(\lambda) = \text{number of FFLV(\( \lambda \))-patterns} \]

- (2010) We found a combinatorial/discrete geometric explanation for the number of GT(\( \lambda \))-patterns = number of FFLV(\( \lambda \))-patterns.

\[ \text{GT and FFLV Patterns} \]

We generalize to the context of marked posets, extending work of Stanley (1986).

A marked poset \( (P, A, \lambda) \) is

\[ \begin{align*}
&\bullet \text{ a poset } P, \\
&\bullet \text{ a subset } A \subseteq P \text{ containing all extremal elements of } P, \text{ and} \\
&\bullet \text{ a vector } \lambda \in \mathbb{Z}^n \text{ such that } \lambda_i \leq \lambda_j \text{ for } p \leq q.
\end{align*} \]

The marked order polytope of \( (P, A, \lambda) \) is

\[ O(P, A, \lambda) = \{ x \in \mathbb{R}^{P-A} | x_p \leq x_q \text{ for } p < q, \ x_a \leq x_p \text{ for } a < p, \ x_p \leq \lambda_x \text{ for } p < a \}, \]

where \( p \) and \( q \) represent elements of \( P-A \), and \( a \) represents an element of \( A \).

The marked chain polytope of \( (P, A, \lambda) \) is

\[ C(P, A, \lambda) = \{ x \in \mathbb{R}_{\geq 0}^{P-A} | x_{p_1} + \cdots + x_{p_k} \leq \lambda - \lambda_0 \text{ for } a < p_1 < \cdots < p_k < b \}, \]

where \( a, b \) represent elements of \( A \), and \( p_1, \ldots, p_k \) represent elements of \( P-A \).

\[ \text{Theorem 2. (ABS '10) The polytopes } \phi(P, A, \lambda) \text{ and } \psi(P, A, \lambda) \text{ have the same Ehrhart polynomial. In particular, they have the same number of lattice points. They are very different combinatorially! We give a piecewise linear bijection.} \]

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\[ \text{The bijection} \]

Stanley (1986) proved the analogous result for the order and chain polytopes of an unmarked poset. Essentially the same bijection works here:

\[ \phi: \phi(P, A, \lambda) \rightarrow \psi(P, A, \lambda), \quad \phi(x) = x_{max} \leq \max\{x_q | q < p\} \]

The bijection shows that \( \psi(P, A, \lambda) \) is integral. (non-trivial!)

\[ \text{Back to representation theory} \]

\[ \begin{align*}
&\bullet \text{ \( \mathfrak{sp}_{2n+1} \): For the marked poset on the right,} \\
&\text{ lattice points of } \phi(P, A, \lambda) = GT(\lambda) \text{ patterns,} \\
&\text{ lattice points of } \psi(P, A, \lambda) = FFLV(\lambda) \text{ patterns.} \\
&\text{So Theorem 2 implies Theorem 1.} \\
&\text{Question 1. What does this bijection say about the change of basis between the} \\
&\text{GT and FFLV bases of } V(\lambda)? \\
&\text{Question 2. The bijection induces natural subdivisions on } \phi(P, A, \lambda) \text{ and } \psi(P, A, \lambda). \\
&\text{Do these have an algebraic meaning?} \\
&\bullet \text{ \( \mathfrak{sp}_{2n+1} \): Berenstein-Zelevinsky (1989) constructed bases for the} \\
&\text{irreducible representations of all semisimple Lie algebras in terms of “generalized Gelfand-Tsetlin patterns”.} \\
&\text{For the marked poset on the right,} \\
&\text{ lattice points of } \phi(P, A, \lambda) = GT(\lambda) \text{ patterns of } \mathfrak{sp}_{2n+1}. \\
&\text{Our bijection gives a definition of “FFLV patterns of } \mathfrak{sp}_{2n+1} \text{”, and:} \\
&\text{Theorem 3. (Feigin-Fourier-Littelmann '11) (Conjecture: ABS '10) } \text{There is a FFLV basis for } \mathfrak{sp}_{2n+1} \text{ indexed by the lattice points of } \psi(P, A, \lambda). \\
&\bullet \text{ \( \mathfrak{so}_{2n+1} \): The } \lambda(\lambda) \text{ patterns of } \mathfrak{so}_{2n+1} \text{ are some (non-lattice) points of } \phi(P, A, \lambda) \text{ as above. This suggests:} \\
&\text{Question 3. Is there a FFLV basis for } \mathfrak{so}_{2n+1} \text{ in terms of } \psi(P, A, \lambda)? \\
&\bullet \text{ \( \mathfrak{so}_{2n} \), etc.: The } \lambda(\lambda) \text{ patterns of type } D_n, E_6, E_7, E_8, F_4 \text{ do not seem to} \\
&\text{correspond to points in a marked order polytope. What can be done here?} \]