\[
\sum_{k=1}^{d-1} A(d-1,k) \left( k z^{k-1} + (d-k) z^k \right) \frac{1}{(1-z)^{d+1}} = \sum_{k=1}^{d} A(d,k) z^{k-1} \frac{1}{(1-z)^{d+1}}
\]

thus:
\[
\sum_{t \geq 0} (t+1)^d z^t = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}
\]

**EXC 3**

Every permutation of $[d]$ is obtained from a permutation of $[d-1]$ by introducing $d$ into one of the $d$ possible positions: at the beginning, at the end or in any of the $d-2$ spaces between the $d-1$ elements of the permutation of $[d-1]$. Since the permutations obtained in this way are $d!$, they are exactly the permutations of $[d]$ (no repetition).

Thus, we can count the number $E(d,k)$ of permutations of $[d]$ with $k$ descents by analyzing their relation to the permutations of $[d-1]$: Let $(n_1, n_2, ..., n_{d-1})$ be a permutation of $[d-1]$ with $t$ descents then there are 4 possible types of locations to introduce $d$: if we insert $d$ at the beginning, we get $(d, n_1, n_2, ..., n_{d-1})$ a permutation of $[d]$ with $t+1$ descents if we insert $d$ at the end, we get $(n_1, n_2, ..., n_{d-1}, d)$ a permutation of $[d]$ with $t$ descents if we insert $d$ in the middle of a descent $(n_i, n_j)$, we get $(d, n_1, n_2, ..., n_i, d, n_j, n_{d-1})$ a permutation of $[d]$ with $t$ descents (the descent $(n_i, n_j)$ converted into a non-descent $(n_i, d)$ and a descent $(d, n_j)$) if we insert $d$ in the middle of a non-descent $(n_i, n_j)$, we get $(d, n_1, n_2, ..., n_i, d, n_j, n_{d-1})$ a permutation of $[d]$ with $t+1$ descents (the non-descent $(n_i, n_j)$ converted into a non-descent $(n_i, d)$ and a descent $(d, n_j)$)

consequently, the permutations of $[d]$ with $k - 1$ descents, are obtained by: introducing $d$ into one of the $E(d-1, k-1)$ permutations of $[d-1]$ with $k-2$ descends in such a way that it rises the number of descents or by introducing $d$ into one of the $E(d-1, k)$ permutations of $[d-1]$ with $k-1$ descends in such a way that it conserves the number of descents.

There are $n-k+1$ ways of rising the number of descents of a permutation of $[d-1]$ with $k-2$ descents (introducing $d$ at the beginning or in the middle of one of the $(n-2) - (k-2)$ non-descents) There are $k$ ways of conserving the number of descents of a permutation of $[d-1]$ with $k-1$ descends (introducing $d$ at the end or in the middle of one of the $k-1$ descents.
since the number of descends either rises or stays the same when obtaining a permutation of \([d]\) from a permutation of \([d - 1]\) by introducing \(d\), these are the only ways of obtaining a permutation of \([d]\) with \(k - 1\) descends.

Thus, \(E(d, k) = (n - k + 1)E(d - 1, k - 1) + kE(d - 1, k)\)

Since \(E(1, 1) = 1 = A(11)\) and since \(E(d, k)\) and \(A(d, k)\) are defined by the same recursion, we have \(E(d, k) = A(d, k)\) for all integers \(1 \leq k \leq d\).

(this can be proofed easily by induction on \(d\), the base case is done, and \(E(d, k), A(d, k)\) only depend on \(d, k\) and the previous cases with a smaller \(d\))

To see \(A(d, k) = A(d, d + 1 - k)\) for all integers \(1 \leq k \leq d\), just notice that \(A(d, d + 1 - k)\) represents the number of permutations of \([d]\) with \(d - k\) descents. Since there is either a descent or an ascend between numbers of a permutation, the permutations with \(d - k\) descents are exactly those with \(k - 1\) ascends. Since there is a bijection ("taking it in reversed order") between the permutations with \(t\) ascends and the permutations with \(t\) descends, the number of the permutations with \(k - 1\) ascends is exactly the number of permutations with \(k - 1\) descends, that is \(A(d, k)\).

**EXC 4**

Let \(f : \mathbb{N} \to \mathbb{R}\) be a function and \(d \in \mathbb{N}\). Write

\[
\sum_{t \geq 0} f(t)z^t = \frac{g(z)}{(1 - z)^{d+1}}
\]

TFAE:

- \(f\) is a polynomial of degree \(d\)

- \(g\) is a polynomial of degree at most \(d\) such that \(g(1) \neq 0\).

First suppose \(f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0\) with \(a_d \neq 0\)

recall that

\[
\sum_{t \geq 0} (t + 1)^dz^t = \frac{A(d, 1)z^0 + \cdots + A(d, d)z^{d-1}}{(1 - z)^{d+1}}
\]

that is

\[
\sum_{t \geq 0} t^dz^t = \frac{A(d, 1)z^1 + \cdots + A(d, d)z^d}{(1 - z)^{d+1}}
\]