1. We have that $P = \triangle_{n-1} \times \triangle_1$ is a n dimension polytope which is constructed by taking two copies of $\triangle_{n-1}$ and connecting corresponding vertices together with a copy of $\triangle_1$. Let’s examine the case for $n = 3$. Let $\triangle_2 = \text{conv}\{v_1, v_2, v_3\}$ such that $v_1, v_2, v_3 \in \text{Vert}(\triangle_2)$ and $\triangle_2^* = \text{conv}\{w_1, w_2, w_3\}$ such that $w_1, w_2, w_3 \in \text{Vert}(\triangle_2^*)$ so each $v_i$ is connected to $w_i$ by a copy of $\triangle_1$. Two form a 3 dimensional simplex we need to choose 4 points of our polytope P such that three points are co-planer. By inspection we see that the simplex choices of the form $v_iv_kw_kw_l$ where each index has to be represented and we are choosing $\frac{4}{2}$ number of vertices from each 2 copy of $\triangle_2$ this forces the remaining simplices to be $v_iv_kw_kw_l$ and $w_l^2v_kw_l^3v_l$. Every index for this combination needs to be represented or all points woul lie on the same hyperplane and since we need four points we know that one index is going to be represented twice and this forces the hyperplane that contains 3 points to contains the points that correspond to this index as well. Once we choose that index value $k$ not equal to 4 for one of the $\triangle_2$ in P then we can’t choose the same index $k$ in the other copy because this would force all of the points to lie on the same hyperplane that contain $v_i$ and $w_i$ and $w_k$ therefore that last coordinate must be $v_l$. We can look at this as there being $\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right) = 3$ ways to pick two points from $\triangle_2$ which leaves $3 - \frac{3}{2} = 1$ remaining indices that have to be selected from $\triangle_2^*$, once we have chosen 3 vertices, 2 from $\triangle_2$ and one from $\triangle_2^*$ this leaves us with 3 - 1 number of vertices remaining in $\triangle_2^*$ that we can choose for our 4th point. So in total we have 3 x 2 = 6 number triangulations.

In general, for $P = \triangle_{n-1} \times \triangle_1$ we have that simplex selection which takes $\left\lfloor \frac{n+1}{2} \right\rfloor$ sets of vertices from $\triangle_{n-1}$ and $(n - \left\lfloor \frac{n+1}{2} \right\rfloor)$ sets of vertices in $\triangle_2$ such that each of the $n$ indices are represented and any single vertex in the remaining $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices of $\triangle_2^*$ will force a triangulation for P. Hence there are in total

$$\left(\frac{n}{2}\right) \times \left\lfloor \frac{n+1}{2} \right\rfloor$$

number of triangulations in total for $P = \triangle_{n-1} \times \triangle_1$. For this problem I worked with Nick.

2. Show that,

$$\sum_{t \geq 0}^{\infty} (t+1)^d z^t = \frac{A(d,1)z^0 + A(d,2)z^1 + \cdots A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

where $A(d,k) = (d-k+1)A(d-1,k-1) + kA(d-1,k)$ and $A(d,k) = 0$ if $k \leq 0$ or $k \geq d+1$ for all $1 \leq k \leq d$

Proof. When $d=0$ we get that $\sum_{t \geq 0}^{\infty} z^t = \frac{1}{(1-z)}$. When $d=1$ we have that

$$\sum_{t \geq 0}^{\infty} (t+1)^1 z^t = \sum_{t \geq 0}^{\infty} d \frac{dz}{dz} z^{t+1} = d \sum_{t \geq 0}^{\infty} z^{t+1} = d \frac{1}{(1-z)} = \frac{1}{(1-z)^2}$$

notice that $A(2,1)z^0 = (2A(1,0) + A(1,1))z^0 = (0 + A(1,1))z^0 = 1z^0 = 1$ which satisfies the equality above. We shall proceed by induction and assume true for the $d-1$ case and now we need to show that equality holds for $d.$

$$\sum_{t \geq 0}^{\infty} (t+1)^d z^t = \sum_{t \geq 0}^{\infty} (t+1)^{d-1} d \frac{dz}{dz} z^{t+1} = d \sum_{t \geq 0}^{\infty} (t+1)^{d-1} z^{t+1}$$
by our induction hypothesis we get, 
\[
\frac{d}{dz} \left( \frac{A(d-1,1)z^1 + A(d-1,2)z^2 + \cdots + A(d-1,d-1)z^{d-1}}{(1-z)^{d+1}} \right) = \frac{d}{dz} \left( \frac{A(d-1,1)z^1}{(1-z)^{d+1}} + \frac{A(d-1,2)z^2}{(1-z)^{d+1}} + \cdots + \frac{A(d-1,d-1)}{(1-z)^{d+1}} \right)
\]
where the derivative of each term is of the form,
\[
\frac{d}{dz} \left( \frac{A(d-1,i)z^i}{(1-z)^{d+1}} \right) = \frac{iA(d-1,i)z^{i-1} + (d-i)A(d-1,i)z^i}{(1-z)^d}
\]
for \(1 \leq i \leq d-1\). so the sum is of the form
\[
A(d-1,i)z^0 + \sum_{i=2}^{d-2} (A(d-1,i-1) + (d-i)A(d-1,i))z^{i-1} + ((d-1)A(d-1,d-1))z^{d-1}
\]
\[= \frac{A(d-1, k) z^{k-1}}{(1-z)^d}
\]
Notice that for \(l = k - 1\) we have that the coefficient of each \(z^i\) for \(1 \leq k \leq d\) is exactly of the form
\[
A(d,k)z^{k-1} = (d-k+1)A(d-1,k-1) + kA(d-1,k)
\]
Note that when \(k = 1, l = 0\) we have \(A(d,1) = dA(d-1,0) + A(d-1,1) = 0 + A(d-1,1)\) and for \(k = d, l = d-1\) we have \(A(d,d) = A(d-1,d-1) + dA(d-1,d) = A(d-1,d-1) + 0\). There we have the exact equality for \(d\) and so we are done. For this problem I worked with Ashley. 

3. Let \(E(d,k)\) be the number of permutations of \([d]\) having exactly \(k - 1\) descents. Prove that, for all \(1 \leq k \leq d\),
\[
E(d,k) = (d-k+1)E(d-1,k-1) + kE(d-1,k)
\]
conclude that \(E(d,k) = A(d,k)\) for all integers \(1 \leq k \leq d\).

\begin{proof}
We will denote a \(d\)-permutation by \(p = p_1p_2\cdots p_d\) with \(p_i\) being the \(i\)th entry in the linear order given by \(p\). There are two ways we can get an \(d\)-permutation \(p\) with \(k - 1\) descents from an \((d-1)\)-permutation \(p'\) by inserting the entry \(d\) into \(p'\). Either \(p\) has \(k - 1\) descents, and the insertion of \(d\) does not form a new descent, or \(p'\) has \(k - 2\) descents, and the insertion of \(d\) does form a new descent.

In the first case, we have to put the entry \(d\) at the end of \(p'\), or we have to insert \(d\) between two entries that form one of the \(d - k\) descents of \(p'\). This means we have \(d - k + 1\) choices for the position of \(d\). As we have \(E(d-1,k-1)\) choices for \(p'\), the first term of the right-hand side is explained.

In the second case we have to put the entry \(d\) at the front of \(p'\), or we have to insert \(d\) between two entries that form one of the \((d-1) - (d - k)\) ascents of \(p'\). This means that we have \(k\) choices for the position of \(d\). As we have \(E(d-1,k)\) choices for \(p'\), the second part of the right-hand side is explained, and the equality is proved.

Note that \(E(d,k) = E(d,d+1-k)\) since if \(p = p_1p_2\cdots p_d\) has \(k - 1\) descents then its reverse \(p^r = p_1^r p_2^r \cdots p_d^r\) has \(d - k + 1\) descents.
\end{proof}

4. Let \(f : \mathbb{N} \rightarrow \mathbb{R}\) be a function and \(d \in \mathbb{N}\). Write
\[
\sum_{t \geq 0} f(t)z^t = \frac{g(z)}{(1 - z)^{d+1}}
\]
Prove that the following are equivalent:
* \(f\) is a polynomial of degree \(d\)
* \(g\) is a polynomial of degree at most \(d\) such that \(g(1) \neq 0\).