i. The edges of the polytope that do not lie in \( x_{d+1} = 0 \) and \( \mathcal{H} \) join vertices of the form \( v \) and \( v' \).

ii. The facets of the new vertices are \( x_{d+1} \), \( \mathcal{H} \) and facets of \( P \) together with their projections except for \( F \).

Thus \( P' \) has \( n + 1 \) facets and is \( d + 1 \) dimensional. Note that the diameter of \( P' \) is at greater or equal than the diameter of \( P \), because in any path between two vertices we can forget about the distinction between \( v \) and \( v' \), and omit their edges, and we obtain a shorter path between corresponding vertices in \( P \). Thus \( \Delta(d, n) \leq \Delta(d + 1, n + 1) \).

(c) Now note that \( \Delta(d, n) \leq \Delta(d + k, n + k) \) for \( k \geq 0 \) by a direct induction and (b). Now \( 2d < n \) so setting \( k = n - 2d > 0 \), we get that \( \Delta(d, n) \leq \Delta(n - d, 2(n - d)) \), and the Hirsch difference is equal for both. We are done.

5. Enumerate the men and the women from 1 to \( n \). We have \( n! \) possible pairing and each pair induces a distinct permutation \( \pi \in S_n \) where \( \pi(i) = j \) if \( m_i \) and \( w_j \) are paired. Now \( S_n \) has a canonical representation as a group of matrices in \( \mathcal{M}_n(\mathbb{R}) \) given by \( a_{ij} = \delta_{\pi(i)j} \), where \( \delta \) is the kronecker function. Denote this matrix by \( A_\pi \) and let \( \mathcal{A} := \{ A_\pi | \pi \in S_n \} \). Let \( P = \text{conv}\mathcal{A} \). This polytope will be the one we use in the linear program. Now let \( C \in \mathcal{M}_{n \times n} \), so that \( C_{ij} \) is the profit of arranging \( m_i \) and \( w_j \). If we view this matrix as a vector in \( R^{n \times n} \), then \( C \cdot A_\pi \) is the profit made by the arrangement given by the permutation \( \pi \) (here the product is the usual coordinate to coordinate dot product, not the matrix product). Thus what we want is to maximize \( C \) in \( P \). The set of vertices of \( P \) is \( \mathcal{A} \), because \( A_\pi \) is maximized by itself in \( P \). Thus we have the \( V \) representation of \( P \).

It remains to find the \( \mathcal{H} \) representation of \( P \). We claim that \( X = (x_{ij}) \in P \) if and only if the following three conditions hold:

i. \( x_{ij} \geq 0 \) for \( i, j \in [n] \)

ii. \( \sum_{j=1}^{n} x_{ij} = 1 \) for every \( i \in [n] \)

iii. \( \sum_{i=1}^{n} x_{ij} = 1 \) for every \( j \in [n] \)

The last two conditions say that the rows and columns add up to 1. That any point in \( P \) satisfies the three conditions is obvious, because the vertices \( A_\pi \) satisfy them and all of them are preserved by convex combinations. To prove the other direction we first prove another result.

**Proposition 1:** Let \( X = (x_{ij}) \in \mathcal{M}_n(\mathbb{R}) \) be a matrix that satisfies i., ii. and iii.. Then it is possible to place \( n \) chess rooks on the board in such a way every rook lies in a coordinate with a positive number and no pair of rooks attack each other.

**Proof:** Consider a bipartite graph \( G \) with left vertices equal to \( m_1, m_2, \ldots, m_n \) and right vertices equal to \( k_1, k_2, \ldots, k_n \) and where an edge is drawn from \( m_i \) to \( k_j \) if and only \( x_{ij} > 0 \). The condition of the roots is equivalent to say that \( G \) has a matching, thus we have to show that Hall’s condition is satisfied. Let \( \mathcal{B} \) be set of left vertices of size \( k \) and let \( \mathcal{C} \) be the set of right vertices that are connected to some element of \( \mathcal{B} \). We have to show that \( |\mathcal{C}| \geq k \). Note that for a vertex \( m_i \) the sum of the numbers of numbers of the coordinates of the implied
edges is 1 (it is a complete row). It follows that the sum of the number of coordinates of the edges in $B$ is $k$. If $C \leq k - 1$ the the sum of the numbers of the edges entering in $C$ is at most $k - 14$ (ever column adds up to 1) and this is impossible since the implied edges are the same as for $B$ and by condition 1. Thus $|C| \geq k$ as desired.

We proceed to show the the assertion by strong induction on the number of positive entries of $X$. For the base case note that there are at least $n$ non-zero entries in the matrix, and if there are $n$ then the matrix is $A_\pi$ for some $\pi$. Assume the result is true for all matrices with less that $k$ positive entries and let $X$ be a matrix that satisfies the three given conditions. By the proposition we can place $n$ non-attacking rooks on positive entries. Assume that $r$ is the least entry under a rook. Thus we have that $X \geq rA_\pi$ where $\pi$ is the permutation associated to the position of the rooks. Also $(1 - r)^{-1}(X - rA_\pi)$ satisfies the three conditions and has less positive entries than $X$ (because the entry with an $r$ goes to zero, and no position with a zero is changed), thus $(1 - r)^{-1}(X - rA_\pi)$ is in $convA$. Also $X = (1 - r)((1 - r)^{-1}(X - rA_\pi)) + rA_\pi$ is a convex combination of elements in $A$ and is therefore a member of $A$, because $A$ is convex.

6. (a) The simples is 2-neighborly, so it’s diameter is 1. If the dimension is $d$ then it has $d + 1$ facets, so the Hirsch conjecture works.

(b) The vertices of the cube may be seen as $\{0, 1\}^d$. Two vertices of the cube are adjacent if and only if they differ by exactly one coordinate. Thus the distance between two vertices is exactly the number of distinct coordinates they have. This difference is at most $d$, because there are $d$ coordinates. The vectors $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ differ in all $d$ positions so it’s distance is $d$. It follows that the diameter of the cube is $d$. Now, the cube has exactly $2d$ facets corresponding to the hyperplanes given by $x_i = 0, 1$ for $i \in [d]$. So the Hirsch hypothesis is satisfy since we are in dimension $d$.

(c) Note that if we remove a vertex $\pm e_i$ from $Q_d$ we get a pyramid that consists of adding $\mp e_i$ to the other points that lie on the hyperplane $x_i = 0$. Thus if we choose to vertices that are not opposite, their distance is 1, because we delete the vertex that is opposite to the first we choose and we get a pyramid. Now the distance between $e_i$ and $-e_i$ is 2 (unless $d = 1$ that is a trivial case) because we can choose $j = i$ and go from $e_i$ to $e_j$ in one step and from $e_j$ to $-e_i$ in another step. $e_i$ and $-e_i$ are not connected by and edge, because their convex hull contains 0 that is an interior point of $Q_d$. Thus the diameter of the polytope is 2. It is easy to see that the the number of faces of $Q_d$ is $2^d$ by induction, thus the Hirsch conjecture is obviously satisfied since $2^d - d \geq 2$ for $d \geq 2$. The one dimensional case holds because it is a simplex.