1. (How neighborly can you be?) If a $d$-dimensional polytope is $\left(\frac{d}{2} + 1\right)$-neighborly, prove that it is a simplex. (Discussed with Nick) (Recall that a polytope is $k$-neighborly if any set of $k$ vertices forms a face.)

**Proof.** Let $P$ be a $d$-dimensional polytope that is $\left(\frac{d}{2} + 1\right)$-neighborly. Suppose for a contradiction that $P$ has a subset $V = \{v_1, \ldots, v_{d+2}\}$ consisting of $d+2$ vertices. Then, $V$ is affinely dependent, so there is $I$ such that $v_i \in V$ and $v_I = \sum_{i=1}^{d+2} \lambda_i v_i$ with $\sum_{i=1}^{d+2} \lambda_i = 1$. Hence, $0 = v_I - \sum_{i=1}^{d+2} \lambda_i v_i = v_I - \lambda_1 v_1 - \ldots - \lambda_{d+2} v_{d+2}$ and $1 - \lambda_1 - \ldots - \lambda_{d+2} = 0$. Since $\sum_{i=1}^{d+2} \lambda_i = 1$, not all $\lambda_i$ are zero. Let $\lambda'_I = 1 - \lambda_i$. Then, $\lambda_1 v_1 + \ldots + \lambda'_I v_I + \ldots + \lambda_{d+2} v_{d+2} = 0$ with $\lambda_1 + \ldots + \lambda'_I + \ldots + \lambda_{d+2} = 0$ and not all $\lambda_i$ are zero. Hence, there are nonzero $\lambda_j$ and $\lambda_k$ with opposite signs. Relabeling the $v_i$ and $\lambda_i$ if necessary, let $\lambda_1, \ldots, \lambda_n$ be nonnegative (i.e. $\geq 0$) and $\lambda_{n+1} \ldots \lambda_{d+2}$ be nonpositive (i.e. $\leq 0$). Since $\lambda_1 + \ldots + \lambda_n + \lambda_{n+1} + \ldots + \lambda_{d+2} = 0$, $\lambda_1 + \ldots + \lambda_n = -\lambda_{n+1} + \ldots + \lambda_{d+2}$. Let $\Lambda = \lambda_1 + \ldots + \lambda_n$. Then, $\Lambda > 0$ because at least one of $\lambda_1, \ldots, \lambda_n$ is nonzero. Hence, $1 = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i = -\frac{1}{\Lambda} \sum_{i=n+1}^{d+2} \lambda_i$, so $v = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} v_i = \sum_{i=n+1}^{d+2} (-\frac{\lambda_i}{\Lambda} v_i)$ is a convex combination of $v_1, \ldots, v_n$ and a convex combination of $v_{n+1}, \ldots, v_{d+2}$, so $v \in \text{conv}(v_1, \ldots, v_n)$ and $v \in \text{conv}(v_{n+1}, \ldots, v_{d+2})$, implying $v \in \text{conv}(v_1, \ldots, v_n) \cap \text{conv}(v_{n+1}, \ldots, v_{d+2})$. Let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{d+2}\}$. Then, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 \neq \emptyset$, and $\text{conv}(V_1) \cap \text{conv}(V_2) \neq \emptyset$. Hence, $V_1 \cap V_2 = \emptyset$, $|V| = |V_1| + |V_2|$, so renaming the sets if necessary, let $|V_1| \leq \left(\frac{d}{2}\right) + 1$. Since $\text{conv}(V_1) \cap \text{conv}(V_2) \neq \emptyset$, every hyperplane $H$ that contains $V_1$ and does not contain points in the interior of $P$, contains points from $\text{conv}(V_2)$, so $H$ also contains at least one vertex from $V_2$. Then, $\text{conv}(V_1)$ has vertices that are not in $V_1$, so $\text{conv}(V_1)$ does not define a face consisting of $|V_1|$ vertices, implying that $P$ is not $|V_1|$-neighborly, which is a contradiction to $P$ being $\left(\frac{d}{2} + 1\right)$-neighborly. Hence, $P$ has less than $d+2$ vertices. Since $P$ is $d$-dimensional, $P$ has $d+1$ vertices, so $P$ is a simplex. Therefore, if a $d$-dimensional polytope is $\left(\frac{d}{2} + 1\right)$-neighborly, then it is a simplex.

\[ \square \]

2. (The spanning tree polytope.) Let $G$ be a graph and $\text{ST}(G)$ its spanning tree polytope.

(a) Prove that $\chi_T$ is a vertex of $\text{ST}(G)$ for every spanning tree $T$ of $G$.

**Proof.** Let $T$ be a spanning tree of a graph $G$. Then, $\text{ST}(G) = \text{conv}(\chi_T \mid T$ is a spanning tree) $\subset \mathbb{R}^E$, where $E$ is the set of edges in $G$. Since $\chi_T \in \{0, 1\}^E$, $\{\chi_T \mid T$ is a spanning tree$\}$ is a subset of vertices of the unit cube $\text{conv}(\{0, 1\}^E)$. Hence, $\text{ST}(G)$ is the convex hull of a subset of vertices of the unit cube, so $\chi_T$ is a vertex of $\text{ST}(G)$. Therefore, $\chi_T$ is a vertex of $\text{ST}(G)$ for every spanning tree $T$ of $G$. \[ \square \]