9 edges In this case we have that \(v + f = 11\). Thus as in the previous cases \(\min\{v, f\} \geq 5\) thus \(v = 5, f = 6\) or \(v = 6, f = 5\). We deal first with the case \(f = 5\) and show that there is a unique 3-polytope. Now

\[
18 = 2e = \sum_f u(f)
\]

This shows that the largest face has more than three sides. Note that the largest size can’t be a hexagon, because we just have 6 vertices and the polytope is three dimensional. If the largest face has five sides, then the polytope is the pyramid of a pentagon that has 10 edges. Thus the maximum side is a quadrilateral. Then by the property of the sum there are three quadrilaterals. Now any to quadrilaterals have to share two vertices, because there are 6 vertices and they can’t share three vertices (or the 6 points would be on same plane). Two quadrilaterals give the six vertices so the resulting polytope is the one in the left hand side of Fig. 3. It is the unique polytopes with 9 edges and 5 faces. It’s dual is the left hand side of the figure and is the unique one with 6 vertices, because \(P^{\Delta^2} = P\) for every polytope, and distinct polytopes have distinct duals.

Figure 3: The 3-Polytopes with nine edges

3. (a) \(f_{\text{pyr}}(P) = (1 + x)f_P(x) + 1\). We assume that \(P \subset \mathbb{R}^e\) and for the construction of \(P\) we embed it into \(x_{e+1} = 0\). Recall that a face of a polytope is the convex hull of the vertices of \(P\) it contains. Let \(v\) be the vertex added to obtain \(\text{pyr}(P)\) and let \(F\) be a face of \(\text{pyr}(P)\). We study two cases:

- If \(v \notin F\) we claim that \(F\) is a face of \(P\). All the vertices of \(F\) lie in \(P\). Let \(c\) be a functional that is maximized by \(F\). Every point on \(P\) has it’s last coordinate equal to zero, so we can omit the last coordinate of \(c\) and view it as a functional in \(\mathbb{R}^e\) acting on \(P\). This shows that \(F\) must be a face of \(P\) since the values of \(c\) and the new functional coincide in all of \(P\).

- If \(v \in F\) we claim that the convex hull of the vertices of \(F\) different from \(v\) are a face of \(P\). We can assume that \(v \in \text{span}\{e_{e+1}\}\) by translating the whole \(\mathbb{R}^{e-1}\) (this gives an isometry, thus the combinatorial structure of \(P\) does not change, because the geometry of the figure does not change at all). Now let \(c\) be vector such that \(c \cdot \cdot\) is maximized by \(F\). Write \(c = c' + c^\perp\), where \(c'\) is in \(x_{e+1} = 0\) and \(c^\perp \in \text{span}(e_{e+1})\). Then for \(u\) a vertex in \(P\) we have that \(c \cdot u = c' \cdot u\) and \(c \cdot v = c^\perp \cdot v\). Thus \(c'\) maximizes exactly the vertices of \(F\) in \(P\) and we can view it as a vector in \(\mathbb{R}^e\) thus if we take the vertices of \(F\) that are different from \(v\), because \(c'\) maximizes them.
We showed that a face of \( \text{pyr}(P) \) is either a face of \( P \) or a pyramid of a face of \( P \). The dimension of a face increases by one if we take the pyramid. Thus - if \( d \geq 1 \) - the \( d \)-dimensional faces of \( \text{pyr}(P) \) are given by \( f_d + f_{d-1} \) where \( f_k \) denotes the \( k \) dimensional faces of \( P \). Since we added exactly one vertex we get that \( f_{\text{pyr}(P)}(x) = 1 + f_P(x) + x f_P(x) = f_P(x)(x + 1) + 1 \).

(b) \( f_{P \times Q}(x) = f_P(x) f_Q(x) \). We claim that \( F \) is a face of \( P \times Q \) if and only if \( F \) is of the form \( G \times H \) where \( G \) is a face of \( P \) and \( H \) is a face of \( Q \). If this is true then we are done, because the dimension of a product is the sum of the dimensions. Thus the answer follows immediately if we prove that the classification is well done.

If \( c \) is a linear functional in the space of \( P \times Q \), then we can view it as \( (c_p, c_q) \) where \( c_p \) is in the space of \( P \) and \( c_q \) is in the space of \( Q \). Thus the elements that maximize \( c \) \( P \times Q \) come from some element that maximizes \( c_p \) in the first coordinate and some element that maximizes \( c_q \) in \( Q \), that is, the face is the product of two faces.

If \( G \) and \( H \) are faces of \( P \) and \( Q \) respectively let \( c_G \) and \( c_H \) be functionals maximized by \( G \) and \( H \) respectively. Then \( (c_G, c_H) \) is maximized by \( G \times H \), since the coordinates in \( c_G \) only act on the part that comes from \( G \) and the same for \( H \) (it is a direct dot product).

4. Let \( c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}^n)^* \). The rearrangement inequality implies directly that if \( \sigma \) is a permutations of \( [n] \) such that \( c_{\sigma^{-1}(1)} \leq c_{\sigma^{-1}(2)} \leq \cdots \leq c_{\sigma^{-1}(n)} \), then it maximizes \( \sum \tau(i) c_i \) where \( \tau \) varies among the permutations of \( [n] \) and the converse is also true. Thus the vertices of the face \( P_c \) correspond to the permutations that order \( c \). Thus two functionals yield the same face if and only if the relative order between their coordinates is exactly the same. Assume now that there are \( k \) different values \( m_1 < m_2 < \cdots < m_k \) in the set that contains the coordinates. The last map induces an ordered partition \( A_1, A_2, \ldots, A_k \) of \( [n] \) given by the relation \( j \in A_i \) if and only if \( c_j = m_i \). We can easily recover the face we have from a given partition. We still have to check that de dimension of the face coincides with de \( n \) minus the size of the partition induced. Given a partition \( A_1, \ldots, A_{n-k} \), we take the vector \( x = (x_1, x_2, \ldots, x_n) \), in such a way that \( x_i = k \) if \( x_i \in A_k \). Let \( b_i = |\bigcup_{k \leq i} A_i| \). The affine hull of of the vertices is given by the vectors that satisfy the system of equations given by

\[
\sum_{j \in A_i} x_j = \sum_{j=b_{i-1}+1}^{b_i} j
\]

because those are exactly the equalities satisfied by the vertices. When we put this system into matrix form it is consistent (every row is non zero), every row has a pivot and every variable appears in exactly one row. Thus the number of free variables is the number of variables minus the number of rows, that is \( n - (n - k) = k \). This is the dimension of the affine space that solves the system. It follows that the face associated to the ordered partition with \( n - k \) elements is \( k \)-dimensional.

5. (a) We construct an inverse and since we are in finite sets the bijection is implied because both sets have the same size (they are equal) and the inverse implies injectivity. For a flag \( F \) in \( P \) consider \( F^{\Delta} \) to be \( F \) in the dual (the same path after inverting the face poset). Let \( Y F = (T F^{\Delta})^{\Delta} \). Then \( Y \) is an inverse for \( T \), because it is defining inductively to complete the intervals of height 2 downwards instead of upwards in the poset.