Lemma: \[ S(\alpha_n) = (-1)^n \alpha_n \] (A much nicer formula!)

Pf: Need \( S \times I = \alpha I = I \times \alpha \). Enough: check \( \alpha = \alpha_n \).

We need \( \sum \epsilon^{(i)} h_r \epsilon_{n-k} = \{0 \text{ if } n \leq 0 \} \) (Exercise) \( \blacksquare \)

Corollary: If \( \lambda \vdash n \), then \( S(h_n) = (-1)^n \alpha_n \)

Pf: \( S^2 = I \) hence. \( \blacksquare \)

Corollary: \( \{ h_n : \lambda \vdash n \} \) is a basis for \( \text{Sym}_n \)

Pf: Spanning: Let \( f \in \text{Sym}_n \)

\[ \Rightarrow S(f) = \sum_{\lambda \vdash n} \alpha_{\lambda} \epsilon_\lambda \] (\( \epsilon \) basis)

\[ \Rightarrow f = (-1)^n \sum_{\lambda \vdash n} \alpha_{\lambda} h_\lambda \]

Lin indep: If \( \sum_{\lambda \vdash n} \alpha_{\lambda} h_\lambda = 0 \) then \( \sum_{\lambda \vdash n} \alpha_{\lambda} \epsilon_\lambda = 0 \) applying \( S \) \( \blacksquare \)

Corollary: \( \text{Sym}_n = \text{IK}[h_1, h_2, \ldots] \)
Prop
\[ \Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k} \]

Pf
\[ \Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k} \]
\[ (-1)^n \Delta(h_n) = \Delta(S(e_n)) \]
\[ = \sum_{k=0}^{n} S(e_{n-k}) \otimes S(e_k) \]
\[ = \sum_{k=0}^{n} h_{n-k} \otimes h_k. \]

Corollary
The map \( e_\lambda \mapsto h_\lambda \) is an automorphism of Sym.

In fact we can say more.
There is a natural inner product on Sym,
\[ \langle e_\lambda, h_\mu \rangle = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases} \]
which allows us to identify Sym* with Sym, regarding \( h_\mu \) as a linear functional on Sym.

Thm Sym is a self-dual Hopf algebra.

Graded Dual
If \( H = \bigoplus_{n \geq 0} H_n \) is a graded Hopf algebra and \( \text{dim}(H_n) \) is finite for all \( n \), let
\[ H^* = \bigoplus_{n \geq 0} H_n^* \]
be the \underline{graded dual} of \( H \). It is a Hopf algebra as defined earlier on in the class.

The inner product on Sym then defines a dual Hopf algebra structure on Sym. But in fact, this precisely sends \( e_\lambda \mapsto h_\lambda \).

Proof
Unravelling defn, this is \( \langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle \)

Enough to show for \( 1 \}\\text{st}: f = M_\lambda, g = h_\mu, h = h_\nu \)
\[ \langle \Delta M_\lambda, h_\mu \otimes h_\nu \rangle = \langle M_\lambda, h_\mu h_\nu \rangle = \text{coeff of } M_\lambda \otimes M_\mu \text{ in } \Delta(h_\nu) \]
\[ = \begin{cases} 1 & \lambda + \mu = \nu \\ 0 & \text{otherwise} \end{cases} \]